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Bounds of Riemann-Liouville fractional integral operators

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Abstract Fractional integral operators play an important role in generalizations and extensions of various subjects of sciences and engineering. This research is the study of bounds of Riemann-Liouville fractional integrals via $(h-m)$-convex functions. The author succeeded to find upper bounds of the sum of left and right fractional integrals for $(h-m)$-convex function as well as for functions which are deducible from aforementioned function (as comprise in Remark 1.2). By using $(h-m)$-convexity of $|f'|$ a modulus inequality is established for bounds of Riemann-Liouville fractional integrals. Moreover, a Hadamard type inequality is obtained by imposing an additional condition. Several special cases of the results of this research are identified.

Keywords. Convex function, $(h-m)$-convex function, Riemann-Liouville fractional integral operators, Bounds.

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1. Introduction

The convexity of functions is a basic concept in mathematics, its extensions and generalizations have been analyzed in various directions. For example the $(h-m)$-convexity is the generalization of convexity that contains h-convexity, m-convexity, s-convexity defined on the right half of real line including zero (see, [14, 19] and references therein).

Definition 1.1. Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. We say that $f : [0,b] \to \mathbb{R}$ is a $(h-m)$-convex function, if f is non-negative and for all $x, y \in [0,b], m \in [0,1]$ and $\alpha \in (0,1)$, one has

$$f(\alpha x + m(1-\alpha)y) \leq h(\alpha)f(x) + mh(1-\alpha)f(y). \tag{1.1}$$

The following Remark 1.2, elaborates all possible outcomes of Definition 1.1.

Remark 1.2. (i) By setting $m = 1$ in (1.1), it reduces to the definition of h-convex function.
(ii) By setting $h(\alpha) = \alpha$ in (1.1), it reduces to the definition of m-convex function.
(iii) By setting $h(\alpha) = \alpha$ and $m = 1$ in (1.1), it reduces to the definition of convex function.

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(iv) By setting $h(\alpha) = 1$ and $m = 1$ in (1.1), it reduces to the definition of p-function.
(v) By setting $h(\alpha) = \alpha^s$, $0 < s \leq 1$ and $m = 1$ in (1.1), it reduces to the definition of s-convex function of second sense.
(vi) By setting $h(\alpha) = \frac{1}{\alpha}$ and $m = 1$ in (1.1), it reduces to the definition of Godunova-Levin function.
(vii) By setting $h(\alpha) = \frac{1}{\alpha^s}$ and $m = 1$ in (1.1), it reduces to the definition of s-Godunova-Levin function of second kind.

Therefore, the results which will be established in this research, are applicable for all functions described in an aforementioned remark.

Inequalities always have been proved important in almost all branches of pure and applied sciences; inequalities are used in establishing the mathematical models of real world problems, and finding their solutions. Convex functions are considered much valuable in enhancement of theory of inequalities, the convexity takes place a very important role in the optimization theory, graph theory, mathematical statistics etc. We suggested the readers to [7, 8, 13, 15, 17].

Fractional calculus, is the study of fractional order derivatives and integrals. Formulation of fractional differentiation and fractional integration operators have generalized the theory of ordinary and partial differential equations. Applications of fractional calculus are present in almost all modern disciplines of science and engineering; for example in rheology, viscoelasticity, acoustics, optics, chemical and statistical physics, robotics, control theory, electrical and mechanical engineering, bioengineering, etc. [10, 11, 12, 16].

Fractional integral and differential inequalities are very useful in the study of uniqueness and boundedness of the solutions of fractional differential equations and fractional boundary value problems. The aim of this paper is to establish fractional inequalities for the Riemann-Liouville fractional integral operators via $(h-m)$-convex functions. The results of this research are much generalized, and contain various fractional inequalities for functions which are; convex, h-convex, m-convex, s-convex of second kind, Godunova-Levin, s-Gounova-Levin of second kind, p-function. For some recent studies on fractional inequalities readers are suggested to [2, 3, 4, 5, 9, 18].
In the following Riemann-Liouville fractional integrals are defined:

Definition 1.3. Let $f \in L_1[a, b]$. Then the Riemann-Liouville fractional integrals of f of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \; x > a$$

and

$$J_{b_-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \; x < b,$$

where $\Gamma(.)$ is the notation of gamma function.

Remark 1.4. Using Riemann-Liouville fractional integrals, Riesz fractional integral $({}^{R}{}_{a}J_{b}^{\alpha}f)(x)$ is defined by [6]

$$({}^{R}{}_{a}J_{b}^{\alpha}f)(x) = \frac{1}{2}\left(J_{a^+}^{\alpha}f(x) + J_{b^-}^{\alpha}f(x)\right). \tag{1.2}$$

The rest of the paper is organized as follows:
In the upcoming section, an upper bound for the sum of left and right Riemann-Liouville fractional integrals has been established by using the definition of $(h-m)$-convex function. Furthermore, a modulus inequality is proved by applying $(h-m)$-convexity of $|f'|$, then by proving an inequality for $(h-m)$-convex function f with the condition $f(x) = f\left(\frac{a+b-x}{m}\right)$, Hadamard type inequality is obtained. All results of this research can be proved independently for the functions deducible from $(h-m)$-convex function. In the last section, applications are discussed at the boundary as well as midpoints of a closed interval of the real line.
In the whole paper, the notation $M_t^{\alpha}(x) := \frac{|x-t|^{\alpha}}{\Gamma(\alpha)}$ has been frequently utilized.

2. Main Results

Firstly, an upper bound for the sum of left and right Riemann-Liouville fractional integrals, by using the definition of $(h-m)$-convex function is established.

Theorem 2.1. *Let $f : [0,\infty) \to \mathbb{R}$ be a positive $(h-m)$-convex function, $m \in (0,1]$. Also, let $h \in L_1[0,1]$. Then for $0 \le a < mb$ and $\alpha, \beta \ge 1$, the following inequality for the Riemann-Liouville fractional integrals holds:*

$$\begin{aligned} & J_{a^+}^{\alpha}f(x) + J_{b^-}^{\beta}f(x) \\ & \le \left(M_a^{\alpha}(x)f(a) + M_b^{\beta}(x)f(b) + mf\left(\frac{x}{m}\right)\left(M_b^{\beta}(x) + M_a^{\alpha}(x)\right)\right)\|h\|_1. \end{aligned} \tag{2.1}$$

Proof. Using definition of $(h-m)$-convex function, we have

$$f(t) \le h\left(\frac{x-t}{x-a}\right)f(a) + mh\left(\frac{t-a}{x-a}\right)f\left(\frac{x}{m}\right), \tag{2.2}$$

Also for $\alpha > 1$ and $t \in [a,x]$, we have

$$(x-t)^{\alpha-1} \le (x-a)^{\alpha-1}. \tag{2.3}$$

Multiplying (2.2) with (2.3), then integrating over $[a,x]$, the following inequality can be obtained:

$$\begin{aligned} & \int_a^x (x-t)^{\alpha-1}f(t)dt \\ & \le (x-a)^{\alpha-1}\left(f(a)\int_a^x h\left(\frac{x-t}{x-a}\right)dt + mf\left(\frac{x}{m}\right)\int_a^x h\left(\frac{t-a}{x-a}\right)dt\right). \end{aligned} \tag{2.4}$$

Now by using definition of Riemann-Liouville fractional integral on left hand side and by change of variables on right hand side, (2.4) takes the form as follows:

$$J_{a^+}^{\alpha}f(x) \le M_a^{\alpha}(x)\left(f(a) + mf\left(\frac{x}{m}\right)\right)\|h\|_1. \tag{2.5}$$

Similarly, using $(t-x)^{\beta} \leq (x-b)^{\beta},\ t \in [x,b]$ and $\beta > 1$, and definition of $(h-m)$-convexity for identity

$$t = \frac{t-x}{b-x}b + \frac{m(b-t)}{b-x}\frac{x}{m},$$

one can have the following inequality:

$$J^{\beta}_{b-}f(x) \leq M^{\beta}_{b}(x)\left(f(b) + mf\left(\frac{x}{m}\right)\right)\|h\|_1. \tag{2.6}$$

From inequalities (2.5) and (2.6), the required inequality (2.1) can be obtained. □

Particularly, the following corollaries and remarks, emphasize the importance of aforementioned result:

Corollary 2.2. *By taking $\alpha = \beta$ in (2.1), we get the following inequality for Riemann-Liouville fractional integrals:*

$$\begin{aligned}&J^{\alpha}_{a+}f(x) + J^{\alpha}_{b-}f(x)\\ &\leq \left(M^{\alpha}_{a}(x)f(a) + M^{\alpha}_{b}(x)f(b) + mf\left(\frac{x}{m}\right)(M^{\alpha}_{b}(x) + M^{\alpha}_{a}(x))\right)\|h\|_1.\end{aligned}$$

Corollary 2.3. *By taking $\alpha = \beta$ in (2.1) and using Remark 1.4, we get the following inequality involving Riesz fractional integral:*

$$({}^{R}{}_{a}J^{\alpha}_{b}f)(x) \leq \frac{1}{2}\left(M^{\alpha}_{a}(x)f(a) + M^{\alpha}_{b}(x)f(b) + mf\left(\frac{x}{m}\right)(M^{\alpha}_{b}(x) + M^{\alpha}_{a}(x))\right)\|h\|_1.$$

Following results can be observed for particular definitions of function h in Theorem 2.1.

Corollary 2.4. *Under the assumptions of Theorem 2.1 , the following results hold:*
(i) For $h(z) = 1$, the following bound is valid:

$$\begin{aligned}&J^{\alpha}_{a+}f(x) + J^{\beta}_{b-}f(x)\\ &\leq M^{\alpha}_{a}(x)f(a) + M^{\beta}_{b}(x)f(b) + mf\left(\frac{x}{m}\right)\left(M^{\beta}_{b}(x) + M^{\alpha}_{a}(x)\right).\end{aligned} \tag{2.7}$$

(ii) For $h(z) = z^p$, the following bound is valid:

$$\begin{aligned}&J^{\alpha}_{a+}f(x) + J^{\beta}_{b-}f(x)\\ &\leq \frac{1}{p+1}\left(M^{\alpha}_{a}(x)f(a) + M^{\beta}_{b}(x)f(b) + mf\left(\frac{x}{m}\right)\left(M^{\beta}_{b}(x) + M^{\alpha}_{a}(x)\right)\right).\end{aligned} \tag{2.8}$$

It is interesting to see, bound in (2.7) can be refined by selecting convenient values of p in (2.8).

Remark 2.5. (i) If we take $m = 1$ and $p = s$ in (2.8), then bound of (RL) fractional integrals for s-convex function of second kind is obtained.
(ii) If we take $m = 1$ and $p = -s$ in (2.8), then bound of (RL) fractional integrals for s-Godunova-Levin function of second kind is obtained.
(iii) If we take $m = 1$ and $p = -1$ in (2.8), then bound of (RL) fractional integrals for Godunova-Levin function is obtained.
(iv) If we take $m = 1$ and $p = 0$ in (2.8), then bound of (RL) fractional integrals for

p-function is obtained.
(v) If we take $m = 1$ and $p = 1$ in (2.8), then [2, Theorem 1] is obtained.

The following result provides the fractional integral inequality that elaborates the bounds of Riemann-Liouville fractional integrals in modulus form:

Theorem 2.6. *Let $f : [0,\infty) \to \mathbb{R}$ be a differentiable function. If $|f'|$ is $(h-m)$-convex, $m \in (0,1]$ and $h \in L_1[0,1]$, then for $0 \le a < mb$ and $\alpha, \beta > 0$, the following inequality for the Riemann-Liouville fractional integrals holds:*

$$\begin{aligned}
&\left| J_{a^+}^{\alpha} f(x) + J_{b^-}^{\beta} f(x) - \left(\frac{1}{\alpha} M_a^{\alpha}(x) f(a) + \frac{1}{\beta} M_a^{\beta}(x) f(b) \right) \right| \\
&\le \Big[M_a^{\alpha+1}(x)|f'(a)| + M_b^{\beta+1}(x)|f'(b)| \\
&+ m \left| f'\left(\frac{x}{m}\right) \right| \left(M_a^{\alpha+1}(x) + M_b^{\beta+1}(x) \right) \Big] \|h\|_1 .
\end{aligned} \tag{2.9}$$

Proof. Since $|f'|$ is $(h-m)$-convex, therefore for $t \in [a,x]$, we have

$$|f'(t)| \le h\left(\frac{x-t}{x-a}\right)|f'(a)| + mh\left(\frac{t-a}{x-a}\right)\left|f'\left(\frac{x}{m}\right)\right|,$$

from which we can write

$$\begin{aligned}
&-\left(h\left(\frac{x-t}{x-a}\right)|f'(a)| + mh\left(\frac{t-a}{x-a}\right)\left|f'\left(\frac{x}{m}\right)\right| \right) \\
&\le f'(t) \le h\left(\frac{x-t}{x-a}\right)|f'(a)| + mh\left(\frac{t-a}{x-a}\right)\left|f'\left(\frac{x}{m}\right)\right| .
\end{aligned} \tag{2.10}$$

We consider the second inequality of inequality (2.10), that is

$$f'(t) \le h\left(\frac{x-t}{x-a}\right)|f'(a)| + mh\left(\frac{t-a}{x-a}\right)\left|f'\left(\frac{x}{m}\right)\right| . \tag{2.11}$$

Now for $\alpha > 0$, we have the following inequality:

$$(x-t)^{\alpha} \le (x-a)^{\alpha}, \quad t \in [a,x]. \tag{2.12}$$

Multiplying the last two inequalities and integrating with respect to t over $[a,x]$, the following inequality can be obtained:

$$\begin{aligned}
&\int_a^x (x-t)^{\alpha} f'(t)dt \\
&\le (x-a)^{\alpha}\left[|f'(a)| \int_a^x h\left(\frac{x-t}{x-a}\right)dt + m\left|f'\left(\frac{x}{m}\right)\right| \int_a^x h\left(\frac{t-a}{x-a}\right)dt \right].
\end{aligned} \tag{2.13}$$

The left hand side is calculated as follows:

$$\int_a^x (x-t)^{\alpha} f'(t)dt = -f(a)(x-a)^{\alpha} + \Gamma(\alpha+1) J_{a^+}^{\alpha} f(x),$$

while using change of variables in the right hand side the resulting inequality takes the form as follows:

$$J_{a^+}^{\alpha} f(x) - \frac{1}{\alpha} M_a^{\alpha}(x) f(a) \le M_a^{\alpha+1}(x)\left(|f'(a)| + m\left|f'\left(\frac{x}{m}\right)\right| \right)\|h\|_1 . \tag{2.14}$$

If we consider from (2.10) the first inequality and proceed as we did for the second, the following inequality is obtained:

$$\frac{1}{\alpha}M_a^\alpha(x)f(a) - J_{a^+}^\alpha f(x) \leq M_a^{\alpha+1}(x)\left(|f'(a)| + m\left|f'\left(\frac{x}{m}\right)\right|\right)\|h\|_1. \tag{2.15}$$

From (2.14) and (2.15), the following modulus inequality holds:

$$\left|J_{a^+}^\alpha f(x) - \frac{1}{\alpha}M_a^\alpha(x)f(a)\right| \leq M_a^{\alpha+1}(x)\left(|f'(a)| + m\left|f'\left(\frac{x}{m}\right)\right|\right)\|h\|_1. \tag{2.16}$$

On the other hand for $t \in [x,b]$ using $(h-m)$-convexity of $|f'|$, we have

$$|f'(t)| \leq h\left(\frac{t-x}{b-x}\right)|f'(b)| + mh\left(\frac{b-t}{b-x}\right)\left|f'\left(\frac{x}{m}\right)\right|. \tag{2.17}$$

Also for $t \in [x,b]$ and $\beta > 0$, we have

$$(t-x)^\beta \leq (b-x)^\beta. \tag{2.18}$$

By adopting the same treatment as we did for (2.10) and (2.12), one can obtain from (2.17) and (2.18), the following inequality:

$$\left|J_{b^-}^\beta f(a) - \frac{1}{\beta}M_a^\beta(x)f(b)\right| \leq M_b^{\beta+1}(x)\left(|f'(b)| + m\left|f'\left(\frac{x}{m}\right)\right|\right)\|h\|_1. \tag{2.19}$$

By combining the inequalities (2.16) and (2.19), via triangular inequality, we get the required inequality (2.9). □

Special cases of above-proved result, are highlighted in the following corollaries and remark.

Corollary 2.7. *By taking $\alpha = \beta$ in (2.9), we get the following fractional integral inequality:*

$$\begin{aligned}&\left|\left(J_{a^+}^\alpha f(x) + J_{b^-}^\alpha f(x)\right) - \frac{1}{\alpha}\left(M_a^\alpha(x)f(a) + M_b^\alpha(x)f(b)\right)\right|\\ &\leq \Big[M_a^{\alpha+1}(x)|f'(a)| + M_b^{\alpha+1}(x)|f'(b)|\\ &+ m\left|f'\left(\frac{x}{m}\right)\right|\left(M_a^{\alpha+1}(x) + M_b^{\alpha+1}(x)\right)\Big]\|h\|_1.\end{aligned}$$

Corollary 2.8. *By taking $\alpha = \beta$ in (2.9) and using Remark 1.4, we get the following inequality for Riesz fractional integral:*

$$\begin{aligned}&\left|\left(({}^R{}_aJ_b^\alpha f)(x)\right) - \frac{1}{2\alpha}\left(M_a^\alpha(x)f(a) + M_b^\alpha(x)f(b)\right)\right| \\ &\leq \frac{1}{2}\Big[M_a^{\alpha+1}(x)|f'(a)| + M_b^{\alpha+1}(x)|f'(b)|\\ &+ m\left|f'\left(\frac{x}{m}\right)\right|\left(M_a^{\alpha+1}(x) + M_b^{\alpha+1}(x)\right)\Big]\|h\|_1.\end{aligned} \tag{2.20}$$

Remark 2.9. (i) Axioms (i)-(iv) of Remark 2.5 about Theorem 2.1, are also valid for Theorem 2.6.
(ii) If we take $m = 1$ and $h(z) = z$ in (2.9), then [2, Theorem 2] is obtained.

The upcoming result is useful to establish the Hadamard type estimates of Riemann-Liouville fractional integrals.

Lemma 2.10. *Let $f : [0, \infty) \to \mathbb{R}$ be a $(h - m)$-convex function where $m \in (0, 1]$, and $a, b \in [0, \infty)$. If $f(x) = f\left(\frac{a+b-x}{m}\right)$, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq (m+1)h\left(\frac{1}{2}\right) f(x) \quad x \in [a, b]. \tag{2.21}$$

Proof. Since f is $(h - m)$-convex, therefore the following inequality is valid:

$$f\left(\frac{a+b}{2}\right) \leq h\left(\frac{1}{2}\right)\left[f\left(\frac{x-a}{b-a}b + \frac{b-x}{b-a}a\right) + mf\left(\frac{\frac{x-a}{b-a}a + \frac{b-x}{b-a}b}{m}\right)\right] \tag{2.22}$$

$$= h\left(\frac{1}{2}\right)\left(f(x) + mf\left(\frac{a+b-x}{m}\right)\right).$$

Using given condition $f(x) = f\left(\frac{a+b-x}{m}\right)$ in above inequality, the inequality in (2.21) holds. □

Theorem 2.11. *Let $f : [0, \infty) \to \mathbb{R}$ be a positive $(h - m)$-convex function, $m \in (0, 1]$ and $h \in L_1[0, 1]$. If $0 \leq a < mb$ and $f(x) = f\left(\frac{a+b-x}{m}\right)$, then for $\alpha, \beta > 0$, the following inequality for the Riemann-Liouville fractional integrals holds:*

$$\frac{1}{2(m+1)h\left(\frac{1}{2}\right)}\left(\frac{1}{\alpha+1} + \frac{1}{\beta+1}\right) f\left(\frac{a+b}{2}\right) \tag{2.23}$$

$$\leq \frac{\Gamma(\beta+1)J_{b^-}^{\beta+1}f(a)}{2(b-a)^{\beta+1}} + \frac{\Gamma(\alpha+1)J_{a^+}^{\alpha+1}f(b)}{2(b-a)^{\alpha+1}}$$

$$\leq \left(f(a) + mf\left(\frac{b}{m}\right)\right)\|h\|_1.$$

Proof. For $x \in [a, b]$, we have

$$(x-a)^\beta \leq (b-a)^\beta, \quad \beta > 0. \tag{2.24}$$

Also f is $(h - m)$-convex function, therefore we have

$$f(x) \leq h\left(\frac{b-x}{b-a}\right) f(a) + mh\left(\frac{x-a}{b-a}\right) f\left(\frac{b}{m}\right). \tag{2.25}$$

Multiplying (2.24) and (2.25) and then integrating with respect to x over $[a, b]$, the following inequality is obtained:

$$\int_a^b (x-a)^\beta f(x)dx$$

$$\leq (b-a)^\beta \left(f(a)\int_a^b h\left(\frac{b-x}{b-a}\right)dx + mf\left(\frac{b}{m}\right)\int_a^b h\left(\frac{x-a}{b-a}\right)dx\right).$$

From which we get

$$\frac{\Gamma(\beta+1)J_{b^-}^{\beta+1}f(a)}{(b-a)^{\beta+1}} \leq \left(f(a) + mf\left(\frac{b}{m}\right)\right)\|h\|_1. \tag{2.26}$$

On the other hand for $x \in [a,b]$, we have

$$(b-x)^{\alpha} \leq (b-a)^{\alpha}, \quad \alpha > 0. \tag{2.27}$$

Multiplying (2.25) and (2.27) and then integrating with respect to x over $[a,b]$, the following inequality is obtained:

$$\begin{aligned}&\int_a^b (b-x)^{\alpha} f(x)dx \\ &\leq (b-a)^{\alpha}\left(f(a)\int_a^b h\left(\frac{b-x}{b-a}\right)dx + mf\left(\frac{b}{m}\right)\int_a^b h\left(\frac{x-a}{b-a}\right)dx\right).\end{aligned}$$

From which we get

$$\frac{\Gamma(\alpha+1)J_{a^+}^{\alpha+1}f(b)}{(b-a)^{\alpha+1}} \leq \left(f(a)+mf\left(\frac{b}{m}\right)\right)\|h\|_1. \tag{2.28}$$

Adding (2.26) and (2.28), the following inequality is established:

$$\frac{\Gamma(\beta+1)J_{b^-}^{\beta+1}f(a)}{2(b-a)^{\beta+1}} + \frac{\Gamma(\alpha+1)J_{a^+}^{\alpha+1}f(b)}{2(b-a)^{\alpha+1}} \leq \left(f(a)+mf\left(\frac{b}{m}\right)\right)\|h\|_1. \tag{2.29}$$

Using Lemma 2.10 and multiplying (2.21) with $(x-a)^{\beta}$, then integrating over $[a,b]$, the following inequality is obtained:

$$f\left(\frac{a+b}{2}\right)\int_a^b (x-a)^{\beta}dx \leq (m+1)h\left(\frac{1}{2}\right)\int_a^b (x-a)^{\beta}f(x)dx$$

$$f\left(\frac{a+b}{2}\right)\frac{(b-a)^{\beta+1}}{(m+1)h\left(\frac{1}{2}\right)(\beta+1)} \leq \Gamma(\beta+1)J_{b^-}^{\beta+1}f(a)$$

$$f\left(\frac{a+b}{2}\right)\frac{1}{2(m+1)h\left(\frac{1}{2}\right)(\beta+1)} \leq \frac{\Gamma(\beta+1)J_{b^-}^{\beta+1}f(a)}{2(b-a)^{\beta+1}}. \tag{2.30}$$

Using Lemma 2.10 and multiplying (2.21) with $(b-x)^{\alpha}$, then integrating over $[a,b]$, the following inequality is obtained:

$$f\left(\frac{a+b}{2}\right)\frac{1}{2(m+1)h\left(\frac{1}{2}\right)(\alpha+1)} \leq \frac{\Gamma(\alpha+1)J_{a^+}^{\alpha+1}f(b)}{2(b-a)^{\alpha+1}}. \tag{2.31}$$

Adding (2.30) and (2.31), the following inequality is established:

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\frac{1}{2(m+1)h\left(\frac{1}{2}\right)}\left(\frac{1}{\alpha+1}+\frac{1}{\beta+1}\right) \\ &\leq \frac{\Gamma(\beta+1)J_{b^-}^{\beta+1}f(a)}{2(b-a)^{\beta+1}} + \frac{\Gamma(\alpha+1)J_{a^+}^{\alpha+1}f(b)}{2(b-a)^{\alpha+1}}.\end{aligned} \tag{2.32}$$

Hence (2.29) and (2.32), establish the required inequality (2.23). □

Furthermore, some interesting facts of above result are elaborated in the following corollaries and remarks:

Corollary 2.12. *By taking $\alpha = \beta$ in (2.23), we get the following fractional integral inequality:*

$$f\left(\frac{a+b}{2}\right)\frac{1}{(m+1)h\left(\frac{1}{2}\right)(\alpha+1)} \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha+1}}\left(J_{b^-}^{\alpha+1}f(a)+J_{a^+}^{\alpha+1}f(b)\right) \tag{2.33}$$
$$\leq \left(f(a)+mf\left(\frac{b}{m}\right)\right)\|h\|_1.$$

Corollary 2.13. *If we take $h(z) = z^p$ in (2.33), then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right)\frac{2^p}{(m+1)(\alpha+1)} \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha+1}}\left[J_{b^-}^{\alpha+1}f(a)+J_{a^+}^{\alpha+1}f(b)\right] \tag{2.34}$$
$$\leq \frac{f(a)+mf\left(\frac{b}{m}\right)}{p+1}.$$

Note that the axioms (i)-(iv) of Remark 2.5 are valid for (2.34).

Corollary 2.14. *If we take $m = 1$ in (2.34), then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right)\frac{2^{p-1}}{\alpha+1} \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha+1}}\left[J_{b^-}^{\alpha+1}f(a)+J_{a^+}^{\alpha+1}f(b)\right] \leq \frac{f(a)+f(b)}{p+1}. \tag{2.35}$$

Corollary 2.15. *If we take $p = 1$ in (2.35), then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right)\frac{1}{\alpha+1} \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha+1}}\left[J_{b^-}^{\alpha+1}f(a)+J_{a^+}^{\alpha+1}f(b)\right] \leq \frac{f(a)+f(b)}{2}.$$

Remark 2.16. If we take $m = 1$ and $h(z) = z$ in (2.23), then we get [2, Theorem 3].

Firstly, by apply Theorem 2.1 at end points of the interval $[a, b]$, the following result is obtained:

Theorem 2.17. *Under the assumptions of Theorem 2.1, the following fractional inequality holds:*

$$J_{a^+}^{\alpha}f(b)+J_{b^-}^{\beta}f(a) \tag{2.36}$$
$$\leq \left(M_a^{\alpha}(b)f(a)+M_a^{\beta}(b)f(b)+m\left(M_a^{\beta}(b)f\left(\frac{a}{m}\right)+M_a^{\alpha}(b)f\left(\frac{b}{m}\right)\right)\right)\|h\|_1.$$

Proof. If we take $x = a$ in (2.1), then the following inequality is obtained:

$$\Gamma(\beta)J_{b^-}^{\beta}f(a) \leq \left((b-a)^{\beta}f(b)+m(b-a)^{\beta}f\left(\frac{a}{m}\right)\right)\|h\|_1. \tag{2.37}$$

If we take $x = b$ in (2.1), then the following inequality is obtained:

$$\Gamma(\alpha)J_{a^+}^{\alpha}f(b) \leq \left((b-a)^{\alpha}f(a)+m(b-a)^{\alpha}f\left(\frac{b}{m}\right)\right)\|h\|_1. \tag{2.38}$$

From inequalities (2.37) and (2.38), the required inequality (2.36) can be obtained. □

Corollary 2.18. *By taking $\alpha = \beta$ in (2.36), the following fractional integral inequality holds:*

$$\begin{aligned}
&J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a) \\
&\leq M_a^{\alpha}(b)\left(f(a) + f(b) + m\left(f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right)\right)\right)\|h\|_1.
\end{aligned} \tag{2.39}$$

Corollary 2.19. *By taking $\alpha = \beta$ in (2.36) and using Remark 1.4, we get the following inequality for Riesz fractional integral:*

$$({}^{R}{}_{a}J_{b}^{\alpha} f)(x) \leq \frac{M_a^{\alpha}(b)}{2}\left(f(a) + f(b) + m\left(f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right)\right)\right)\|h\|_1. \tag{2.40}$$

Corollary 2.20. *If we take $\alpha = m = 1$ and $h(z) = z$ in (2.39), then the following inequality for convex function is obtained:*

$$\frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a) + f(b)}{2}. \tag{2.41}$$

By applying Theorem 2.6 at mid point $\frac{a+b}{2}$ of $[a, b]$, the upcoming result is obtained.

Theorem 2.21. *Under the assumptions of Theorem 2.6, the following fractional integral inequality holds:*

$$\begin{aligned}
&\left|J_{a^+}^{\alpha} f\left(\frac{a+b}{2}\right) + J_{b^-}^{\beta} f\left(\frac{a+b}{2}\right) - \left(\frac{M_a^{\alpha}(b)}{\alpha 2^{\alpha}} f(a) + \frac{M_a^{\beta}(b)}{\beta 2^{\beta}} f(b)\right)\right| \\
&\leq \left(\frac{M_a^{\alpha+1}(b)}{2^{\alpha+1}}|f'(a)| + \frac{M_a^{\beta+1}(b)}{2^{\beta+1}}|f'(b)|\right. \\
&\left. + m\left|f'\left(\frac{a+b}{2m}\right)\right|\left(\frac{M_a^{\alpha+1}(b)}{2^{\alpha+1}} + \frac{M_a^{\beta+1}(b)}{2^{\beta+1}}\right)\right)\|h\|_1.
\end{aligned} \tag{2.42}$$

Proof. If we take $x = \frac{a+b}{2}$ in (2.9), then the required inequality (2.42) can be obtained. □

Corollary 2.22. *By taking $\alpha = \beta$ in (2.42), the following fractional integral inequality holds:*

$$\begin{aligned}
&\left|J_{a^+}^{\alpha} f\left(\frac{a+b}{2}\right) + J_{b^-}^{\alpha} f\left(\frac{a+b}{2}\right) - \frac{M_a^{\alpha}(b)}{\alpha 2^{\alpha}}\left(f(a) + f(b)\right)\right| \\
&\leq \frac{M_a^{\alpha+1}(b)}{2^{\alpha+1}}\left(|f'(a)| + |f'(b)| + 2m\left|f'\left(\frac{a+b}{2m}\right)\right|\right)\|h\|_1.
\end{aligned} \tag{2.43}$$

Corollary 2.23. *By taking $\alpha = \beta$ in (2.42) and using Remark 1.4, we get the following inequality for Riesz fractional integral:*

$$\left| \left({}^{R}{}_{a}J_{b}^{\alpha} f \right) \left(\frac{a+b}{2} \right) - \frac{M_a^{\alpha}(b)}{\alpha 2^{\alpha+1}} \left(f(a) + f(b) \right) \right| \tag{2.44}$$
$$\leq \frac{M_a^{\alpha+1}(b)}{2^{\alpha+2}} \left(|f'(a)| + |f'(b)| + 2m \left| f' \left(\frac{a+b}{2m} \right) \right| \right) \|h\|_1 .$$

Corollary 2.24. [2] *If we take $\alpha = m = 1$ and $h(z) = z$ in (2.43), then we get*

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{f(a)+f(b)}{2} \right| \leq \frac{b-a}{8} \left[|f'(a)| + |f'(b)| + 2 \left| f' \left(\frac{a+b}{2} \right) \right| \right] . \tag{2.45}$$

Remark 2.25. If $f'\left(\frac{a+b}{2}\right) = 0$, then from (2.45), we get [1, Theorem 2.2]. If $f'(x) \leq 0$, then (2.45) gives the refinement of [1, Theorem 2.2].

By applying Theorem 2.11, similar results can be established; these are left for the readers.

Concluding remarks

This paper investigates Riemann-Liouville fractional integral inequalities via a class of real valued functions which are $(h - m)$-convex. It is remarkable to elaborate that, the presented results contain fractional inequalities for h-convex functions, m-convex functions, convex functions, Godunova-Levin functions, p-functions and s-convex functions in the second sense, on the closed interval of non-negative real numbers.

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