Local existence and blow up of solutions for a logarithmic nonlinear viscoelastic wave equation with delay

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Abstract
In this work, we consider a logarithmic nonlinear viscoelastic wave equation with a delay term in a bounded domain. We obtain the local existence of solution by using the Faedo-Galerkin approximation. Then, under suitable conditions, we prove the blow up of solutions in finite time.

Keywords. Local existence, Blow-up, Logarithmic nonlinearity, Delay term.

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1. INTRODUCTION

In this paper, we are concerned with the following the logarithmic nonlinear viscoelastic wave equation with delay term

\[
\begin{aligned}
& u_{tt} - \Delta u + \int_0^t g(t-s) \Delta u(s) \, ds + \mu_1 u_t(x, t) + \mu_2 u_t(x, t-\tau) \\
& = u|u|^{p-2} \ln |u|^k, \quad x \in \Omega, \quad t > 0, \\
& u(x, t) = 0, \quad x \in \partial \Omega, \\
& u_t(x, t-\tau) - f_0(x, t-\tau), \quad \text{in } (0, \tau), \\
& u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega,
\end{aligned}
\]

(1.1)

where \( \Omega \) is a bounded domain of \( \mathbb{R}^n \), with a smooth boundary \( \partial \Omega \). \( p \geq 2, \ k, \ \mu_1 \) are positive constants, \( \mu_2 \) is a real number, \( \tau > 0 \) represents the time delay and the functions \( u_0, u_1, f_0 \) are the initial data to be specified later. This type of source term \( (u|u|^{p-2} \ln |u|^k) \) appears naturally in nuclear physics, optics, geophysics, supersymmetric and inflation cosmology (see [1, 3]).

In the absence of the source term \( (u|u|^{p-2} \ln |u|^k) \), the equation (1.1) reduces to the following viscoelastic wave equation

\[
u_{tt} - \Delta u + \int_0^t g(t-s) \Delta u(s) \, ds + \mu_1 u_t(x, t) + \mu_2 u_t(x, t-\tau) = 0.
\]

(1.2)
Kirane and Houari [5] studied the equation (1.2) with suitable initial-boundary value conditions. They obtained the well-posedness and the energy decay of solutions for the concerned problem, under the restriction $0 < \mu_2 < \mu_1$. Later, Dai and Yang [2] improved the result of [5] under weaker conditions.

In the absence of the viscoelastic term ($g = 0$), the equation (1.1) reduces to the following logarithmic nonlinear wave equation

$$u_{tt} - \Delta u + \mu_1 u_t(x, t) + \mu_2 u_t(x, t - \tau) = u |u|^{p-2} \ln |u|^k. \quad (1.3)$$

In [4], Kafini and Messaoudi studied the local existence and the blow up result of the equation (1.3). In [6], Nicaise and Pignotti considered the following wave equation with a linear damping and delay term inside the domain

$$u_{tt} - \Delta u + \mu_1 u_t(x, t) + \mu_2 u_t(x, t - \tau) = 0.$$

They obtained some stability results in the case $0 < \mu_2 < \mu_1$.

Yang et al. [9] introduced the following equation

$$u_{tt} - \Delta u + \int_0^t g(t - s) \Delta u(x, s) \, ds + \mu_1 u_t(x, t) + \mu_2 u_t(x, t - \tau(t)) = h(x), \quad (1.4)$$

to prove the global well-posedness and stability which generates a gradient system. Moreover, they presented the effect and balance between damping and time-varying delay.

The main purpose of this paper is to establish the local existence of solution by using the Faedo-Galerkin method and the sufficient conditions for the blow up to the logarithmic nonlinear viscoelastic wave equation with delay term.

This paper is organized as follows: In section 2, we recall some notations and hypotheses. In section 3 and 4, we state and prove our main result.

2. Preliminaries

In this section, we present some material for the proof of our result. As usual, $(\ldots)$ and $\|\ldots\|_p$ show the inner product in the space $L^2(\Omega)$ and the norm of the space $L^p(\Omega)$, respectively. For brevity, we denote $\|\ldots\|_2$ by $\|\ldots\|$.

The memory kernel $g(t) : [0, \infty) \to [0, \infty)$ is a nonincreasing and differentiable $C^1$ function satisfying

$$1 - \int_0^\infty g(s) \, ds = l > 0, \quad (2.1)$$

and that $\mu_1$ and $\mu_2$ satisfy

$$\tau |\mu_2| < \xi < \tau (2\mu_1 - |\mu_2|), \quad |\mu_2| < \mu_1. \quad (2.2)$$

We make the following extra assumptions on $g$

$$g(s) \geq 0, \quad g'(s) \leq 0, \quad \int_0^t g(s) \, ds < \frac{C|\mu_2|^2}{4kC_0} + \frac{p(1-a)-2}{2} \frac{2}{1 - \frac{1}{4\eta} - \frac{p(1-a)}{2}} \quad (2.3)$$
Let $B_p > 0$ be the constant satisfying
\[ \|v\|_p \leq B_p \|\nabla v\|_p, \quad \text{for } v \in H^1_0(\Omega). \]  
(2.4)

By using the direct calculations, we have
\[
\int_0^t g(t-s) \langle \nabla u(s), \nabla u_t(t) \rangle \, ds 
= -\frac{1}{2} g(t) \|\nabla u(t)\|^2 + \frac{1}{2} \left( g'(t) + \int_0^t g(s) \, ds \right) \|\nabla u(t)\|^2, 
\]
(2.5)

where
\[ (g\nabla u)(t) = \int_0^t g(t-s) \|\nabla u(t) - \nabla u(s)\|^2 \, ds. \]

As in [7], let us introduce the function
\[ z(x, \rho, t) = u_t(x, t-\tau \rho), \quad x \in \Omega, \ \rho \in (0, 1), \ t > 0. \]

So, we have
\[ \tau z_t(x, \rho, t) + z_p(x, \rho, t) = 0, \quad x \in \Omega, \ \rho \in (0, 1), \ t > 0. \]

Then, the problem (1.1) can be transformed as follows
\[
\begin{cases}
  u_{tt}(x, t) - \Delta u(x, t) + \int_0^t g(t-s) \Delta u(x, s) \, ds \\
  \quad + \mu_1 u_t(x, t) + \mu_2 z(x, 1, t) = 0, \quad x \in \Omega \times (0, 1) \\
  \tau z_t(x, \rho, t) + z_p(x, \rho, t) = 0, \quad x \in \Omega \times (0, 1) \\
  z(x, \rho, 0) = f_0(x, -\rho \tau), \quad x \in \Omega \times (0, 1) \\
  u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega.
\end{cases}
\]
(2.6)

For any regular solution of (2.6), we define the energy as
\[ E(t) = \frac{1}{2} \|u_t\|^2 + \frac{k}{p^2} \|u\|_p^p + \frac{1}{2} \left( g\nabla u)(t) + \frac{1}{2} \left( 1 - \int_0^t g(s) \, ds \right) \|\nabla u\|^2 \\
+ \frac{\xi}{2} \int_\Omega \int_0^1 |z(x, \rho, t)|^2 \, d\rho dx - \frac{1}{p} \int_\Omega |u|^p \ln |u|^k \, dx, \]
(2.7)

where
\[ (g\nabla u)(t) = \int_0^t g(t-\tau) \|v(t) - v(\tau)\|^2 \, d\tau. \]
We also set
\[ H(t) = -E(t) = \frac{1}{p} \int_{\Omega} |u|^p \ln |u|^k \, dx - \frac{1}{2} \|u_t\|^2 \]
\[ - \frac{1}{2} \left( 1 - \int_0^t g(s) \, ds \right) \|\nabla u\|^2 - \frac{k}{p} \|u\|^p_p \]
\[ - \frac{1}{2} \left( \frac{1}{2} \frac{1}{2} \int_0^t \int_{1}^{1} z(x, \rho, t)^2 \, d\rho \, dx \right) \]
to prove our main result.

The following lemma shows that the associated energy of the problem under the condition \( \mu_1 > |\mu_2| \) is decreasing.

**Lemma 2.1.** Let \( u \) be the solution of (2.6). Then, for some \( C_0 \geq 0 \),
\[ E'(t) \leq -C_0 \left( \int_{\Omega} \left( |u_t|^2 + |z(x, 1, t)|^2 \right) \, dx - (g' \circ \nabla u) (t) + g(t) \|\nabla u\|^2 \right) \leq 0, \tag{2.8} \]

**Proof.** Multiplying the first equation in (2.6) by \( u_t \) and integrating over \( \Omega \) and multiplying the second equation in (2.6) by \( (\xi/\tau) z \) and integrating over \((0, 1) \times \Omega \) with respect to \( \rho \) and \( x \) summing up, we get
\[ \frac{d}{dt} \left( \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \left( 1 - \int_0^t g(s) \, ds \right) \|\nabla u\|^2 \right) \]
\[ + \frac{\xi}{2} \int_{\Omega} \int_{0}^{1} z(x, \rho, t)^2 \, d\rho \, dx \]
\[ = -\mu_1 \int_{\Omega} |u_t|^2 \, dx + \frac{1}{2} (g' \circ \nabla u) (t) - \frac{1}{2} g(t) \|\nabla u\|^2 \]
\[ - \frac{\xi}{\tau} \int_{\Omega} \int_{0}^{1} zz_{\rho}(x, \rho, t) \, d\rho \, dx - \mu_2 \int_{\Omega} u_t z(x, 1, t) \, dx. \tag{2.9} \]

Now, we estimate the last two terms of the right-hand side in (2.9) as follows:
\[ -\frac{\xi}{\tau} \int_{\Omega} \int_{0}^{1} zz_{\rho}(x, \rho, t) \, d\rho \, dx = -\frac{\xi}{2\tau} \int_{\Omega} \int_{0}^{1} \frac{\partial}{\partial \rho} z^2 (x, \rho, t) \, d\rho \, dx \]
\[ = \frac{\xi}{2\tau} \int_{\Omega} \left( z^2 (x, 0, t) - z^2 (x, 1, t) \right) \, dx \]
\[ = \frac{\xi}{2\tau} \left( \int_{\Omega} |u_t|^2 \, dx - \int_{\Omega} z^2 (x, 1, t) \, dx \right) \]
and
\[ \mu_2 \int_{\Omega} u_t z(x, 1, t) \, dx \leq \frac{|\mu_2|}{2} \left( \int_{\Omega} |u_t|^2 \, dx + \int_{\Omega} |z(x, 1, t)|^2 \, dx \right). \]
Hence, we get
\[
\frac{dE(t)}{dt} \leq - \left( \mu_1 - \frac{\xi}{2\tau} - \frac{\|\mu_2\|}{2} \right) \int_{\Omega} |u_t|^2 \, dx
\]
\[
- \left( \frac{\xi}{2\tau} - \frac{\|\mu_2\|}{2} \right) \int_{\Omega} |z(x,1,t)|^2 \, dx
\]
\[
+ (g'\varphi u) (t) - g (t) \|\nabla u\|^2 .
\] (2.10)

By (2.2), for some \( C_0 > 0 \), we have
\[
E' (t) \leq - C_0 \left[ \int_{\Omega} \left( |u_t|^2 + |z(x,1,t)|^2 \right) \, dx \right] \leq 0.
\]

**Lemma 2.2.** [4]. There exists a positive constant \( C > 0 \) depending on \( \Omega \) only such that
\[
\left( \int_{\Omega} |u|^p \ln |u|^k \, dx \right)^{s/p} \leq C \left[ \int_{\Omega} |u|^p \ln |u|^k \, dx + \|\nabla u\|_2^2 \right],
\]
for any \( u \in \text{L}^{p+1} (\Omega) \) and \( 2 \leq s \leq p \), provided that \( \int_{\Omega} |u|^p \ln |u|^k \, dx \geq 0 \).

**Lemma 2.3.** [4]. There exists a positive constant \( C > 0 \) depending on \( \Omega \) only such that
\[
\|u\|_2^2 \leq C \left[ \left( \int_{\Omega} |u|^p \ln |u|^k \, dx \right)^{2/p} + \|\nabla u\|_2^2 \right],
\] (2.11)
provided that \( \int_{\Omega} |u|^p \ln |u|^k \, dx \geq 0 \).

**Lemma 2.4.** [4]. There exists a positive constant \( C > 0 \) depending on \( \Omega \) only such that
\[
\|u\|_p^s \leq C \left[ \|u\|_p^p + \|\nabla u\|_2^2 \right],
\] (2.12)
for any \( u \in \text{L}^p (\Omega) \) and \( 2 \leq s \leq p \).

### 3. Local existence

In this section, we show the following local existence of solution for the problem \((2.6)\).

**Theorem 3.1.** Assume that
\[
\mu_1 > \|\mu_2\|, u_0 \in H^2 (\Omega) \cap W, v_0 \in W \text{ and } f_0 \in \text{L}^2 (\Omega \times (0,1)),
\]
then there exists a unique solution \((u, z)\) of problem \((2.6)\) defined on \( \Omega \times (0,T) \) for some constant \( T > 0 \) satisfying
\[
u \in L^\infty (0,T; H^2 (\Omega) \cap W), \quad u_t \in L^\infty (0,T; W)
\]
where \((u,v) = \int_{\Omega} u(x) v(x) \, dx\) the scalar product in \( L^2 (\Omega) \) and
\[
W = \{ u \in H^2 (\Omega) ; u (0) = u_t (0) = 0 \}.
\]
We prove this theorem by Faedo-Galerkin’s method. We will give sufficient estimates for the solution of the problem (2.6) by using Faedo-Galerkin procedure. In the next step, we obtain approximate solution of the problem (2.6).

**The first estimate:** Let \( \{ \phi_y \}_{y=1}^{\infty} \) be a complete orthogonal system of \( W \) and \( W_m = \text{span} \{ \phi_1, \cdots, \phi_n \} \), for each \( m \in N \). Moreover, we define \( V_m = \text{span} \{ \psi_1, \cdots, \psi_m \} \), \( m \in N \) and we can find a set of bases \( \{ \psi_r(x, \rho) \}_{r=1}^{m} \), which is a subset of \( L^2(\Omega \times (0,1)) \) such that

\[
\psi_r(x, 0) = \phi_r(x), \quad 1 \leq r \leq m.
\]

Choosing \( \{ u_m \} \) and \( \{ v_m \} \) in \( W_m \) and \( \{ z_m \} \) in \( V_m \) such that \( u_m \to u_0 \) strongly in \( W \), \( v_m \to v_0 \) strongly in \( W \), and \( z_m \to f_0 \) strongly in \( L^2(\Omega \times (0,1)) \). We will seek approximates solution in the form

\[
u_m(x, t) = \sum_{r=1}^{m} \phi_r(x) g_r(t), \]

\[
z_m(x, \rho, t) = \sum_{r=1}^{m} \psi_k(x, \rho) h_r(t).
\]

We say that \( (u_m(t), z_m(t)) \) are solutions of the following problem,

\[
\begin{cases}
\int_{\Omega} u_{mt}(x) \phi_t dx - \int_{\Omega} \Delta u_m \phi_r dx + \int_{\Omega} \int_{0}^{t} g(t-s) \Delta u_m(x, s) \phi_r dsdx \\
+ \int_{\Omega} \mu_1 u_{mt} (x, t) \phi_r dx + \int_{\Omega} \mu_2 z_m(x, t) \phi_r dx \\
= \int_{\Omega} u_m |u_m|^{p-2} \ln |u_m| \phi_r dx, \quad \text{in} \quad \Omega \times (0, T) \\
\int_{\Omega} [\tau z_m(x, \rho, t) + \zeta_m(t)] \phi_r dx = 0, \quad \text{in} \quad \Omega \times (0, 1) \times (0, T) \\
z_m(x, \rho, 0) = f_0m(x, -\rho t), \quad \text{in} \quad \Omega \times (0, 1) \\
u_m(x, t) = 0, \quad \text{on} \quad \partial \Omega \times (0, 1) \\
u_m(x, 0) = u_{0m}(x), \quad u_{mt}(x, 0) = u_{1m}(x), \quad \text{in} \quad \Omega.
\end{cases}
\]

We obtain (3.1) has a unique solution \( g_r(t), h_r(t) \) defined on \( (0, T) \) by using the theories of ordinary differential equation. In the next step, we obtain a priori estimates for the solution of the problem (2.6).

**The second estimate:** Taking into consideration the initial boundary value conditions and multiplying the first equation of (3.1) by \( g'_{rm}(t) \), integrating over \( (0, t) \)
and using integration by parts, we get

\[
\frac{1}{2} \int_\Omega |u_{mt}|^2 \, dx + \frac{1}{2} \int_\Omega (g o \nabla u_m) (t) \, dx \\
+ \frac{1}{2} \int_\Omega \left( 1 - \int_0^t g (s) \, ds \right) |\nabla u_m|^2 \, dx \\
+ \int \frac{k}{\rho^2} |u_m|^p \, dx - \frac{1}{p} \int_\Omega |u_m|^p \ln |u_m|^k \, dt \\
+ \mu_1 \int_0^t |u_{mt}|^2 \, ds + \mu_2 \int_0^t \int_\Omega z_m (x, 1, s) u_{mt} (x, s) \, ds \, dx \\
- \frac{1}{2} \int_\Omega (g' o \nabla u_m) (t) \, dx + \frac{1}{2} \int_\Omega g (t) |\nabla u_m|^2 \, dx \\
= \frac{1}{2} \|u_{0m}\|^2 + \frac{k}{\rho^2} |u_{1m}|^p + \frac{1}{2} (g o \nabla u_{1m}) (t) \\
+ \frac{1}{2} \left( 1 - \int_0^t g (s) \, ds \right) |\nabla u_{1m}|^2 - \frac{1}{p} \int_\Omega |u_{1m}|^p \ln |u_{1m}|^k \, dx.
\] (3.2)

We suppose that the constant $\xi > 0$, multiplying the second equation of (3.1) by $(\xi/\tau) h_{rm} (t)$ and integrating over $(0, t) \times (0, 1)$, we obtain

\[
\frac{\xi}{2} \int_\Omega \int_0^1 |z (x, \rho, t)|^2 \, d\rho dx + \frac{\xi}{2} \int_0^t \int_\Omega \int_0^1 z_{m \rho} z_m (x, \rho, s) \, d\rho dx ds \\
= \frac{\xi}{2} |z_{0m}|^2_{L^2 (\Omega \times (0, 1))}.
\] (3.3)

Using the second term in the left-hand side of (3.3), we have

\[
\int_0^t \int_\Omega \int_0^1 z_{m \rho} z_m (x, \rho, s) \, d\rho dx ds \\
= \frac{1}{2} \int_0^t \int_\Omega \int_0^1 \frac{\partial}{\partial \rho} z_m^2 (x, \rho, s) \, d\rho dx ds \\
= \frac{1}{2} \int_0^t \int_\Omega [z_m^2 (x, 1, s) - z_m^2 (x, 0, s)] \, dx ds.
\] (3.4)

Both (3.2) and (3.3) adding, using (3.4), we get

\[
E_m (0) = E_m (t) + \mu_1 \int_0^t |u_{mt}|^2 \, ds + \mu_2 \int_0^t \int_\Omega z_m (x, 1, s) u_{mt} (x, s) \, ds \, dx \\
- \frac{1}{2} \int_\Omega (g' o \nabla u_m) (t) \, dx + \frac{1}{2} \int_\Omega g (t) |\nabla u_m|^2 \, dx \\
+ \frac{\xi}{2} \int_0^t \int_\Omega [z_m^2 (x, 1, s) - z_m^2 (x, 0, s)] \, dx ds.
\] (3.5)
where
\[ E_m(t) = \frac{1}{2} \int_{\Omega} |u_m|^2 \, dx + \frac{1}{2} \int_{\Omega} (g' \nabla u_m) (t) \, dx \]
\[ + \frac{1}{2} \int_{\Omega} \left( 1 - \int_0^t g(s) \, ds \right) |\nabla u_m|^2 \, dx \]
\[ + \int_{\Omega} \frac{k}{p} |u_m|^p \, dx - \frac{1}{p} \int_{\Omega} |u_m|^p \ln |u_m|^k \, dx + \frac{\xi}{2} \|z_{0m}\|^2_{L^2(\Omega \times (0,1))}. \] (3.6)

Application the Young inequality and Sobolev Poincare inequality, we get
\[ E_m(t) + \left( \mu_1 - \frac{\xi}{2\tau} - \frac{1}{2} |\mu_2| \right) \int_{\Omega} |u_m (x,t)|^2 \, dx \]
\[ + \left( \frac{\xi}{2\tau} - \frac{1}{2} |\mu_2| \right) \int_{\Omega} |z_m (x,1,t)|^2 \, dx \]
\[ - \frac{1}{2} (g' \nabla u_m) (t) + \frac{1}{2} g(t) \|\nabla u_m\|^2 \]
\[ \leq E_m(0). \]

Moreover by choosing \( \tau |\mu_2| < \xi < \tau (2\mu_1 - |\mu_2|) \), we obtain
\[ D_0 = \mu_1 - \frac{\xi}{2\tau} - \frac{1}{2} |\mu_2| > 0, \quad D_1 = \frac{\xi}{2\tau} - \frac{1}{2} |\mu_2| > 0; \]
by this way, we get
\[ E_m(t) + D_0 \int_{\Omega} |u_m (x,t)|^2 \, dx + D_1 \int_{\Omega} |z_m (x,1,t)|^2 \, dx \]
\[ - \frac{1}{2} (g' \nabla u_m) (t) + \frac{1}{2} g(t) \|\nabla u_m\|^2 \]
\[ \leq E_m(0). \] (3.7)

Because of the sequence \( \{u_{0m}\}, \{u_{0m}\}, \) and \( \{z_{0m}\} \) are convergent, we can get some positive constant \( K_\ast \) independent of \( m \) such that
\[ E_m(t) \leq K_\ast. \] (3.8)

By combining (3.6) and (3.8), we find
\[ \{u_m\} \quad \text{is bounded in} \quad L^\infty (0,T;W), \]
\[ \{u_{mt}\} \quad \text{is bounded in} \quad L^\infty (0,T;W), \]
\[ \{z_m\} \quad \text{is bounded in} \quad L^\infty (0,T;L^2 (\Omega) \times (0,1)). \]

For this reason, we finalize that
\[ \{u_m\} \to u \quad \text{weak star in} \quad L^\infty (0,T;W), \]
\[ \{u_{mt}\} \to u_t \quad \text{weak star in} \quad L^\infty (0,T;W), \]
\[ \{z_m\} \to z \quad \text{weak star in} \quad L^\infty (0,T;L^2 (\Omega) \times (0,1)). \]
We know that the embeddings $H^2(\Omega) \hookrightarrow H^1(\Omega) \hookrightarrow L^2(\Omega)$ are compact, from Aubin-Lions theorem (see [8]), we arrive that there exists a subsequence $\{u_i\}$ of $\{u_m\}$ such that

$$\{u_i\} \to u \text{ strongly in } L^2(0,T;H^1(\Omega)).$$

So, we get

$$\{u_i\} \to u \text{ strongly and a.e. on } \Omega \times (0,T).$$

Thus, the proof is completed. (For more details see [10]) □

4. Blow up

In this section, we investigate the blow up of the solutions in a finite time for the problem (2.6). For the blow up of solutions, we modified the method of [4].

**Theorem 4.1.** Assume that (2.2) and (2.3) hold. Let

$$\begin{align*}
2 < p < \frac{2(n-1)}{n-2}, & \quad \text{if } n \geq 3 \\
p > 2, & \quad \text{if } n = 1, 2,
\end{align*}$$

and $\eta < \frac{p(1-a)}{2}$. Suppose further that

$$E(0) = \frac{1}{2} \left( 1 - \int_0^t g(s) \, ds \right) \left\| \nabla u_0 \right\|^2 + \frac{1}{2} \left\| u_1 \right\|^2 + \frac{k}{p^2} \left\| u_0 \right\|_p^p - \frac{1}{p} \int_\Omega \left| u_0 \right|^{p \ln \left| u_0 \right|} \, dx$$

$$+ \frac{1}{2} \left( g(\nabla u_0)(t) + \frac{\eta}{2} \int_0^t \int_\Omega f_0^2(x,-\rho \tau) \, d\rho \, dx \right) < 0. \tag{4.1}$$

Then the solution of (2.6) blows up in finite time.

**Proof.** Recalling (2.8), we obtain

$$E(t) < E(0) < 0.$$ 

Therefore,

$$H'(t) = -E'(t) \geq C_0 \left[ \int_\Omega \left( |u_t|^2 + |z(x,1,t)|^2 \right) \, dx \right]$$

$$- (g'\nabla u)(t) + g(t) \| \nabla u \|^2 \geq C_0 \int_0^1 z^2(x,1,t) \, dx \geq 0, \tag{4.2}$$

and

$$0 < H(0) \leq H(t) \leq \frac{1}{p} \int_\Omega |u|^p \ln |u|^k \, dx \quad \text{for } t \geq 0, \tag{4.3}$$

We set

$$L(t) = H^{1-\alpha}(t) + \varepsilon \int_\Omega uu_t \, dx + \frac{\mu \varepsilon}{2} \int_\Omega u^2 \, dx, \quad t \geq 0,$$
where \( \varepsilon > 0 \) to be specified later and
\[
\frac{2(p-2)}{p^2} < \alpha < \frac{p-2}{2p} < 1.
\] (4.4)

A direct differentiation of \( L(t) \) gives
\[
L'(t) = (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon \| u_t \|^2 - \varepsilon \| \nabla u \|^2
\]
\[-\varepsilon \mu_2 \int \Omega uz(x,1,t) \, dx
\]
\[+\varepsilon \left( \int_0^t \int_0^t g(t-s) \nabla u(s) \nabla u(t) \, ds \, dx \right)
\]
\[+\varepsilon \int_0^t \| u_t \|^2 \ln |u|^k \, dx.
\] (4.5)

By \( \forall \delta > 0 \)
\[-\varepsilon \mu_2 \int \Omega uz(x,1,t) \, dx \leq \varepsilon |\mu_2| \left( \delta \int \Omega u^2 \, dx + \frac{1}{4\delta} \int \Omega z^2(x,1,t) \, dx \right),
\] (4.6)

and for some number \( \eta > 0 \),
\[
\int_0^t g(t-s) (\nabla u(s), \nabla u(t)) \, ds
\]
\[= \int_0^t g(t-s) (\nabla u(s) - \nabla u(t), \nabla u(t)) \, ds
\]
\[+ \int_0^t g(t-s) \| \nabla u(t) \|^2 \, ds
\]
\[\geq \left( 1 - \frac{1}{4\eta} \right) \int_0^t g(s) \, ds \| \nabla u(t) \|^2 - \eta (g \nabla u)(t),
\] (4.7)

by (4.5), we get
\[
L'(t) \geq \left[ (1-\alpha)H^{-\alpha}(t) - \frac{\varepsilon |\mu_2|}{4\delta C_0} \right] H'(t) + \varepsilon \| u_t \|^2
\]
\[-\varepsilon \left( 1 - \frac{1}{4\eta} \right) \int_0^t g(s) \, ds \| \nabla u \|^2
\]
\[-\varepsilon \eta (g \nabla u)(t) + \varepsilon \int_\Omega |u|^p \ln |u|^k \, dx - \varepsilon \delta |\mu_2| \| u \|^2.
\] (4.8)

By taking \( \delta \) so that \( |\mu_2|/4\delta C_0 = kH^{-\alpha}(t) \), for large \( k \) to be specified later and substituting in (4.8), we obtain
\[
L'(t) \geq [(1-\alpha) - \varepsilon k] H^{-\alpha}(t) H'(t) + \varepsilon \| u_t \|^2
\]
\[-\varepsilon \left[ 1 - \left( 1 - \frac{1}{4\eta} \right) \int_0^t g(s) \, ds \right] \| \nabla u \|^2
\]
\[-\varepsilon \eta (g \nabla u)(t) - \frac{\varepsilon |\mu_2|^2}{4kC_0} H^{\alpha}(t) \| u \|^2 + \varepsilon \int_\Omega |u|^p \ln |u|^k \, dx.
\]
For $0 < a < 1$, we get

$$L'(t) \geq [(1-a) - \varepsilon k] H^{-\alpha}(t) H'(t) + \varepsilon a \int_{\Omega} |u|^p \ln|u|^k \, dx$$

$$+ \varepsilon \frac{p(1-a)}{2} \|u_t\|^2$$

$$+ \varepsilon \left( \frac{p(1-a) - 2}{2} + \left( 1 - \frac{1}{4\eta} - \frac{p(1-a)}{2} \right) \int_0^t g(s) \, ds \right) \|\nabla u\|^2$$

$$+ \frac{\varepsilon k (1-a)}{p} \|u\|^p_p - \frac{\varepsilon |\mu_2|^2}{4kC_0} H^\alpha(t) \|u\|^2 + \varepsilon p(1-a) H(t)$$

$$+ \varepsilon \left( -\eta + \frac{p(1-a)}{2} \right) (g_0 \nabla u)(t)$$

$$+ \frac{\varepsilon (1-a) p \xi}{2} \int_0^1 (x, \rho, t) \, d\rho \, dx. \quad (4.9)$$

Using (2.11), (4.3) and Young’s inequality, we obtain

$$H^\alpha(t) \|u\|^2_2 \leq \left( \int_{\Omega} |u|^p \ln|u|^k \, dx \right)^\alpha \|u\|^2_2$$

$$\leq C \left[ \left( \int_{\Omega} |u|^p \ln|u|^k \, dx \right)^{\alpha^2/2/p} + \left( \int_{\Omega} |u|^p \ln|u|^k \, dx \right) \|\nabla u\|^4/2 \right]$$

$$\leq C \left[ \left( \int_{\Omega} |u|^p \ln|u|^k \, dx \right)^{(\alpha p + 2)/p} + \|\nabla u\|^2_2 + \left( \int_{\Omega} |u|^p \ln|u|^k \, dx \right)^{\alpha/(p-2)} \right]$$

By (4.4), we get

$$2 < \alpha p + 2 \leq p \text{ and } 2 < \frac{\alpha p^2}{p-2} \leq p.$$
Combining (4.9) and (4.10), we get

\[ L'(t) \geq [ (1 - \alpha) - \varepsilon k ] H^{-\alpha} (t) H'(t) + \varepsilon \left( a - \frac{\varepsilon |\mu_2|^2}{4kC_0} \right) \int_{\Omega} |u|^p \ln |u|^k \ dx \\
+ \varepsilon \left( \frac{p(1-a)}{2} - \frac{C|\mu_2|^2}{4kC_0} + \left( 1 - \frac{1}{4\eta} \right) \frac{p(1-a)}{2} \right) \int_{0}^{t} g(s) \ ds \| \nabla u \|^2 \\
+ \varepsilon \left( -\eta + \frac{p(1-a)}{2} \right) (go\nabla u)(t) + \frac{\varepsilon k(1-a)}{p} \| u \|^p_p \\
+ \frac{\varepsilon (1-a)}{2} \| u_t \|^2 + \varepsilon p(1-a) H(t) \\
+ \frac{\varepsilon (1-a) p\xi}{2} \int_{0}^{1} \int_{0}^{1} z^2 (x, \rho, t) \ d\rho dx. \]

(4.11)

Here, we choose \( a > 0 \) so small that

\[ \frac{p(1-a)}{2} > 0 \text{ and } \eta < \frac{p(1-a)}{2} \]

and \( k \) so large that

\[ \left\{ \begin{array}{l}
\frac{p(1-a)}{2} - \frac{C|\mu_2|^2}{4kC_0} > 0, \\
a - \frac{\varepsilon |\mu_2|^2}{4kC_0} > 0, \\
\int_{0}^{t} g(s) \ ds < \frac{C|\mu_2|^2}{4kC_0 - \frac{p(1-a)}{2}} \end{array} \right\}. \]

Once \( k \) and \( a \) are fixed, we choose \( \varepsilon \) so small so that

\[ (1 - \alpha) - \varepsilon k > 0, \]

\[ H(0) + \varepsilon \int_{\Omega} u_0 u_1 \ dx > 0. \]

Hence, for some \( \lambda > 0 \), the estimate (4.11) becomes

\[ L'(t) \geq \lambda \left[ H(t) + \| u \|^2 + \| \nabla u \|^2 + \| u \|^p_p + (go\nabla u)(t) \right] \\
+ \lambda \left[ \int_{\Omega} \int_{0}^{1} z^2 (x, \rho, t) \ d\rho dx + \int_{\Omega} |u|^p \ln |u|^k \ dx \right] \]

and

\[ L(t) \geq L(0) > 0, \ t \geq 0. \]

(4.13)

Next, using Hölder’s inequality and the embedding \( \| u \|^2 \leq C \| u \|^p \), we get

\[ \int_{\Omega} uu_t \ dx \leq \| u \|^2 \| u_t \|^2 \\
\leq C \| u \|^p_p \| u_t \|^2 \]

and exploiting Young’s inequality, we get

\[ \left| \int_{\Omega} uu_t \ dx \right|^{1/(1-\alpha)} \leq C \left( \| u \|^p/(1-\alpha) + \| u_t \|^\theta/(1-\alpha) \right), \]

(4.14)
for $1/\mu + 1/\theta = 1$. To be able to use Lemma 2.4, we take $\theta = 2 \left(1 - \alpha \right)$ which satisfies $
mu / \left(1 - \alpha \right) = 2 / \left(1 - 2\alpha \right) \leq p$. Thus, for $s = 2 / \left(1 - 2\alpha \right)$, estimate (4.14) yields

$$\left| \int_{\Omega} uu_t dx \right|^{1/(1-\alpha)} \leq C \left( \|u\|_p^s + \|u_t\|_2^2 \right).$$

Therefore, Lemma 2.4 gives

$$\left| \int_{\Omega} uu_t dx \right|^{1/(1-\alpha)} \leq C \left[ \|\nabla u\|_2^2 + \|u_t\|_2^2 + \|u\|_p^p \right]. \tag{4.15}$$

Hence,

$$L^{1/(1-\alpha)}(t) = \left( H^{1-\alpha}(t) + \varepsilon \int_{\Omega} uu_t dx + \frac{H_1\varepsilon}{2} \int_{\Omega} u^2 dx \right)^{1/(1-\alpha)} \leq C \left[ H(t) + \left| \int_{\Omega} uu_t dx \right|^{1/(1-\alpha)} + \|u\|_2^{2/(1-\alpha)} \right] \leq C \left[ H(t) + \left| \int_{\Omega} uu_t dx \right|^{1/(1-\alpha)} + \|u\|_p^{2/(1-\alpha)} \right] \leq C \left[ H(t) + \|\nabla u\|_2^2 + \|u_t\|_2^2 + \|u\|_p^p \right], \tag{4.16}$$

Combining (4.12) and (4.16), we obtain

$$L'(t) \geq \Lambda L^{1/(1-\alpha)}(t), \quad t \geq 0. \tag{4.17}$$

where $\Lambda$ is a positive constant depending only on $\lambda$ and $C$.

A simple integration of (4.17) over $(0, t)$ yields

$$L^{\alpha/(1-\alpha)}(t) \geq \frac{1}{L^{\alpha/(1-\alpha)}(0) - \Lambda \alpha t / (1-\alpha)}.$$

Hence, $L(t)$ blows up in time

$$T \leq T^* = \frac{1 - \alpha}{\Lambda \alpha L^{\alpha/(1-\alpha)}(0)}. \tag{5.1}$$

Consequently, the solution of problem (1.1) blows up in finite time $T^*$ and $T^* \leq \frac{1 - \alpha}{\Lambda \alpha L^{\alpha/(1-\alpha)}(0)}$. $\square$

5. Conclusions

In recent years, there has been published much work concerning the wave equation with constant delay or time-varying delay. However, to the best of our knowledge, there was no blow-up result for the logarithmic nonlinear viscoelastic wave equation with delay term. Firstly, we have been obtained the local existence result by using the Faedo-Galerkin approximation. Later, we have been proved that blow-up of solutions for problem (1.1) under the sufficient conditions in a bounded domain.
References