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Local existence and blow up of solutions for a logarithmic nonlinear viscoelastic wave equation with delay

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Abstract In this work, we consider a logarithmic nonlinear viscoelastic wave equation with a delay term in a bounded domain. We obtain the local existence of the solution by using the Faedo-Galerkin approximation. Then, under suitable conditions, we prove the blow up of solutions in finite time.

Keywords. Local existence, Blow-up, Logarithmic nonlinearity, Delay term.

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1. Introduction

In this paper, we are concerned with the following the logarithmic nonlinear viscoelastic wave equation with delay term

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\,\Delta u(s)\,ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) \\ \qquad\qquad = u|u|^{p-2}\ln|u|^k, \; x \in \Omega, \; t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), & \text{in } (0,\tau), \\ u(x,0) = u_0(x), \; u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of R^n, with a smooth boundary $\partial\Omega$. $p \geq 2$, k, μ_1 are positive constants, μ_2 is a real number, $\tau > 0$ represents the time delay and the functions u_0, u_1, and f_0 are the initial data to be specified later. This type of source term $(u|u|^{p-2}\ln|u|^k)$ appears naturally in nuclear physics, optics, geophysics, supersymmetric and inflation cosmology (see [1, 3]).

In the absence of the source term $(u|u|^{p-2}\ln|u|^k)$, the equation (1.1) reduces to the following viscoelastic wave equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\,\Delta u(s)\,ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0. \tag{1.2}$$

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Kirane and Houari [5] studied the equation (1.2) with suitable initial-boundary value conditions. They obtained the well-posedness and the energy decay of solutions for the concerned problem, under the restriction $0 < \mu_2 \leq \mu_1$. Later, Dai and Yang [2] improved the result of [5] under weaker conditions.

In the absence of the viscoelastic term ($g = 0$), the equation (1.1) reduces to the following logarithmic nonlinear wave equation

$$u_{tt} - \Delta u + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau) = u|u|^{p-2} \ln |u|^k. \tag{1.3}$$

In [4], Kafini and Messaoudi studied the local existence and the blow up result of the equation (1.3). In [7], Nicaise and Pignotti considered the following wave equation with a linear damping and delay term inside the domain

$$u_{tt} - \Delta u + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau) = 0.$$

They obtained some stability results in the case $0 < \mu_2 < \mu_1$.

Yang et al. [10] introduced the following equation

$$u_{tt} - \Delta u + \int_0^t g(t-s) \Delta u(x,s)\, ds + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau(t)) = h(x), \tag{1.4}$$

to prove the global well-posedness and stability which generates a gradient system. Moreover, they presented the effect and balance between damping and time-varying delay.

The main purpose of this paper is to establish the local existence of solution by using the Faedo-Galerkin method and the sufficient conditions for the blow up to the logarithmic nonlinear viscoelastic wave equation with delay term.

This paper is organized as follows: In section 2, we recall some notations and hypotheses. In sections 3 and 4, we state and prove our main result.

2. Preliminaries

In this section, we present some material for the proof of our result. As usual, $(.,.)$ and $\|.\|_p$ show the inner product in the space $L^2(\Omega)$ and the norm of the space $L^p(\Omega)$, respectively. For brevity, we denote $\|.\|_2$ by $\|.\|$.

The memory kernel $g(t) : [0,\infty) \to [0,\infty)$ is a nonincreasing and differentiable C^1 function satisfying

$$1 - \int_0^\infty g(s)\, ds = l > 0, \tag{2.1}$$

and that μ_1 and μ_2 satisfy

$$\tau|\mu_2| < \xi < \tau(2\mu_1 - |\mu_2|), \quad |\mu_2| < \mu_1. \tag{2.2}$$

We make the following extra assumptions on g

$$g(s) \geq 0,\ g'(s) \leq 0,\ \int_0^t g(s)\, ds < \frac{\frac{C|\mu_2|^2}{4kC_0} - \frac{p(1-a)-2}{2}}{1 - \frac{1}{4\eta} - \frac{p(1-a)}{2}}. \tag{2.3}$$

Let $B_p > 0$ be the constant satisfying

$$\|v\|_p \le B_p \|\nabla v\|_p, \text{ for } v \in H_0^1(\Omega). \tag{2.4}$$

By using the direct calculations, we have

$$\begin{aligned} & \int_0^t g(t-s)(\nabla u(s), \nabla u_t(t))\, ds \\ = & -\frac{1}{2} g(t) \|\nabla u(t)\|^2 + \frac{1}{2}(g' o \nabla u)(t) \\ & -\frac{1}{2}\frac{d}{dt}\left[(g o \nabla u)(t) - \left(\int_0^t g(s)\, ds\right)\|\nabla u(t)\|^2\right], \end{aligned} \tag{2.5}$$

where

$$(g o \nabla u)(t) = \int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|^2\, ds.$$

As in [8], let us introduce the function

$$z(x,\rho,t) = u_t(x, t-\tau\rho),\ x \in \Omega,\ \rho \in (0,1),\ t > 0.$$

So, we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0,\ x \in \Omega,\ \rho \in (0,1),\ t > 0.$$

Then, the problem (1.1) can be transformed as follows

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)\, ds \\ \quad + \mu_1 u_t(x,t) + \mu_2 z(x,1,t) \\ \quad = u(x,t)|u(x,t)|^{p-2}\ln|u(x,t)|^k, \text{ in } \Omega \times (0,\infty) \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad \text{in } \Omega \times (0,1) \times (0,\infty) \\ z(x,\rho,0) = f_0(x,-\rho\tau), \quad \text{in } \Omega \times (0,1) \\ u(x,t) = 0, \quad \text{on } \partial\Omega \times [0,1) \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad \text{in } \Omega. \end{cases} \tag{2.6}$$

For any regular solution of (2.6), we define the energy as

$$\begin{aligned} E(t) = & \frac{1}{2}\|u_t\|^2 + \frac{k}{p^2}\|u\|_p^p + \frac{1}{2}(g o \nabla u)(t) + \frac{1}{2}\left(1 - \int_0^t g(s)\, ds\right)\|\nabla u\|^2 \\ & + \frac{\xi}{2}\int_\Omega \int_0^1 |z(x,\rho,t)|^2\, d\rho dx - \frac{1}{p}\int_\Omega |u|^p \ln|u|^k\, dx, \end{aligned} \tag{2.7}$$

where

$$(g o v)(t) = \int_0^t g(t-\tau)\|v(t) - v(\tau)\|_2^2\, d\tau.$$

We also set

$$\begin{aligned} H(t) &= -E(t) = \frac{1}{p}\int_{\Omega} |u|^{p} \ln |u|^{k}\, dx - \frac{1}{2}\|u_t\|^2 \\ &\quad -\frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|^2 - \frac{k}{p^2}\|u\|_p^p \\ &\quad -\frac{1}{2}(go\nabla u)(t) - \frac{\xi}{2}\int_{\Omega}\int_0^1 |z(x,\rho,t)|^2\, d\rho dx \end{aligned}$$

to prove our main result.

The following lemma shows that the associated energy of the problem under the condition $\mu_1 > |\mu_2|$ is decreasing.

Lemma 2.1. *Let u be the solution of (2.6). Then, for some $C_0 \geq 0$,*

$$E'(t) \leq -C_0\left[\int_{\Omega}\left(|u_t|^2 + |z(x,1,t)|^2\right) dx - (g' o\nabla u)(t) + g(t)\|\nabla u\|^2\right] \leq 0. \tag{2.8}$$

Proof. Multiplying the first equation in (2.6) by u_t and integrating over Ω and multiplying the second equation in (2.6) by $(\xi/\tau)\, z$ and integrating over $(0,1)\times\Omega$ with respect to ρ and x summing up, we get

$$\begin{aligned} &\frac{d}{dt}\left(\begin{array}{c} \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|^2 \\ +\frac{1}{2}(go\nabla u)(t) + \frac{k}{p^2}\|u\|_p^p - \frac{1}{p}\int_{\Omega}|u|^p \ln|u|^k\, dx \end{array}\right) \\ &+\frac{\xi}{2}\frac{d}{dt}\int_{\Omega}\int_0^1 |z(x,\rho,t)|^2\, d\rho dx \\ = \; & -\mu_1\int_{\Omega}|u_t|^2\, dx + \frac{1}{2}(g' o\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u\|^2 \\ & -\frac{\xi}{\tau}\int_{\Omega}\int_0^1 z z_{\rho}(x,\rho,t)\, d\rho dx - \mu_2\int_{\Omega} u_t z(x,1,t)\, dx. \end{aligned} \tag{2.9}$$

Now, we estimate the last two terms of the right-hand side in (2.9) as follows:

$$\begin{aligned} -\frac{\xi}{\tau}\int_{\Omega}\int_0^1 z z_{\rho}(x,\rho,t)\, d\rho dx &= -\frac{\xi}{2\tau}\int_{\Omega}\int_0^1 \frac{\partial}{\partial\rho} z^2(x,\rho,t)\, d\rho dx \\ &= \frac{\xi}{2\tau}\int_{\Omega}\left(z^2(x,0,t) - z^2(x,1,t)\right) dx \\ &= \frac{\xi}{2\tau}\left(\int_{\Omega}|u_t|^2\, dx - \int_{\Omega} z^2(x,1,t)\, dx\right) \end{aligned}$$

and

$$-\mu_2\int_{\Omega} u_t z(x,1,t)\, dx \leq \frac{|\mu_2|}{2}\left(\int_{\Omega}|u_t|^2\, dx + \int_{\Omega}|z(x,1,t)|^2\, dx\right).$$

Hence, we get

$$\begin{aligned}\frac{dE(t)}{dt} \leq & -\left(\mu_1 - \frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\int_\Omega |u_t|^2\,dx \\ & -\left(\frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\int_\Omega |z(x,1,t)|^2\,dx \\ & +(g'o\nabla u)(t) - g(t)\|\nabla u\|^2 . \end{aligned} \tag{2.10}$$

By (2.2), for some $C_0 > 0$, we have

$$E'(t) \leq -C_0 \begin{bmatrix} \int_\Omega \left(|u_t|^2 + |z(x,1,t)|^2\,dx\right)dx \\ -(g'o\nabla u)(t) + g(t)\|\nabla u\|^2 \end{bmatrix} \leq 0.$$

□

Lemma 2.2. [4]. *There exists a positive constant $C > 0$ depending on Ω only such that*

$$\left(\int_\Omega |u|^p \ln|u|^k\,dx\right)^{s/p} \leq C\left[\int_\Omega |u|^p \ln|u|^k\,dx + \|\nabla u\|_2^2\right],$$

for any $u \in L^{p+1}(\Omega)$ and $2 \leq s \leq p$, provided that $\int_\Omega |u|^p \ln|u|^k\,dx \geq 0$.

Lemma 2.3. [4]. *There exists a positive constant $C > 0$ depending on Ω only such that*

$$\|u\|_2^2 \leq C\left[\left(\int_\Omega |u|^p \ln|u|^k\,dx\right)^{2/p} + \|\nabla u\|_2^{4/p}\right], \tag{2.11}$$

provided that $\int_\Omega |u|^p \ln|u|^k\,dx \geq 0$.

Lemma 2.4. [4]. *There exists a positive constant $C > 0$ depending on Ω only such that*

$$\|u\|_p^s \leq C\left[\|u\|_p^p + \|\nabla u\|_2^2\right], \tag{2.12}$$

for any $u \in L^p(\Omega)$ and $2 \leq s \leq p$.

3. Local existence

In this section, we show the following local existence of solution for the problem (2.6).

Theorem 3.1. *Assume that*

$$\mu_1 > |\mu_2|, u_0 \in H^2(\Omega) \cap W, v_0 \in W \quad \text{and} \quad f_0 \in L^2(\Omega \times (0,1)),$$

then there exists a unique solution (u, z) of problem (2.6) defined on $\Omega \times (0, T)$ for some constant $T > 0$ satisfying

$$u \in L^\infty\left(0, T; H^2(\Omega) \cap W\right), \quad u_t \in L^\infty(0, T; W)$$

where $(u, v) = \int_\Omega u(x)\,v(x)\,dx$ is the scalar product in $L^2(\Omega)$ and

$$W = \left\{u \in H^2(\Omega)\,; u(0) = u_t(0) = 0\right\}$$

is the closed subspace of $H^2(\Omega)$ endowed with the norm equivalent to the usual norm in $H^2(\Omega)$. The Poincare inequality; there exists some positive constant B such that, $\|u\|_p \leq B\|u_t\|$, $p \geq 2$, holds in W.

Proof. We prove this theorem by Faedo-Galerkin's method. We will give sufficient conditions that guarantee the local existence of the problem (2.6) by using Faedo-Galerkin procedure. In the next step, we obtain an approximate solution to the problem (2.6).

The first estimate: Let $\{\phi_y\}_{y=1}^{\infty}$ be a complete orthogonal system of W and $W_m = span\{\phi_1, \cdots, \phi_n\}$, for each $m \in N$.
Moreover, we define $V_m = span\{\psi_1, \cdots, \psi_m\}$, $m \in N$ and we can find a set of bases $\{\psi_r(x,\rho)\}_{r=1}^{m}$, which is a subset of $L^2(\Omega \times (0,1))$ such that

$$\psi_r(x,0) = \phi_r(x), \ 1 \leq r \leq m.$$

Choosing $\{u_{0m}\}$ and $\{v_{0m}\}$ in W_m and $\{z_{0m}\}$ in V_m such that $u_{0m} \to u_0$ strongly in W, $v_{0m} \to v_0$ strongly in W, and $z_{0n} \to f_0$ strongly in $L^2(\Omega \times (0,1))$. We will seek approximates solution in the form

$$u_m(x,t) = \sum_{r=1}^{m} \phi_r(x) g_{rm}(t),$$

$$z_m(x,\rho,t) = \sum_{r=1}^{m} \psi_k(x,\rho) h_{rm}(t).$$

We say that $(u_m(t), z_m(t))$ are solutions of the following problem,

$$\begin{cases} \int_\Omega u_{mtt}\phi_r dx - \int_\Omega \Delta u_m \phi_r dx + \int_\Omega \int_0^t g(t-s) \Delta u_m(x,s) \phi_r ds dx \\ \quad + \int_\Omega \mu_1 u_{mt}(x,t) \phi_r dx + \int_\Omega \mu_2 z_m(x,1,t) \phi_r dx \\ \quad = \int_\Omega u_m |u_m|^{p-2} \ln |u_m|^k \phi_r dx, \ \text{in } \Omega \times (0,T) \\ \int_\Omega [\tau z_{mt}(x,\rho,t) + z_{m\rho}(x,\rho,t)] \phi_r dx = 0, \quad \text{in } (0,1) \times (0,T) \\ z_m(x,\rho,0) = f_{0m}(x,-\rho\tau), & \text{in } \Omega \times (0,1) \\ u_m(x,t) = 0, & \text{on } \partial\Omega \times [0,1) \\ u_m(x,0) = u_{0m}(x), \ u_{mt}(x,0) = u_{1m}(x), & \text{in } \Omega. \end{cases} \tag{3.1}$$

We obtain (3.1) has a unique solution $(g_{rm}(t), h_{rm}(t))_{r=1}^{m}$ defined on $(0,T)$ by using the theories of ordinary differential equations. In the next step, we obtain a priori estimates for the solution of the problem (2.6).

The second estimate: Taking into consideration the initial boundary value conditions and multiplying the first equation of (3.1) by $g'_{rm}(t)$, integrating over $(0,t)$

and using integration by parts, we get

$$\begin{aligned}
& \frac{1}{2}\int_{\Omega}|u_{mt}|^{2}\,dx+\frac{1}{2}\int_{\Omega}(go\nabla u_{m})(t)\,dx \\
& +\frac{1}{2}\int_{\Omega}\left(1-\int_{0}^{t}g(s)\,ds\right)|\nabla u_{m}|^{2}\,dx \\
& +\frac{k}{p^{2}}\|u_{m}\|_{p}^{p}\,dx-\frac{1}{p}\int_{\Omega}|u_{m}|^{p}\ln|u_{m}|^{k}\,dx \\
& +\mu_{1}\int_{0}^{t}\|u_{mt}\|^{2}\,ds+\mu_{2}\int_{\Omega}\int_{0}^{t}z_{m}(x,1,s)\,u_{mt}(x,s)\,dsdx \\
& -\frac{1}{2}\int_{0}^{t}(g'o\nabla u_{m})(s)\,ds+\frac{1}{2}\int_{0}^{t}g(s)\|\nabla u_{m}\|^{2}\,ds \\
= & \frac{1}{2}\|u_{0m}\|^{2}+\frac{k}{p^{2}}\|u_{1m}\|_{p}^{p}+\frac{1}{2}(go\nabla u_{1m})(t) \\
& +\frac{1}{2}\left(1-\int_{0}^{t}g(s)\,ds\right)\|\nabla u_{1m}\|^{2}-\frac{1}{p}\int_{\Omega}|u_{1m}|^{p}\ln|u_{1m}|^{k}\,dx. \qquad (3.2)
\end{aligned}$$

For more details about logarithmic source terms, please see the work of Nadia et al [6] We suppose that the constant $\xi>0$, multiplying the second equation of (3.1) by $(\xi/\tau)\,h_{rm}(t)$ and integrating over $(0,t)\times(0,1)$, we obtain

$$\begin{aligned}
& \frac{\xi}{2}\int_{\Omega}\int_{0}^{1}|z_{m}(x,\rho,t)|^{2}\,d\rho dx+\frac{\xi}{\tau}\int_{0}^{t}\int_{\Omega}\int_{0}^{1}z_{m\rho}z_{m}(x,\rho,s)\,d\rho dxds \\
= & \frac{\xi}{2}\|z_{0m}\|_{L^{2}(\Omega\times(0,1))}^{2} \qquad (3.3)
\end{aligned}$$

Using the second term in the left-hand side of (3.3), we have

$$\begin{aligned}
& \int_{0}^{t}\int_{\Omega}\int_{0}^{1}z_{m\rho}z_{m}(x,\rho,s)\,d\rho dxds \\
= & \frac{1}{2}\int_{0}^{t}\int_{\Omega}\int_{0}^{1}\frac{\partial}{\partial\rho}z_{m}^{2}(x,\rho,s)\,d\rho dxds \\
= & \frac{1}{2}\int_{0}^{t}\int_{\Omega}\left[z_{m}^{2}(x,1,s)-z_{m}^{2}(x,0,s)\right]dxds. \qquad (3.4)
\end{aligned}$$

Adding (3.2) and (3.3) and using (3.4), we get

$$\begin{aligned}
E_{m}(0) = & E_{m}(t)+\mu_{1}\int_{0}^{t}\|u_{mt}\|^{2}\,ds+\mu_{2}\int_{\Omega}\int_{0}^{t}z_{m}(x,1,s)\,u_{mt}(x,s)\,dsdx \\
& -\frac{1}{2}\int_{0}^{t}(g'o\nabla u_{m})(s)\,ds+\frac{1}{2}\int_{0}^{t}g(s)\|\nabla u_{m}\|^{2}\,ds \\
& +\frac{\xi}{2\tau}\int_{0}^{t}\int_{\Omega}\left[z_{m}^{2}(x,1,s)-z_{m}^{2}(x,0,s)\right]dxds, \qquad (3.5)
\end{aligned}$$

where

$$\begin{aligned} E_m(t) &= \frac{1}{2}\int_\Omega |u_{mt}|^2 dx + \frac{1}{2}(go\nabla u_m)(t) \\ &+\frac{1}{2}\int_\Omega \left(1-\int_0^t g(s)\,ds\right)|\nabla u_m|^2 dx \\ &+\frac{k}{p^2}\|u_m\|_p^p - \frac{1}{p}\int_\Omega |u_m|^p \ln|u_m|^k dx + \frac{\xi}{2}\|z_m\|^2_{L^2(\Omega\times(0,1))}. \end{aligned} \tag{3.6}$$

Application the Young inequality and Sobolev Poincare inequality, we get

$$\begin{aligned} & E_m(t) + \left(\mu_1 - \frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\int_0^t |u_{mt}|^2 ds \\ & +\left(\frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\int_0^t\int_\Omega |z_m(x,1,s)|^2 dxds \\ & -\frac{1}{2}\int_0^t (g' o\nabla u_m)(s)\,ds + \frac{1}{2}\int_0^t g(s)\|\nabla u_m(s)\|^2 ds \\ \leq\; & E_m(0). \end{aligned}$$

Moreover by choosing $\tau|\mu_2| < \xi < \tau(2\mu_1 - |\mu_2|)$, we obtain

$$D_0 = \mu_1 - \frac{\xi}{2\tau} - \frac{|\mu_2|}{2} > 0, \;\; D_1 = \frac{\xi}{2\tau} - \frac{|\mu_2|}{2} > 0;$$

by this way, we get

$$\begin{aligned} & E_m(t) + D_0\int_0^t |u_{mt}|^2 ds + D_1\int_0^t\int_\Omega |z_m(x,1,s)|^2 dxds \\ & -\frac{1}{2}\int_0^t (g' o\nabla u_m)(s)\,ds + \frac{1}{2}\int_0^t g(s)\|\nabla u_m(s)\|^2 ds \\ \leq\; & E_m(0). \end{aligned} \tag{3.7}$$

Because the sequence $\{u_{0m}\}$, $\{v_{0m}\}$, and $\{z_{0m}\}$ are convergent, we can get some positive constant K_* independent of m such that

$$E_m(t) \leq K_*. \tag{3.8}$$

By combining (3.6) and (3.8), we find

$$\begin{aligned} &\{u_m\} \;\; \text{is bounded in} \;\; L^\infty(0,T;W), \\ &\{u_{mt}\} \;\; \text{is bounded in} \;\; L^\infty(0,T;W), \\ &\{z_m\} \;\; \text{is bounded in} \;\; L^\infty\left(0,T;L^2(\Omega)\times(0,1)\right). \end{aligned}$$

For this reason, we finalize that

$$\begin{aligned} &\{u_m\} \to u \;\; \text{weak star in } L^\infty(0,T;W), \\ &\{u_{mt}\} \to u_t \;\; \text{weak star in } L^\infty(0,T;W), \\ &\{z_m\} \to z \;\; \text{weak star in } L^\infty\left(0,T;L^2(\Omega)\times(0,1)\right). \end{aligned}$$

We know that the embeddings $H^2(\Omega) \hookrightarrow H^1(\Omega) \hookrightarrow L^2(\Omega)$ are compact, from Aubin-Lions theorem (see [9]), we arrive that there exists a subsequence $\{u_i\}$ of $\{u_m\}$ such that

$$\{u_i\} \to u \text{ strongly in } L^2\left(0,T;H^1(\Omega)\right).$$

So, we get

$$\{u_i\} \to u \text{ strongly and a.e. on } \Omega \times (0,T).$$

Thus, the proof is completed. (For more details see [11]) □

4. Blow up

In this section, we investigate the blow up of the solutions in a finite time for the problem (2.6). For the blow up of solutions, we modified the method of [4].

Theorem 4.1. *Assume that (2.2) and (2.3) hold. Let*

$$\begin{cases} 2 < p < \frac{2(n-1)}{n-2}, & \text{if } n \geq 3 \\ p > 2, & \text{if } n = 1,2, \end{cases}$$

and $\eta < \frac{p(1-a)}{2}$. Suppose further that

$$\begin{aligned} E(0) &= \frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\nabla u_0\|^2 + \frac{1}{2}\|u_1\|^2 \\ &\quad + \frac{k}{p^2}\|u_0\|_p^p - \frac{1}{p}\int_\Omega |u_0|^p \ln|u_0|^k\,dx \\ &\quad + \frac{1}{2}(go\nabla u_0)(t) + \frac{\xi}{2}\int_\Omega\int_0^1 f_0^2(x,-\rho\tau)\,d\rho dx \\ &< 0. \end{aligned} \tag{4.1}$$

Then the solution of (2.6) blows up in finite time.

Proof. Recalling (2.8), we obtain

$$E(t) \leq E(0) < 0.$$

Therefore,

$$\begin{aligned} H'(t) &= -E'(t) \geq C_0 \left[\begin{array}{c} \int_\Omega \left(|u_t|^2 + |z(x,1,t)|^2\right) dx \\ -(g'o\nabla u)(t) + g(t)\|\nabla u\|^2 \end{array} \right] \\ &\geq C_0 \int_0^1 z^2(x,1,t)\,dx \geq 0, \end{aligned} \tag{4.2}$$

and

$$0 < H(0) \leq H(t) \leq \frac{1}{p}\int_\Omega |u|^p \ln|u|^k\,dx. \tag{4.3}$$

We set

$$L(t) = H^{1-\alpha}(t) + \varepsilon\int_\Omega u u_t dx + \frac{\mu_1\varepsilon}{2}\int_\Omega u^2 dx,\ t \geq 0,$$

where $\varepsilon > 0$ to be specified later and

$$\frac{2(p-2)}{p^2} < \alpha < \frac{p-2}{2p} < 1. \tag{4.4}$$

A direct differentiation of $L(t)$ gives

$$\begin{aligned} L'(t) &= (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \|u_t\|^2 - \varepsilon \|\nabla u\|^2 \\ &\quad -\varepsilon\mu_2 \int_\Omega uz(x,1,t)\,dx \\ &\quad +\varepsilon \left(\int_\Omega \int_0^t g(t-s) \nabla u(s) \nabla u(t)\, ds dx \right) \\ &\quad +\varepsilon \int_\Omega |u|^p \ln |u|^k \,dx. \end{aligned} \tag{4.5}$$

Using $\forall \delta > 0$

$$-\varepsilon\mu_2 \int_\Omega uz(x,1,t)\,dx \le \varepsilon |\mu_2| \left(\delta \int_\Omega u^2 dx + \frac{1}{4\delta} \int_\Omega z^2(x,1,t)\,dx \right), \tag{4.6}$$

and for some number $\eta > 0$,

$$\begin{aligned} &\int_0^t g(t-s)(\nabla u(s), \nabla u(t))\,ds \\ = &\int_0^t g(t-s)(\nabla u(s) - \nabla u(t), \nabla u(t))\,ds \\ &+ \int_0^t g(t-s) \|\nabla u(t)\|^2 \,ds \\ \ge &\left(1 - \frac{1}{4\eta}\right) \int_0^t g(s)\,ds \|\nabla u(t)\|^2 - \eta (go\nabla u)(t), \end{aligned} \tag{4.7}$$

by (4.5), we get

$$\begin{aligned} L'(t) &\ge \left[(1-\alpha) H^{-\alpha}(t) - \frac{\varepsilon |\mu_2|}{4\delta C_0} \right] H'(t) + \varepsilon \|u_t\|_2^2 \\ &\quad -\varepsilon \left(1 - \left(1 - \frac{1}{4\eta}\right) \int_0^t g(s)\,ds \right) \|\nabla u\|^2 \\ &\quad -\varepsilon\eta (go\nabla u)(t) + \varepsilon \int_\Omega |u|^p \ln |u|^k \,dx - \varepsilon\delta |\mu_2| \|u\|^2. \end{aligned} \tag{4.8}$$

By taking δ so that $|\mu_2| / 4\delta C_0 = kH^{-\alpha}(t)$, for large k to be specified later and substituting in (4.8), we obtain

$$\begin{aligned} L'(t) &\ge [(1-\alpha) - \varepsilon k] H^{-\alpha}(t) H'(t) + \varepsilon \|u_t\|^2 \\ &\quad -\varepsilon \left[1 - \left(1 - \frac{1}{4\eta}\right) \int_0^t g(s)\,ds \right] \|\nabla u\|^2 \\ &\quad -\varepsilon\eta (go\nabla u)(t) - \frac{\varepsilon |\mu_2|^2}{4kC_0} H^\alpha(t) \|u\|^2 + \varepsilon \int_\Omega |u|^p \ln |u|^k \,dx. \end{aligned}$$

For $0 < a < 1$, we get

$$\begin{aligned} L'(t) &\geq [(1-\alpha)-\varepsilon k] H^{-\alpha}(t) H'(t) + \varepsilon a \int_\Omega |u|^p \ln |u|^k \, dx \\ &+\varepsilon \frac{p(1-a)+2}{2} \|u_t\|^2 \\ &+\varepsilon \left(\frac{p(1-a)-2}{2} + \left(1 - \frac{1}{4\eta} - \frac{p(1-a)}{2}\right) \int_0^t g(s)\, ds \right) \|\nabla u\|^2 \\ &+ \frac{\varepsilon k (1-a)}{p} \|u\|_p^p - \frac{\varepsilon |\mu_2|^2}{4kC_0} H^\alpha(t) \|u\|^2 + \varepsilon p (1-a) H(t) \\ &+\varepsilon \left(-\eta + \frac{p(1-a)}{2} \right) (go\nabla u)(t) \\ &+ \frac{\varepsilon (1-a) p\xi}{2} \int_\Omega \int_0^1 z^2(x,\rho,t)\, d\rho dx. \end{aligned} \tag{4.9}$$

Using (2.11), (4.3) and Young's inequality, we obtain

$$\begin{aligned} H^\alpha(t) \|u\|_2^2 &\leq \left(\int_\Omega |u|^p \ln |u|^k \, dx \right)^\alpha \|u\|_2^2 \\ &\leq C \left[\left(\int_\Omega |u|^p \ln |u|^k \, dx \right)^{\alpha + 2/p} + \left(\int_\Omega |u|^p \ln |u|^k \, dx \right)^\alpha \|\nabla u\|_2^{4/p} \right] \\ &\leq C \left[\left(\int_\Omega |u|^p \ln |u|^k \, dx \right)^{(p\alpha+2)/p} + \|\nabla u\|_2^2 + \left(\int_\Omega |u|^p \ln |u|^k \, dx \right)^{\alpha p/(p-2)} \right]. \end{aligned}$$

By (4.4), we get

$$2 < \alpha p + 2 \leq p \text{ and } 2 < \frac{\alpha p^2}{p-2} \leq p.$$

Therefore, Lemma 2.2 provides

$$H^\alpha(t) \|u\|_2^2 \leq C \left(\int_\Omega |u|^p \ln |u|^k \, dx + \|\nabla u\|_2^2 \right) \tag{4.10}$$

Combining (4.9) and (4.10), we get

$$\begin{aligned} L'(t) &\geq [(1-\alpha)-\varepsilon k] H^{-\alpha}(t) H'(t) + \varepsilon \left(a - \frac{\varepsilon |\mu_2|^2}{4kC_0} \right) \int_\Omega |u|^p \ln |u|^k \, dx \\ &+\varepsilon \left(\frac{p(1-a)-2}{2} - \frac{C|\mu_2|^2}{4kC_0} + \left(1 - \frac{1}{4\eta} - \frac{p(1-a)}{2}\right) \int_0^t g(s)\, ds \right) \|\nabla u\|^2 \\ &+\varepsilon \left(-\eta + \frac{p(1-a)}{2} \right) (go\nabla u)(t) + \frac{\varepsilon k (1-a)}{p} \|u\|_p^p \\ &+\varepsilon \frac{p(1-a)+2}{2} \|u_t\|^2 + \varepsilon p (1-a) H(t) \\ &+ \frac{\varepsilon (1-a) p\xi}{2} \int_\Omega \int_0^1 z^2(x,\rho,t)\, d\rho dx. \end{aligned} \tag{4.11}$$

Here, we choose $a > 0$ so small that

$$\frac{p(1-a)-2}{2} > 0 \text{ and } \eta < \frac{p(1-a)}{2}$$

and k so large that

$$\begin{cases} \frac{p(1-a)-2}{2} - \frac{C|\mu_2|^2}{4kC_0} > 0, \\ a - \frac{\varepsilon|\mu_2|^2}{4kC_0} > 0, \\ \int_0^t g(s)\,ds < \frac{\frac{C|\mu_2|^2}{4kC_0} - \frac{p(1-a)-2}{2}}{1 - \frac{1}{4\eta} - \frac{p(1-a)}{2}}. \end{cases}$$

Once k and a are fixed, we choose ε so small so that

$$(1-\alpha) - \varepsilon k > 0,$$

$$H(0) + \varepsilon \int_\Omega u_0 u_1 dx > 0.$$

Hence, for some $\lambda > 0$, the estimate (4.11) becomes

$$\begin{aligned} L'(t) \geq\ & \lambda \left[H(t) + \|u_t\|^2 + \|\nabla u\|^2 + \|u\|_p^p + (go\nabla u)(t) \right] \\ & + \lambda \left[\int_\Omega \int_0^1 z^2(x,\rho,t)\,d\rho dx + \int_\Omega |u|^p \ln |u|^k\,dx \right] \end{aligned} \tag{4.12}$$

and

$$L(t) \geq L(0) > 0, \ t \geq 0. \tag{4.13}$$

Next, using Hölder's inequality and the embedding $\|u\|_2 \leq C\|u\|_p$, we get

$$\begin{aligned} \int_\Omega u u_t dx \leq\ & \|u\|_2 \|u_t\|_2 \\ \leq\ & C\|u\|_p \|u_t\|_2 \end{aligned}$$

and exploiting Young's inequality, we get

$$\left| \int_\Omega u u_t dx \right|^{1/(1-\alpha)} \leq C\left(\|u\|_p^{\mu/(1-\alpha)} + \|u_t\|_2^{\theta/(1-\alpha)} \right), \tag{4.14}$$

for $1/\mu + 1/\theta = 1$. To be able to use Lemma 2.4, we take $\theta = 2(1-\alpha)$ which satisfies $\mu/(1-\alpha) = 2/(1-2\alpha) \leq p$. Thus, for $s = 2/(1-2\alpha)$, estimate (4.14) yields

$$\left| \int_\Omega u u_t dx \right|^{1/(1-\alpha)} \leq C\left(\|u\|_p^s + \|u_t\|_2^2 \right).$$

Therefore, Lemma 2.4 gives

$$\left| \int_\Omega u u_t dx \right|^{1/(1-\alpha)} \leq C\left[\|\nabla u\|_2^2 + \|u_t\|_2^2 + \|u\|_p^p \right]. \tag{4.15}$$

Hence,

$$\begin{aligned} L^{1/(1-\alpha)}(t) &= \left(H^{1-\alpha}(t) + \varepsilon \int_{\Omega} u u_t dx + \frac{\mu_1 \varepsilon}{2} \int_{\Omega} u^2 dx\right)^{1/(1-\alpha)} \\ &\leq C\left[H(t) + \left|\int_{\Omega} u u_t dx\right|^{1/(1-\alpha)} + \|u\|_2^{2/(1-\alpha)}\right] \\ &\leq C\left[H(t) + \left|\int_{\Omega} u u_t dx\right|^{1/(1-\alpha)} + \|u\|_p^{2/(1-\alpha)}\right] \\ &\leq C\left[H(t) + \|\nabla u\|_2^2 + \|u_t\|_2^2 + \|u\|_p^p\right], \ t \geq 0. \end{aligned} \tag{4.16}$$

Combining (4.12) and (4.16), we obtain

$$L'(t) \geq \Lambda L^{1/(1-\alpha)}(t), \ t \geq 0, \tag{4.17}$$

where Λ is a positive constant depending only on λ and C.

A simple integration of (4.17) over $(0, t)$ yields

$$L^{\alpha/(1-\alpha)}(t) \geq \frac{1}{L^{-\alpha/(1-\alpha)}(0) - \Lambda\alpha t/(1-\alpha)}.$$

Hence, $L(t)$ blows up in time

$$T \leq T^* = \frac{1-\alpha}{\Lambda\alpha L^{\alpha/(1-\alpha)}(0)}.$$

Consequently, the solution of problem (1.1) blows up in finite time T^* and $T^* \leq \frac{1-\alpha}{\Lambda\alpha L^{\alpha/(1-\alpha)}(0)}$. □

5. Conclusions

In recent years, there has been published much work concerning the wave equation with constant delay or time-varying delay. However, to the best of our knowledge, there was no blow-up result for the logarithmic nonlinear viscoelastic wave equation with delay term. Firstly, we have been obtained the local existence result by using the Faedo-Galerkin approximation. Later, we have been proved that blow-up of solutions for the problem (1.1) under the sufficient conditions in a bounded domain.

References

[1] K. Bartkowski and P. Gorka, *One dimensional Klein-Gordon equation with logarithmic nonlinearities,* J. Phys., *41* (2008).

[2] Q. Dai and Z. Yang, *Global existence and exponential decay of the solution for a viscoelastic wave equation with a delay,* Z. Angew. Math. Phys., *65* (2014), 885-903.

[3] P. Gorka, *Logarithmic Klein-Gordon equation,* Acta Phys. Polon., *B40* (2009), 59–66.

[4] M. Kafini and S.A. Messaoudi, *Local existence and blow up of solutions to a logarithmic nonlinear wave equation with delay,* Appl. Anal., *99*(3) (2020), 530–547.

[5] M. Kirane and B.S. Houari, *Existence and asymptotic stability of a viscoelastic wave equation with a delay,* Z. Angew. Math. Phys., *62* (2011), 1065-1082.

[6] N. Mezouar, S. M. Boulaaras and A. Allahem, Global existence of solutions for the viscoelastic Kirchhoff equation with logarithmic source terms, J. Complex, (2020), 1-25.

[7] S. Nicaise and C. Pignotti, *Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks,* SIAM J. Control Optim, *45*(5) (2006), 1561-1585.

[8] S. Nicaise and C. Pignotti, *Stabilization of the wave equation with boundary or internal distributed delay,* Differ. Integral Equ., *21* (2008), 935-958.

[9] Z. F. Yang, *Existence and energy decay of solutions for the Euler–Bernoulli viscoelastic equation with a delay,* Z. Angew. Math. Phys., *66*(3) (2015), 727-745.

[10] X. Yang, J. Zhang, and S. Wang, *Stability and dynamics of a weak viscoelastic system with memory and nonlinear time-varying delay,* Discrete Cont. Dyn-A, *40*(3) (2020), 1493-1515.

[11] Y. Zhang and X. Liu, *Blow-up of solution for beam equation with delay and dynamic boundary conditions,* Math. Meth. Appl. Sci., (2020), 1-11.