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Stability in distribution of neutral stochastic functional differential equations with infinite delay

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Abstract In this paper, we investigate stability in distribution of neutral stochastic functional differential equations with infinite delay (NSFDEwID) at the state space C_r. We drive a sufficient strong monotone condition for the existence and uniqueness of the global solutions of NSFDEwID in the state space C_r. We also address the stability of the solution map x_t and illustrate the theory with an example.

Keywords. Neutral stochastic functional differential equations, Infinite delay, Solution map, Stability in distribution.

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1. Introduction

Recently, the asymptotic properties of neutral stochastic functional differential equations (NSFDEs) have addressed by many authors. Kolmanovskii and Nosov [17] studied the theory of existence and uniqueness, and the stability of the solutions of these equations. Moreover Mao [20, 22] has investigated the moment exponential stability and the almost sure exponential stability. Furthermore, Bao and Hou [2], under non-Lipschitz condition with weakened linear growth condition, the existence and uniqueness of mild solutions to neutral partial functional SDEs have been investigated. Tan et al. [26] used the weak convergence approach, and have reviewed stability in distribution for NSFDEs. Huang and Deng [13] have studied Razumikhin-type theorems on general p-th moment asymptotic stability. Kolmanovskii et al. [16] have dealt with NSDDE with Markovian switching. Liu and Xia [19] have established some results which are more effective and relatively easy to verify to obtain the required stability, Hu and Wang [11] were concerned with the NSFDEs with Markovian switching and derived some sufficient conditions for stability in distribution, to name a few.

On the other hand, infinite NSFDEs have also received considerable attention. As well as, many remarkable results from their work on the existence and uniqueness and the asymptotic behaviour of NSFDEwID. Ren and Xia [25], for example, have studied existence and uniqueness of solutions with infinite delay at phase space

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$BC((-\infty;0];R^d)$ which denotes the family of bounded continuous R^d-value functions with norm $\|\varphi\|= \sup_{-\infty<\theta\leq 0}|\varphi(\theta)|$ under non-Lipschitz condition. Da Prato and Zabczyk [10] devoted their work to asymptotic properties of the solution. Boufoussi and Hajji [6, 7], by using successive approximation, have proved existence and uniqueness result for a class of NFSDEs in Hilbert spaces. Bao [1] has considered the existence and uniqueness of solutions in $L^p(\Omega, C_h)$ space. Caraballo et al. [8] have analysed the almost sure exponential stability and ultimate boundedness of the solutions. Wei and Cai [27] by choosing C_g as phase space, showed the existence and uniqueness of the solutions to neutral stochastic functional differential equations with infinite delay have been obtained under non-Lipschitz condition, weakened linear growth condition and contractive condition. Chen and Banas [**?**], have devoted their work to obtain some sufficient conditions for the exponential stability as well as almost surely exponential stability for the mild solution of neutral stochastic partial differential equations with delays by establishing an integral-inequality. Zhou, Y. [29] have studied the stability property of stochastic differential equations in Hilbert spaces. Bao and Cao [3] have examined the existence and uniqueness of mild solutions.
Moreover, Wu et al. [28] have studied the ergodicity of underlying processes and establishes the existence of the invariant measure for SFDEs with infinite delay at the state space C_r. Also Asker and Sari [14] have addressed the existence and uniqueness of solutions to NSFDEwID at the same state space under the local Lipschitz condition and Linear growth condition.
The state-space for functional differential equations (FDEs) with infinite delay plays a crucial role in solving a specific problem. In general, the state space will be Banach space of functions or equivalence class of functions, see [15]. Mohammed [24] has examined solution maps of SFDEs with finite delay on appropriate phase spaces and proved that the solution maps have Markov property. Based on the Markov property of solution maps of SFDEs with finite delay, Bao et al.[4, 5] have examined the ergodicity, while Wu et al. [28] investigates existence and uniqueness of solutions, Markov properties, and ergodicity of SFDEs with infinite delay by using phase state C_r.
Motivated by the discussion above, this chapter is developed and extends the results of [14, 28]. To guarantee the well-posedness of solutions of NSFDEwID, boundedness of solutions from different initial data and further asymptotic properties including the mean-square boundedness and convergence of the solutions from different initial data, we have imposed the appropriate strong monotone condition. We have also studied, the stability in distribution of the solution data. Finally, we have introduced an example to describe the theory that we addressed.

2. Preliminary

Throughout this thesis, unless otherwise specified, we use the following notation. R^d denotes the usual d-dimensional Euclidean space, $|\cdot|$ norm in R^d. If A is a vector or a matrix, its transpose is denoted by A^T; and $|A| = \sqrt{trace(A^T A)}$ its trace norm. Denote by $X^T Y$ the inner product of $X, Y \in R^d$. We choose the state space with the fading memory denoted by C_r defined as follows: for given positive number r,

$$C_r = \Big\{\varphi \in C((-\infty,0];R^d) : \|\varphi\|_r = \sup_{-\infty<\theta\le 0} e^{r\theta}|\varphi(\theta)| < \infty\Big\},$$

where $C((-\infty,0];R^d)$ denotes the family of all continuous R^d -value functions φ defined on $(-\infty,0]$ to R^d with the norm $\|\varphi\|_r$. C_r is a Banach space with norm $\|\varphi\|_r = \sup_{-\infty<\theta\le 0} e^{r\theta}|\varphi(\theta)| < \infty$, see [15, 28]. Let $(\Omega,\mathcal{F},\mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\in[0,+\infty)}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all $\mathcal{P}$-null sets). Let I_B denote the indicator function of a set B. The notation $\mathcal{P}(C_r)$ denotes the family of all probability measures on $(C_r,\mathcal{B}(C_r))$. Denote by $\mathcal{C}_b(C_r)$ the set of all bounded continuous functional. For any $F \in \mathcal{C}_b(C_r)$, $F : C_r \to R$ and $\pi(\cdot) \in \mathcal{P}(C_r)$, let $\pi(F) := \int_{C_r} F(\phi)\pi(d\phi)$. M_0 stands for the set of probability measures on $(-\infty,0]$, namely, for any $\mu \in M_0$, $\int_{-\infty}^0 \mu(d\theta) = 1$. For any $r > 0$, let us further define M_r as follows, see [28]:

$$M_r := \Big\{\mu \in M_0; \mu^{(r)} := \int_{-\infty}^0 e^{-r\theta}\mu(d\theta) < \infty\Big\}. \tag{2.1}$$

Obviously, there exist many such probability measures and here we supply an example: let $\mu(d\theta) = e^{\beta\theta}d\theta$. Clearly, for any $q < \beta$,

$$\mu^{(q)} = \int_{-\infty}^0 e^{-q\theta}e^{\beta\theta}d\theta = \frac{1}{\beta-q}\int_{-\infty}^0 (\beta-q)e^{\theta(\beta-q)}d\theta = \frac{1}{\beta-q} < \infty, \tag{2.2}$$

which implies $\mu^{(q)} \in M_q$ for any $q < \beta$.
Consider a d-dimensional neutral stochastic functional differential equations with infinite delay

$$d\{x(t) - D(x_t)\} = b(x_t)dt + \sigma(x_t)dw(t), \quad \text{on} \quad t \ge 0, \tag{2.3}$$

with the initial data

$$x_0 = \xi = \{\xi(\theta) : -\infty < \theta \le 0\} \in C_r, \tag{2.4}$$

where

$$x_t(\theta) = x(t+\theta) : -\infty < \theta \le 0, \quad x(t) = \xi(t) \quad \text{for} \quad t < 0$$

and $b, D : C_r \to \mathbb{R}^d$; $\sigma : C_r \to \mathbb{R}^{d\times m}$ are Borel measurable, $w(t)$ is an m-dimensional Brownian motion. It should be pointed out that $x(t) \in R^d$ is a point, while $x_t \in C_r$ is a continuous function on the interval $(-\infty,t]$ taking values in R^d. Now, we give the definition of the solutions for the equation (2.3).

Definition 2.1. [21, 28] Let $(\Omega,\mathcal{F},\mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t>o}$. Set $\mathcal{F}_t = \mathcal{F}_0$ for $-\infty < t \le 0$ and for a given stopping time τ_e let x(t), $-\infty < t \le \tau_e$ be continuous R^d-value and $\mathcal{F}_t$-adapted process. $x(t)$ is called a **local strong solution** of (2.3) with initial data $\xi \in C_r$ if $x(t) = \xi(t) \quad on -\infty < t \le 0$ and for all $t \ge 0$,

$$x(t\wedge\tau_\ell) = \xi(0) + D(x_{t\wedge\tau_\ell}) - D(\xi) + \int_0^{t\wedge\tau_\ell} b(x_s)ds + \int_0^{t\wedge\tau_\ell} \sigma(x_s)dw(s) \quad \text{a.s.},$$

for each $\ell \geq 1$, where $\{\tau_\ell\}_{\ell\geq 1}$ is a non-decreasing sequence of stopping times such that $\tau_\ell \longrightarrow \tau_e$ almost surely as $\ell \longrightarrow \infty$. If moreover, $\limsup_{t\longrightarrow\tau_e} |x(t)| = \infty$ is satisfied a.s. when $\tau_e < \infty$ a.s., x(t) $(-\infty < t < \tau_e)$ is called a **maximal local strong solution** and τ_e is called the explosion time. It is called a **global solution** when $\tau_e = \infty$. A maximal local strong solution x(t), $-\infty < t < \tau_e$ is said to be unique if for any other maximal local strong solution $\overline{x}(t)$, $-\infty < t < \overline{\tau}_e$, we have $\tau_e = \overline{\tau}_e$ and $x(t) = \overline{x}(t)$ for $-\infty < t < \tau_e$ almost surely.

3. Existence and uniqueness of global solutions

In this section, we study the existence and uniqueness of the global solutions of the NSFDEwID (2.3), mean-square boundedness and convergence of the solutions from different initial data. To examine the stability in distribution of the equation (2.3), we assume the following conditions.

(H1): For $\mu_1 \in M_{2r}$ and $\varphi \in C_r$ there exist $k \in (0,1)$ with $\mu_1^{(2r)} < 1$ such that

$$|D(\varphi) - D(\phi)|^2 \leq k \int_{-\infty}^0 |\varphi(\theta) - \phi(\theta)|^2 \mu_1(d\theta) \quad \text{and} \quad D(0) = 0. \tag{3.1}$$

(H2): Let b be a continuous function. Assume there exist constants $\lambda_1, \lambda_2, \lambda_3, \lambda_4 > 0$, and probability measures $\mu_2, \mu_3 \in M_{2r}$ such that for any $\varphi, \phi \in C_r$

$$\begin{aligned}\Big[\varphi(0) - \phi(0) - \big(D(\varphi) - D(\phi)\big)\Big]^T \Big[b(\varphi) - b(\phi)\Big] &\leq -\lambda_1|\varphi(0) - \phi(0)|^2 \\ &\quad + \lambda_2 \int_{-\infty}^0 |\varphi(\theta) - \phi(\theta)|^2 \mu_2(d\theta),\end{aligned} \tag{3.2}$$

and for a function σ

$$\begin{aligned}\big|\sigma(\varphi) - \sigma(\phi)\big|^2 &\leq \lambda_3\big|\varphi(0) - \phi(0)\big|^2 \\ &\quad + \lambda_4 \int_{-\infty}^0 \big|\varphi(\theta) - \phi(\theta)\big|^2 \mu_3(d\theta).\end{aligned} \tag{3.3}$$

Remark 3.1. For simplicity, we suppose that $\mu_1 = \mu_2 = \mu_3 = \mu$.

Remark 3.2. It is easy to get from (3.1), (3.2) and (3.3), that

i):

$$|D(\varphi) - D(\phi)|^2 \leq k\mu^{(2r)}\|\varphi - \phi\|_r^2. \tag{3.4}$$

ii):

$$\begin{aligned}\Big[\varphi(0) - \phi(0) - \big(D(\varphi) - D(\phi)\big)\Big]^T &\Big[b(\varphi) - b(\phi)\Big] \\ &\leq (-\lambda_1 + \lambda_2\mu^{(2r)})\|\varphi - \phi\|_r^2.\end{aligned} \tag{3.5}$$

iii):

$$\big|\sigma(\varphi) - \sigma(\phi)\big|^2 \leq (\lambda_3 + \lambda_4\mu^{(2r)})\|\varphi - \phi\|_r^2. \tag{3.6}$$

So, the assumptions **(H1)** and **(H2)** guarantee that the equation (2.3) admits a unique local strong solution $\{x(t;\xi)\}_{t>-\infty}$ with initial data $\xi \in C_r$. So, for global solution and further asymptotic properties including the mean-square boundedness and convergence of the solutions from different initial data, we impose those conditions.

Remark 3.3. When $\varphi \equiv 0, D(\varphi) \equiv 0$, by the equation (3.1) from the assumption **(H1)**, we have:

$$|D(\phi)|^2 \leq k \int_{-\infty}^{0} |\phi(\theta)|^2 \mu(d\theta). \tag{3.7}$$

Lemma 3.4. *Let $p > 1$, $\varepsilon > 0$ and $a, b \in R$. Then*

$$|a+b|^p \leq \left[1+\varepsilon^{\frac{1}{p-1}}\right]^{p-1}\Big(|a|^p + \frac{|b|^p}{\varepsilon}\Big).$$

Remark 3.5. For any $\epsilon_1 > 0$ and $\epsilon_2 \in (0,1)$, by the equations (3.2) and (3.3) from the assumption **(H2)**, one can get the following strong monotone condition:

$$\begin{aligned} 2\Big[\phi(0) - D(\phi)\Big]^T \Big[b(\phi)\Big] + \Big|\sigma(\phi)\Big|^2 \\ \leq -\alpha_1|\phi(0)|^2 + \alpha_2 \int_{-\infty}^{0} |\phi(\theta)|^2 \mu(d\theta) + M_1, \end{aligned} \tag{3.8}$$

where, $\alpha_1 = 2\lambda_1 - 2\epsilon_1 - \dfrac{\lambda_3}{1-\epsilon_2}$, $\alpha_2 = 2\lambda_2 + 2k\epsilon_1 + \dfrac{\lambda_4}{1-\epsilon_2}$ and
$M_1 = \dfrac{1}{\epsilon_1}|b(0)|^2 + \dfrac{1}{\epsilon_2}|\sigma(0)|^2$.

Lemma 3.6. *For any $\xi \in C_r$, under the assumption* (**H1**) *we have,*

$$|\xi(0) - D(\xi))|^2 \leq M\|\xi\|_r^2,$$

where $M = (1+k)(1+\mu^{(2r)})$.

Proof: By the Lemma 3.4 and the assumption (**H1**) we have,

$$\begin{aligned}
|\xi(0) - D(\xi))|^2 &\leq (1+\varepsilon)|\xi(0)|^2 + (1+\frac{1}{\varepsilon})|D(\xi)|^2 \\
&\leq (1+\varepsilon) \sup_{-\infty<\theta\leq 0} e^{2r\theta}|\xi(\theta)|^2 + k(1+\frac{1}{\varepsilon})\int_{-\infty}^{0} e^{2r\theta}|\xi(\theta)|^2 e^{-2r\theta}\mu(d\theta) \\
&\leq (1+\varepsilon) \sup_{-\infty<\theta\leq 0} e^{2r\theta}|\xi(\theta)|^2 \\
&\quad + k(1+\frac{1}{\varepsilon})\int_{-\infty}^{0} \sup_{-\infty<\theta\leq 0} e^{2r\theta}|\xi(\theta)|^2 e^{-2r\theta}\mu(d\theta) \\
&\leq (1+\varepsilon)\|\xi\|_r^2 + k(1+\frac{1}{\varepsilon})\int_{-\infty}^{0} \|\xi\|_r^2 e^{-2r\theta}\mu(d\theta) \\
&= (1+\varepsilon)\|\xi\|_r^2 + k(1+\frac{1}{\varepsilon})\|\xi\|_r^2\mu^{(2r)}.
\end{aligned}$$

Set $\varepsilon = k$, so that

$$|\xi(0) - D(\xi))|^2 \leq (1+k)(1+\mu^{(2r)})\|\xi\|_r^2 = M\|\xi\|_r^2. \tag{3.9}$$

Lemma 3.7. *Under* (**H1**) *let* $\xi, \eta \in C_r$, *then*

$$|\xi(0) - \eta(0) - (D(\xi) - D(\eta))|^2 \leq M\|\xi - \eta\|_r^2, \tag{3.10}$$

where, $M = (1+k)(1+\mu^{(2r)})$.
The proof is similar to that of Lemma 3.6, we omit it here.

Lemma 3.8. *Let* $\xi \in C_r((-\infty, 0]; R^d)$ *and* (**H1**) *holds, then for* $t > 0$ *there exist positive constants* k_1 *and* k_2 *such that*

$$\sup_{0<s\leq t} |\xi(s)|^2 \leq k_1\|\xi\|_r^2 + k_2 \sup_{0<s\leq t} |\xi(s) - D(\xi_s)|^2, \tag{3.11}$$

where $k_1 = \frac{k\mu^{(2r)}}{1-k}$ *and* $k_2 = \frac{1}{(1-k)^2}$.

Proof: By the lemma 3.4 and condition (**H1**), for any $\varepsilon > 0$,

$$\begin{aligned}
|\xi(t)|^2 &= |D(\xi_t) + \xi(t) - D(\xi_t)|^2 \\
&\leq (1+\varepsilon)\Big(\frac{|D(\xi_t)|^2}{\varepsilon} + |\xi(t) - D(\xi_t)|^2\Big) \\
&\leq (1+\varepsilon)\Big(\frac{k^2}{\varepsilon}\int_{-\infty}^0 |\xi(t+\theta)|^2\mu(d\theta) + |\xi(t) - D(\xi_t)|^2\Big).
\end{aligned}$$

Taking $\varepsilon = \frac{k}{1-k}$ and using the definition of norm in the phase space $C_r((-\infty, 0]; R^d)$, we obtain

$$\begin{aligned}
\sup_{0<s\leq t} |\xi(s)|^2 &\leq \sup_{0<s\leq t}\left[k\int_{-\infty}^0 |\xi(s+\theta)|^2\mu(d\theta) + \frac{1}{(1-k)}|\xi(s) - D(\xi_s)|^2\right] \\
&= \sup_{0<s\leq t}\Bigg[k\int_{-\infty}^{-s} e^{2r(s+\theta)}|\xi(s+\theta)|^2 e^{-2r(s+\theta)}\mu(d\theta) + k\int_{-s}^0 |\xi(s+\theta)|^2\mu(d\theta) \\
&\quad + \frac{1}{(1-k)}|\xi(s) - D(\xi_s)|^2\Bigg] \\
&\leq \sup_{0<s\leq t}\Bigg[ke^{-2rs}\|\xi\|_r^2\int_{-\infty}^0 e^{-2r\theta}\mu(d\theta) + k\sup_{0<u\leq s}|\xi(u)|^2\int_{-\infty}^0 \mu(d\theta) \\
&\quad + \frac{1}{(1-k)}|\xi(s) - D(\xi_s)||^2\Bigg] \\
&\leq k\|\xi\|_r^2\int_{-\infty}^0 e^{-2r\theta}\mu(d\theta) + k\sup_{0<s\leq t}|\xi(s)|^2\int_{-\infty}^0 \mu(d\theta) \\
&\quad + \frac{1}{(1-k)}\sup_{0<s\leq t}|\xi(s) - D(\xi_s)|^2 \\
&= k\mu^{(2r)}\|\xi\|_r^2 + k\sup_{0<s\leq t}|\xi(s)|^2 + \frac{1}{(1-k)}\sup_{0<s\leq t}|\xi(s) - D(\xi_s)|^2.
\end{aligned}$$

Subsequently,

$$\begin{aligned}\sup_{0<s\le t}|\xi(s)|^2 &\le \frac{k\mu^{(2r)}}{1-k}\|\xi\|_r^2+\frac{1}{(1-k)^2}\sup_{0<s\le t}|\xi(s)-D(\xi_s)|^2\\ &= k_1\|\xi\|_r^2+k_2\sup_{0<s\le t}|\xi(s)-D(\xi_s)|^2,\end{aligned}$$

where $k_1=\frac{k\mu^{(2r)}}{1-k}$ and $k_2=\frac{1}{(1-k)^2}$. The proof is complete. □

Lemma 3.9. *Let $r>0$ and $\xi,\eta\in C_r((-\infty,0];R^d)$. Let condition* (**H1**) *hold. Then there exist positive constants k_3,k_4, such that*

$$\sup_{0\le s\le t}|\xi(s)-\eta(s)|^2\le k_3\|\xi-\eta\|_r^2+k_4\sup_{0\le s\le t}|\xi(s)-\eta(s)-D(z_s)+D(\eta_s)|^2,$$

where $k_3=\frac{k\mu^{(2r)}}{(1-k)}$ and $k_4=\frac{1}{(1-k)^2}$.

Since the proof is similar to that of Lemma 3.9, we will not give details here.

Remark 3.10. In the next theorem, by the initial data ξ, we mean the initial function or initial segment process. Thus ξ is a function, not a fixed constant. To highlight the initial segment process, we denote by $x(t;\xi)$ and $x_t(\xi)$ the solution and the solution map of (2.3), respectively.

Theorem 3.11. *Under assumptions* (**H1**) *and* (**H2**),

- **(i):** *For any initial data $\xi\in C_r$ the NSFDEwID* (2.3) *has a global solution $x(t)$ almost surely, which is continuous and $\mathcal{F}_t$-adapted.*
- **(ii):** *If $\lambda_1,\lambda_2,\lambda_3$ and λ_4 satisfy $2\lambda_1>73\lambda_3+2\lambda_2\mu^{(2r)}+73\lambda_4\mu^{(2r)}$, then there exist constants $C_1,C_2>0$ and $\lambda\in\big(0,\frac{1}{M}\big[2\lambda_1-73\lambda_3-2\lambda_2\mu^{(2r)}-73\lambda_4\mu^{(2r)}\big]\wedge 2r\big)$ such that for any initial data $\xi\in C_r$,*

$$\mathbb{E}\big(|x(t;\xi)|^2\big)\le C_1+C_2\mathbb{E}\|\xi\|_r^2e^{-\lambda t},\tag{3.12}$$

where $C_1=\frac{2k_2}{\lambda}\Big(\frac{73|\sigma(0)|^2}{\varepsilon_2}+\frac{|b(0)|^2}{\varepsilon_1}\Big)$,

$C_2=\bigg\{k_1+2k_2\bigg[M+\frac{\mu^{(2r)}}{2r-\lambda}\Big((1+k)\lambda+2\lambda_2+2k\varepsilon_1+\frac{73\lambda_4}{1-\varepsilon_2}\Big)\bigg]\bigg\}$ *and $\varepsilon_1,\varepsilon_2$ are both sufficiently small constants such that*

$$\bigg[2\lambda_1-M\lambda-2\varepsilon_1-\frac{73\lambda_3}{1-\varepsilon_2}-\Big(2\lambda_2+2k\varepsilon_1+\frac{73\lambda_4}{1-\varepsilon_2}\Big)\mu^{(2r)}\bigg]>0,$$

namely, solution $x(t;\xi)$ is mean-square bounded.
- **(iii):** *Under conditions in* (**ii**), *for different initial data ξ and η, the corresponding solution $x(t;\xi)$ and $x(t;\eta)$ satisfy*

$$\mathbb{E}\Big(\sup_{0<s\le t}|x(s;\xi)-x(s;\eta)|^2\Big)\le C_3\mathbb{E}\|\xi-\eta\|_r^2e^{-\lambda t},\tag{3.13}$$

where $C_3=\bigg\{k_3+2k_4\bigg[M+\frac{\mu^{(2r)}}{2r-\lambda}\Big((1+k)\lambda+2\lambda_2+73\lambda_4\Big)\bigg]\bigg\}.$

Proof: We divide the proof into three steps:
Step 1 (Proof of (i).): The proof is similar to that of [28, Theorem 3.2], we here only highlight the difficulty from the neutral term. By the conditions (**H1**) and (**H2**), the coefficients are local Lipschitz continuous, then there exists a unique maximal local strong solution $x(t)$ to (2.3) on $t \in (-\infty, \tau_\ell)$. In order to prove that $x(t)$ is a global solution, we need only prove that $\tau_e = \infty$ almost surely. Define $\tau_\ell = \inf\{t \geq 0 : |x(t)| \geq \ell\}$, then τ_ℓ is increasing as $\ell \to \infty$, let $\lim_{\ell\to\infty} \tau_\ell = \tau_\infty \leq \tau_e$ almost surly. If we can show $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s., which implies that $x(t)$ is actually global. This is equivalent to prove that for any $T > 0$, $\mathbb{P}(\tau_\ell \leq T) \to 0$ as $\ell \to \infty$.
Using Lemma 3.4 and Assumption (**H1**), one can derive that

$$
\begin{aligned}
\mathbb{E}|x(t\wedge\tau_\ell)|^2 &\leq (1+\varepsilon)\Big(\mathbb{E}|x(t\wedge\tau_\ell) - D(x_{t\wedge\tau_\ell})|^2\Big) \\
&\quad + k(1+\frac{1}{\varepsilon})\Big(\mathbb{E}\int_{-\infty}^0 |x(t\wedge\tau_\ell+\theta)|^2\mu(d\theta)\Big) \\
&\leq (1+\varepsilon)\Big(\mathbb{E}|x(t\wedge\tau_\ell) - D(x_{t\wedge\tau_\ell})|^2\Big) \\
&\quad + k(1+\frac{1}{\varepsilon})\mathbb{E}\Big[\int_{-\infty}^{-t\wedge\tau_\ell} |x(t\wedge\tau_\ell+\theta)|^2\mu(d\theta) \\
&\quad + \int_{-t\wedge\tau_\ell}^0 |x(t\wedge\tau_\ell+\theta)|^2\mu(d\theta)\Big] \\
&\leq (1+\varepsilon)\Big(\mathbb{E}|x(t\wedge\tau_\ell) - D(x_{t\wedge\tau_\ell})|^2\Big) \\
&\quad + k(1+\frac{1}{\varepsilon})\mathbb{E}\Big[\sup_{-\infty<t\wedge\tau_\ell+\theta\leq 0}\Big(e^{2r(t\wedge\tau_\ell+\theta)}|x(t\wedge\tau_\ell+\theta)|^2\Big) \\
&\quad \int_{-\infty}^0 e^{-2r(t\wedge\tau_\ell+\theta)}\mu(d\theta) + \Big(|x(t\wedge\tau_\ell)|^2\Big)\int_{-\infty}^0 \mu(d\theta)\Big] \\
&= (1+\varepsilon)\Big(\mathbb{E}|x(t\wedge\tau_\ell) - D(x_{t\wedge\tau_\ell})|^2\Big) \\
&\quad + ke^{-2r(t\wedge\tau_\ell)}(1+\frac{1}{\varepsilon})\mu^{(2r)}\mathbb{E}\|\xi\|_r^2 + k(1+\frac{1}{\varepsilon})\mathbb{E}|x(t\wedge\tau_\ell)|^2.
\end{aligned}
$$

Choosing $\varepsilon > \dfrac{k}{1-k}$ implies $\gamma = k(1+\dfrac{1}{\varepsilon}) < 1$, we arrive at

$$
\begin{aligned}
\Big(\mathbb{E}|x(t\wedge\tau_\ell)|^2\Big) \leq &\frac{1+\varepsilon}{(1-\gamma)}\Big(\mathbb{E}|x(t\wedge\tau_\ell) - D(x_{t\wedge\tau_\ell})|^2\Big) \\
&+ \frac{\gamma e^{-2r(t\wedge\tau_\ell)}\mu^{(2r)}}{1-\gamma}\mathbb{E}\|\xi\|_r^2.
\end{aligned} \tag{3.14}
$$

Applying the Itô formula to $|x(t)-D(x_t)|^2$, by (3.9) and the monotone condition (3.8) yields for any $t\in[0,T]$,

$$\begin{aligned}\mathbb{E}|x(t\wedge\tau_\ell)-D(x_{t\wedge\tau_\ell})|^2 &= \mathbb{E}|x(0)-D(\xi)|^2\\ &\quad+2\mathbb{E}\int_0^{t\wedge\tau_\ell}[x(s)-D(x_s)]^T b(x_s)ds+\mathbb{E}\int_0^{t\wedge\tau_\ell}|\sigma(x_s)|^2ds\\ &\le M\mathbb{E}\|\xi\|_r^2-\alpha_1\mathbb{E}\int_0^{t\wedge\tau_\ell}|x(s)|^2ds\\ &\quad+\alpha_2\mathbb{E}\int_0^{t\wedge\tau_\ell}\int_{-\infty}^0|x(s+\theta)|^2\mu(d\theta)ds+N.\end{aligned}\tag{3.15}$$

By the fact [(3.12), from [28]], we have

$$\begin{aligned}&\int_0^{t\wedge\tau_\ell}\int_{-\infty}^0|x(s+\theta)|^2\mu(d\theta)ds\\ &\qquad\le\frac{1}{2r}\|\xi\|_r^2\mu^{(2r)}+\int_0^t|x(s\wedge\tau_\ell)|^2ds.\end{aligned}\tag{3.16}$$

Substituting (3.16) into (3.15), one has

$$\mathbb{E}|x(t\wedge\tau_\ell)-D(x_{t\wedge\tau_\ell})|^2\le L_1+J_1\int_0^t\mathbb{E}|x(s\wedge\tau_\ell)|^2ds,\tag{3.17}$$

where, $L_1=\Big[\Big(M+\frac{\alpha_2\mu^{(2r)}}{2r}\Big)\mathbb{E}\|\xi\|_r^2+N\Big]$ and $J_1=-\alpha_1+\alpha_2$. Therefore, by substituting (3.17) into (3.14) we obtain

$$\begin{aligned}\mathbb{E}|x(t\wedge\tau_\ell)|^2&\le\frac{(1+\varepsilon)L_1}{(1-\gamma)}+\frac{J_1(1+\varepsilon)}{(1-\gamma)}\mathbb{E}\int_0^t|x(s\wedge\tau_\ell)|^2ds\\ &\qquad+\frac{\gamma e^{-2r(t\wedge\tau_\ell)}\mu^{(2r)}}{1-\gamma}\mathbb{E}\|\xi\|_r^2.\end{aligned}\tag{3.18}$$

Hence, applying Gronwall's inequality yields

$$\mathbb{E}|x(t\wedge\tau_\ell)|^2\le L_2e^{J_2t},\tag{3.19}$$

where, $L_2=\frac{(1+\varepsilon)L_1}{(1-\gamma)}+\frac{\gamma e^{-2r(t\wedge\tau_\ell)}\mu^{(2r)}}{1-\gamma}\mathbb{E}\|\xi\|_r^2$, $J_2=\frac{J_1(1+\varepsilon)}{(1-\gamma)}$. According to the definition of τ_ℓ, we have

$$\begin{aligned}\mathbb{E}\Big(|x(T\wedge\tau_\ell)|^2\Big)&=\mathbb{E}\Big(|x(T\wedge\tau_\ell)|^2I_{\{T\le\tau_\ell\}}\Big)+\mathbb{E}\Big(|x(T\wedge\tau_\ell)|^2I_{\{T\ge\tau_\ell\}}\Big)\\ &\ge\ell^2\mathbb{P}\big(\tau_\ell\le T\big).\end{aligned}\tag{3.20}$$

Consequently, by (3.19)

$$\mathbb{P}(\tau_\ell\le T)\le\frac{L_2e^{J_2t}}{\ell^2}.$$

This implies

$$\limsup_{\ell\to\infty}\mathbb{P}(\tau_\ell\le T)=0,$$

which means that (2.3) has a unique global solution $x(t)$ on $[0,\infty)$ almost surely.

Step 2 (Proof of (ii).): For any $\lambda \in \big(0, \frac{1}{M}\big[2\lambda_1 - 73\lambda_3 - 2\lambda_2\mu^{(2r)} - 73\lambda_4\mu^{(2r)}\big] \wedge 2r\big)$ applying the Itô formula to $e^{\lambda t}|x(t) - D(x_t)|^2$, together with the (3.9), the Lemma 3.4 with $\varepsilon = k$, the monotone condition (3.8) and the assumption **(H1)**, yields for any $t \in [0,T]$ that,

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{0<s\le t} e^{\lambda s}|x(s) - D(x_s)|^2\Big) \le \mathbb{E}|x(0) - D(\xi)|^2\\
&\quad + \mathbb{E}\Big(\sup_{0<s\le t}\int_0^s e^{\lambda u}\Big[\lambda|x(u) - D(x_u)|^2\\
&\quad + 2[x(u) - D(x_u)]^T b(x_u) + |\sigma(x_u)|^2\Big]du\Big)\\
&\quad + 2\mathbb{E}\Big(\sup_{0<s\le t}\int_0^s e^{\lambda u}[x(u) - D(x_u)]^T\sigma(x_u)dw(u)\Big)\\
&\le M\mathbb{E}\|\xi\|_r^2 + \lambda(1+k)\mathbb{E}\Big(\sup_{0<s\le t}\int_0^s e^{\lambda u}|x(u)|^2du\Big)\\
&\quad + \lambda(1+\frac{1}{k})\mathbb{E}\Big(\sup_{0<s\le t}\int_0^s e^{\lambda u}|D(x_u)|^2du\Big)\\
&\quad + \mathbb{E}\Big(\sup_{0<s\le t}\int_0^s e^{\lambda u}\Big[-\alpha_1\big|x(u)\big|^2 + \alpha_2\int_{-\infty}^0\big|x(u+\theta)\big|^2\mu(d\theta) + N\Big]du\Big)\\
&\quad + 2\mathbb{E}\Big(\sup_{0<s\le t}\int_0^s e^{\lambda u}[x(u) - D(x_u)]^T\sigma(x_u)dw(u)\Big)\\
&\le M\mathbb{E}\|\xi\|_r^2 + ((1+k)\lambda - \alpha_1)\mathbb{E}\Big(\sup_{0<s\le t}\int_0^s e^{\lambda u}|x(u)|^2du\Big)\\
&\quad + ((1+k)\lambda + \alpha_2)\mathbb{E}\Big(\sup_{0<s\le t}\int_0^s e^{\lambda u}\int_{-\infty}^0\big|x(u+\theta)\big|^2\mu(d\theta)du\Big)\\
&+ \frac{N}{\lambda}(e^{\lambda t} - 1) \quad + 2\mathbb{E}\Big(\sup_{0<s\le t}\int_0^s e^{\lambda u}[x(u) - D(x_u)]^T\sigma(x_u)dw(u)\Big).
\end{aligned}
\tag{3.21}
$$

By [(3.13), [28]], we have

$$
\begin{aligned}
&\int_0^t\int_{-\infty}^0 e^{\lambda s}|x(s+\theta)|^2\mu(d\theta)ds\\
&\qquad \le \frac{1}{2r-\lambda}\|\xi\|_r^2\mu^{(2r)} + \mu^{(2r)}\int_0^t e^{\lambda s}|x(s)|^2ds.
\end{aligned}
\tag{3.22}
$$

Now, using the Burkholder-Davis-Gundy inequality, Lemma 3.4, **(H1)** and condition (3.3), we obtain

$$
\begin{aligned}
&2\mathbb{E}\Big(\sup_{0<s\le t}\int_0^s e^{\lambda u}[x(u)-D(x_u)]^T\sigma(x_u)dw(u)\Big)\\
&\le 8\sqrt{2}\mathbb{E}\Big(\int_0^t e^{2\lambda s}|[x(s)-D(x_s)]^T\sigma(x_s)|^2ds\Big)^{\frac{1}{2}}\\
&\le 12\mathbb{E}\Big(\int_0^t e^{2\lambda s}|x(s)-D(x_s)|^2|\sigma(x_s)|^2ds\Big)^{\frac{1}{2}}\\
&\le 12\mathbb{E}\Big[\Big(\sup_{0<s\le t}e^{\lambda s}|x(s)-D(x_s)|^2\Big)^{\frac{1}{2}}\Big(\int_0^t e^{\lambda s}|\sigma(x_s)|^2ds\Big)^{\frac{1}{2}}\Big]\\
&\le \frac{1}{2}\mathbb{E}\Big(\sup_{0<s\le t}e^{\lambda s}|x(s)-D(x_s)|^2\Big)+72\mathbb{E}\int_0^t e^{\lambda s}|\sigma(x_s)|^2ds\\
&\le \frac{1}{2}\mathbb{E}\Big(\sup_{0<s\le t}e^{\lambda s}|x(s)-D(x_s)|^2\Big)+\frac{72\lambda_3}{1-\varepsilon_2}\mathbb{E}\int_0^t e^{\lambda s}\big|x(s)\big|^2ds\\
&\quad+\frac{72\lambda_4}{1-\varepsilon_2}\mathbb{E}\int_0^t\int_{-\infty}^0 e^{\lambda s}\big|x(s+\theta)\big|^2\mu(d\theta)ds\\
&\quad+\frac{72}{\lambda\varepsilon_2}|\sigma(0)|^2(e^{\lambda t}-1).
\end{aligned}
\tag{3.23}
$$

By taking in account, the value of N, α_1 and α_2, substitute (3.22) and (3.23) into (3.21) with $\lambda \le 2r$, we arrive at

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{0<s\le t}e^{\lambda s}|x(s)-D(x_s)|^2\Big)\\
&\le 2\Bigg[M+\frac{\mu^{(2r)}}{2r-\lambda}\Big((1+k)\lambda+2\lambda_2+2k\varepsilon_1+\frac{73\lambda_4}{1-\varepsilon_2}\Big)\Bigg]\mathbb{E}\|\xi\|_r^2\\
&\quad+\frac{2}{\lambda}\Big(\frac{73|\sigma(0)|^2}{\varepsilon_2}+\frac{|b(0)|^2}{\varepsilon_1}\Big)e^{\lambda t}+2\Bigg[(1+k)\lambda-2\lambda_1+2\varepsilon_1+\frac{73\lambda_3}{1-\varepsilon_2}\\
&\quad+\Big((1+k)\lambda+2\lambda_2+2k\varepsilon_1+\frac{73\lambda_4}{1-\varepsilon_2}\Big)\mu^{(2r)}\Bigg]\mathbb{E}\Big(\sup_{0<s\le t}\int_0^s e^{\lambda u}\big|x(u)\big|^2du\Big).
\end{aligned}
$$

Consequently, by the Lemma 3.8, we have

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0<s\le t} e^{\lambda s}|x(s)|^2\Big)\\
&\le \left\{k_1 + 2k_2\left[M + \frac{\mu^{(2r)}}{2r-\lambda}\Big((1+k)\lambda + 2\lambda_2 + 2k\varepsilon_1 + \frac{73\lambda_4}{1-\varepsilon_2}\Big)\right]\right\}\mathbb{E}\|\xi\|_r^2\\
&+ \frac{2k_2}{\lambda}\Big(\frac{73|\sigma(0)|^2}{\varepsilon_2} + \frac{|b(0)|^2}{\varepsilon_1}\Big)e^{\lambda t} - 2k_2\Bigg[2\lambda_1 - M\lambda - 2\varepsilon_1 - \frac{73\lambda_3}{1-\varepsilon_2}\\
&- \Big(2\lambda_2 + 2k\varepsilon_1 + \frac{73\lambda_4}{1-\varepsilon_2}\Big)\mu^{(2r)}\Bigg]\mathbb{E}\Big(\sup_{0<s\le t}\int_0^s e^{\lambda u}\big|x(u)\big|^2 du\Big).
\end{aligned}$$

Note that $2\lambda_1 > 73\lambda_3 + 2\lambda_2\mu^{(2r)} + 73\lambda_4\mu^{(2r)}$, $k \in (0,1)$ and

$$\lambda \in \Big(0, \frac{1}{M}\big[2\lambda_1 - 73\lambda_3 - 2\lambda_2\mu^{(2r)} - 73\lambda_4\mu^{(2r)}\big] \wedge 2r\Big),$$

we can choose $\varepsilon_1, \varepsilon_2$ sufficiently small such that:

$\left[2\lambda_1 - M\lambda - 2\varepsilon_1 - \frac{73\lambda_3}{1-\varepsilon_2} - \Big(2\lambda_2 + 2k\varepsilon_1 + \frac{73\lambda_4}{1-\varepsilon_2}\Big)\mu^{(2r)}\right] > 0$, thus

$$\mathbb{E}\Big(\sup_{0<s\le t}|x(s)|^2\Big) \le C_1 + C_2\mathbb{E}\|\xi\|_r^2 e^{-\lambda t}, \tag{3.24}$$

where, $C_1 = \frac{2k_2}{\lambda}\Big(\frac{73|\sigma(0)|^2}{\varepsilon_2} + \frac{|b(0)|^2}{\varepsilon_1}\Big)$ and

$C_2 = \left\{k_1 + 2k_2\left[M + \frac{\mu^{(2r)}}{2r-\lambda}\Big((1+k)\lambda + 2\lambda_2 + 2k\varepsilon_1 + \frac{73\lambda_4}{1-\varepsilon_2}\Big)\right]\right\}.$

This means that the solution $x(t;\xi)$ is mean-square bounded. □

Step 3 (Proof of (iii).): Let $x(t;\xi)$ and $x(t;\eta)$ be two different solutions with two different initial date ξ, η to the equation (2.3), then:

$$\begin{aligned}
d(x(t;\xi) - x(t;\eta) - D(x_t(\xi)) + D(x_t(\eta))) &= \{b(x_t(\xi)) - b(x_t(\eta))\}dt\\
&+ \{\sigma(x_t(\xi)) - \sigma(x_t(\eta))\}dw(t).
\end{aligned} \tag{3.25}$$

For simplicity, let $z(t) = x(t;\xi) - x(t;\eta)$, $\bar{D}(t) = D(x_t(\xi)) - D(x_t(\eta))$, $\bar{b}(t) = b(x_t(\xi)) - b(x_t(\eta))$ and $\bar{\sigma}(t) = \sigma(x_t(\xi)) - \sigma(x_t(\eta))$ with the initial data $\xi - \eta$. For $\lambda > 0$ defined in (ii), applying the Itô formula to $e^{\lambda t}\mathbb{E}|z(t) - \bar{D}(u)|^2$ with (3.10) gives:

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0<s\le t} e^{\lambda s}|z(s) - \bar{D}(s)|^2\Big) \le M\mathbb{E}\|\xi - \eta\|_r^2\\
&\quad + \mathbb{E}\Big(\sup_{0<s\le t}\int_0^s e^{\lambda u}\big[\lambda|z(u) - \bar{D}(u)|^2 + 2[z(u) - \bar{D}(u)]^T\bar{b}(z_u) + |\bar{\sigma}(z_u)|^2\big]du\\
&\quad + 2\mathbb{E}\Big(\sup_{0<s\le t}\int_0^s e^{\lambda u}[z(u) - \bar{D}(t)]^T\bar{\sigma}(z_u)dw(u)\Big).
\end{aligned}$$

By the Lemma 3.4 and the assumption (**H1**), with $\varepsilon = k$, we have

$$|z(u) - \bar{D}(u)|^2 \leq (1+k)|z(u)|^2 + (1+k)\int_{-\infty}^{0} |z(u+\theta)|^2 \mu(d\theta), \tag{3.26}$$

and by assumption (**H2**),

$$2[z(u) - \bar{D}(u)]^T \bar{b}(z_u) \leq -2\lambda_1 |z(u)|^2 + 2\lambda_2 \int_{-\infty}^{0} |z(u+\theta)|^2 \mu(d\theta), \tag{3.27}$$

$$|\bar{\sigma}(z_u)|^2 \leq \lambda_3 |z(u)|^2 + \lambda_4 \int_{-\infty}^{0} |z(u+\theta)|^2 \mu(d\theta), \tag{3.28}$$

similar to (3.23), with the condition (3.3) we have

$$\begin{aligned} &2\mathbb{E}\Big(\sup_{0<s\leq t} \int_0^s e^{\lambda u}[z(u) - \bar{D}(t)]^T \bar{\sigma}(z_u) dw(u)\Big) \\ &\leq \frac{1}{2}\mathbb{E}\Big(\sup_{0<s\leq t} e^{\lambda s}|z(s) - \bar{D}(s)|^2\Big) \\ &\quad + 72\lambda_3 \mathbb{E}\Big(\sup_{0<s\leq t} \int_0^s e^{\lambda u}|z(u)|^2 du\Big) \\ &\quad + 72\lambda_4 \mathbb{E}\Big(\sup_{0<s\leq t} \int_0^s \int_{-\infty}^0 e^{\lambda u}|z(u+\theta)|^2 \mu(d\theta) du\Big). \end{aligned}$$

Consequently, by considering the fact (3.22) and the Lemma 3.9, we get that $\mathbb{E}\Big(\sup_{0<s\leq t} e^{\lambda s}|z(s)|^2\Big) \leq \Bigg\{k_3 + 2k_4\Bigg[M + \frac{\mu^{(2r)}}{2r-\lambda}\Big((1+k)\lambda + 2\lambda_2 + 73\lambda_4\Big)\Bigg]\Bigg\}\mathbb{E}\|\xi - \eta\|_r^2 - 2k_4\Bigg[2\lambda_1 - 73\lambda_3 - M\lambda - 2\lambda_2\mu^{(2r)} - 73\lambda_4\mu^{(2r)}\Bigg]\mathbb{E}\int_0^t e^{\lambda s}|z(s)|^2 ds.$

Note that $2\lambda_1 > 73\lambda_3 + 2\lambda_2\mu^{(2r)} + 73\lambda_4\mu^{(2r)}$ and $\lambda \in \big(0, \frac{1}{M}\big[2\lambda_1 - 73\lambda_3 - 2\lambda_2\mu^{(2r)} - 73\lambda_4\mu^{(2r)}\big] \wedge 2r\big)$, $k \in (0,1)$, yields that: $\Big[2\lambda_1 - 73\lambda_3 - M\lambda - 2\lambda_2\mu^{(2r)} - 73\lambda_4\mu^{(2r)}\Big] > 0$, implies to

$$\mathbb{E}\Big(\sup_{0<s\leq t} |x(s;\xi) - x(s;\eta)|^2\Big) \leq C_3 \mathbb{E}\|\xi - \eta\|_r^2 e^{-\lambda t}, \tag{3.29}$$

where, $C_3 = \Bigg\{k_3 + 2k_4\Bigg[M + \frac{\mu^{(2r)}}{2r-\lambda}\Big((1+k)\lambda + 2\lambda_2 + 73\lambda_4\Big)\Bigg]\Bigg\}$.

The proof is therefore complete. □

4. Stability in Distribution

In this section, we shall study the stability in distribution for the segment process $\{x_t\}_{t\geq 0}$. We need the following lemma.

Lemma 4.1. *If the process* $\{x(t)\}_{t\geq 0}$ *is the unique solution of the equation* (2.3)*, then the segment process* $\{x_t\}_{t\geq 0}$ *is a strong homogeneous Markov process on* C_r*: for any finite* $\mathcal{F}_t$*-stopping time* $\tau > 0$*,* $t \geq 0$ *and a Borel set* $A \in \mathcal{B}(C_r)$

$$\mathbb{P}(x_{t+\tau} \in A) \mid \mathcal{F}_\tau) = \mathbb{P}(x_{t+\tau} \in A) \mid x_\tau) P\text{-}a.s..$$

Proof: We divide the proof into two steps (the proof is similer to (Theorem 4.2, [28])).

Step 1: Strong Markov Property: For all $0 \leq s \leq t < \infty$ and a finite stopping time $\tau > 0$ be, we consider the equation

$$\begin{aligned} x(t+\tau) &= x(\tau) + \big(D(x_{t+\tau}) - D(x_\tau)\big) \\ &\qquad + \int_\tau^t b(x_{s+\tau})ds + \int_\tau^t \sigma(x_{s+\tau})dw(s) \\ &= x(\tau) + \big(D(x_{t+\tau}) - D(x_\tau)\big) \\ &\qquad + \int_\tau^{t+\tau} b(x_s)ds + \int_\tau^{t+\tau} \sigma(x_s)dw(s), \end{aligned}$$

By the definition of x_t, we have $x_t(t_0, x_{t_0}) = x(t+\theta, t_0; x(t_0+\theta))$ with $x_{t_0}(t_0, x_{t_0}) = x_{t_0}$, for any $t_0 \leq t$. Note that $w(t)$ is a strong Markov process with independent increment. It follows that $\mathcal{F}_\tau$ is independent of $\mathcal{G}_\tau = \sigma\{w(\tau+s) - w(\tau)\}$ for any $s > 0$. Also, note that $x_t(\tau, \xi)$ depends completely on the increments $w(\tau+s) - w(\tau)$ and so is $\mathcal{G}_\tau$-measurable when $x_\tau = \xi$ is given. Hence $x_t(\tau, \xi)$ is independent of $\mathcal{F}_\tau$ for any $t > 0$. For any $A \subset C_r$, we therefore have

$$\begin{aligned} \mathbb{E}(1_{\{x_{t+\tau}(0,\xi)\}\in A} \mid \mathcal{F}_\tau) &= \mathbb{E}(1_{\{x_t(\tau,x_\tau)\}\in A} \mid \mathcal{F}_\tau) \\ &= \mathbb{E}(1_{\{x_t(\tau,\xi)\}\in A} | \xi = x_\tau) \\ &= \mathbb{P}(x_\tau = \xi, x_t(\tau,\xi) \in A | \xi = x_t) \\ &= \mathbb{E}(1_{\{x_t(\tau,\xi)\}\in A} \mid x_\tau) \end{aligned}$$

Toward that standard technique, it takes after that for at whatever bounded Borel measurable function $\varphi : C_r \to R$,

$$\mathbb{E}(\varphi(x_{t+\tau}) \mid \mathcal{F}_\tau] = \mathbb{E}(\varphi(x_{t+\tau}) \mid x_\tau),$$

which ends the proof and yields that x_t is a strong Markov process. □

Step 2: Homogeneity: According to the definition of transition probability:

$$P(\xi, u; A, t+u) = \mathbb{P}(x_{t+u}(u,\xi) \in A),$$

where $x_{t+u}(u,\xi)$ is determined by the solution $x(t)$:

$$x(t+u) = \xi(0) + D(x_{t+u}) - D(\xi) + \int_u^{t+u} b(x_s)ds + \int_u^{t+u} \sigma(x_s)dw(s).$$

This equation is equivalent to

$$x(t+u) = \xi(0) + D(x_{t+u}) - D(\xi) + \int_u^t b(x_{s+u})ds + \int_u^t \sigma(x_{s+u})d\widetilde{w}(s), \quad (4.1)$$

where $\widetilde{w}(s) = w(s+u) - w(u)$ is clearly an d-dimensional Brownian motion. So , we have

$$x(t) = \xi(0) + D(x_t) - D(\xi) + \int_0^t b(x_s)ds + \int_0^t \sigma(x_s)dw(s), \tag{4.2}$$

with $x_0 = \xi$. Comparing equations (4.1) with (4.2) and noting that x_{t+u} and x_t completely depend on $x(t+u)$ and $x(t)$ and their history, we see by the week uniqueness that $\{x_{t+u}\}_{t\geq 0}$ are identical in probability law. Consequently,

$$\mathbb{P}(x_{t+u}(u,\xi) \in A) = \mathbb{P}(x_t(0,\xi) \in A),$$

namely,

$$P(\xi, u; A, t+u) = P(\xi, 0; A, t).$$

This complete the proof. □

4.1. **Stability in Distribution of x_t.** To address the stability of x_t in this subsection, let us highlight some notations, see [26] for details. For $t \geq 0$, denoted by $P(\xi; t, \cdot)$ the transition probability of $\{x_t\}_{t\geq 0}$ and for any $P_1, P_2 \in \mathcal{P}(C_r)$, define the metric $d_{\mathbb{L}}$ by

$$d_{\mathbb{L}}(P_1, P_2) = \sup_{g\in\mathbb{L}} \left| \int_{C_r} g(\xi)P_1(d\xi) - \int_{C_r} g(\eta)P_2(d\eta) \right|,$$

where $\mathbb{L} = \{g : C_r \to R : |g(\xi) - g(\eta)| \leq \|\xi - \eta\|_r \quad \text{and} \quad |g(\cdot)| \leq 1 \quad \text{for} \quad \xi, \eta \in C_r\}$.

Definition 4.2. [26]. The process $x_t(\xi)$ is said to be stable in distribution if there exists a probability measure $\pi \in \mathcal{P}(C_r)$ such that $P(\xi; t, \cdot)$ converges weakly to π as $t \to \infty$ for any $\xi \in C_r$, that is,

$$\lim_{t\to\infty} d_{\mathbb{L}}(P(\xi; t, \cdot), \pi(\cdot)) = 0, \quad \text{for all} \quad \xi \in C_r.$$

In this case, (2.3) is said to be stable in distribution.

We start by investigating some sufficient conditions on the stability in distribution for the segment process x_t on $t \geq 0$.

Lemma 4.3. *Under assumptions* (**H1**) *and* (**H2**), *if* $\lambda_1, \lambda_2, \lambda_3$ *and* λ_4 *satisfy* $2\lambda_1 > 73\lambda_3 + 2\lambda_2\mu^{(2r)} + 73\lambda_4\mu^{(2r)}$, *then there exist constants* $C_4, C_5 > 0$ *and* $\lambda \in \big(0, \frac{1}{M}\big[2\lambda_1 - 73\lambda_3 - 2\lambda_2\mu^{(2r)} - 73\lambda_4\mu^{(2r)}\big] \wedge 2r\big)$ *such that for any initial data* $\xi \in C_r$,

$$\mathbb{E}\|x_t\|_r^2 \leq C_4 + C_5 e^{-\lambda t}, \tag{4.3}$$

where $C_4 = \dfrac{2k_2}{\lambda}\Big(\dfrac{73|\sigma(0)|^2}{\varepsilon_2} + \dfrac{|b(0)|^2}{\varepsilon_1}\Big)$,

$$C_5 = \left\{1 + k_1 + 2k_2\left[M + \frac{\mu^{(2r)}}{2r - \lambda}\Big((1+k)\lambda + 2\lambda_2 + 2k\varepsilon_1 + \frac{73\lambda_4}{1 - \varepsilon_2}\Big)\right]\right\}\mathbb{E}\|\xi\|_r^2.$$

Proof: Noting that $\lambda < 2r$, therefore, correspond to the definition of the norm $\|\cdot\|_r$, it is easy to see that:

$$\mathbb{E}\|x_t\|_r^2 = e^{-\lambda t}\mathbb{E}\|\xi\|_r^2 + \mathbb{E}\Big(\sup_{0<s\leq t}|x(s)|^2\Big). \tag{4.4}$$

From (3.24), we have

$$\mathbb{E}\Big(\sup_{0<s\leq t}|x(s)|^2\Big) \leq \frac{2k_2}{\lambda}\Big(\frac{73|\sigma(0)|^2}{\varepsilon_2} + \frac{|b(0)|^2}{\varepsilon_1}\Big) + \Bigg\{k_1 + 2k_2 \Bigg[M + \frac{\mu^{(2r)}}{2r-\lambda}\Big((1+k)\lambda + 2\lambda_2 + 2k\varepsilon_1 + \frac{73\lambda_4}{1-\varepsilon_2}\Big)\Bigg]\Bigg\}e^{-\lambda t}\mathbb{E}\|\xi\|_r^2. \tag{4.5}$$

By substituting (4.5) into (4.4), we get

$$\mathbb{E}\|x_t\|_r^2 \leq C_4 + C_5 e^{-\lambda t}, \tag{4.6}$$

where $C_4 = \dfrac{2k_2}{\lambda}\Big(\dfrac{73|\sigma(0)|^2}{\varepsilon_2} + \dfrac{|b(0)|^2}{\varepsilon_1}\Big)$,
$C_5 = \Bigg\{1 + k_1 + 2k_2\Bigg[M + \dfrac{\mu^{(2r)}}{2r-\lambda}\Big((1+k)\lambda + 2\lambda_2 + 2k\varepsilon_1 + \dfrac{73\lambda_4}{1-\varepsilon_2}\Big)\Bigg]\Bigg\}\mathbb{E}\|\xi\|_r^2$.
This means that, the solution map x_t is mean-square bounded and giving the desired assertion (4.3). □

Lemma 4.4. *Under assumptions* (**H1**) *and* (**H2**)*, if* $\lambda_1, \lambda_2, \lambda_3$ *and* λ_4 *satisfy* $2\lambda_1 > 73\lambda_3 + 2\lambda_2\mu^{(2r)} + 73\lambda_4\mu^{(2r)}$*, then there exist constant* $C_6 > 0$ *and* $\lambda \in \big(0, \frac{1}{M}\big[2\lambda_1 - 73\lambda_3 - 2\lambda_2\mu^{(2r)} - 73\lambda_4\mu^{(2r)}\big] \wedge 2r\big)$ *such that for the different initial data* ξ *and* $\eta \in C_r$*, the corresponding solution maps* $x_t(\xi)$ *and* $x_t(\eta)$ *satisfy,*

$$\mathbb{E}\|x_t(\xi) - x_t(\eta)\|_r^2 \leq C_6\mathbb{E}\|\xi - \eta\|_r^2 e^{-\lambda t}, \tag{4.7}$$

where

$$C_6 = \Bigg\{1 + k_3 + 2k_4\Bigg[M + \frac{\mu^{(2r)}}{2r-\lambda}\Big((1+k)\lambda + 2\lambda_2 + 73\lambda_4\Big)\Bigg]\Bigg\}.$$

Namely, the solution map x_t *is mean-square bounded and solution maps from different initial data are convergent.*

Since the proof is comparable to that of Lemma 4.3, we will not illustrate here.

Theorem 4.5. *Under assumptions* (**H1**) *and* (**H2**)*,* (2.3) *is stable in distribution.*

Prove: Since (4.6) and (4.7) hold, the proof is standard, we omit here.

5. Example

In this section, we give an example to verify the theory that we studied. For $t \geq 0$, consider the one-dimensional neutral stochastic differential equation with infinity delay:

$$d[x(t) - D(x_t)] = -cx(t)dt + \Big(cos(x(t)) + \int_{-\infty}^{0} x_t(\theta)\mu(d\theta)\Big)dw(t) \tag{5.1}$$

with initial value $x(t) = \xi$ when $t \in (-\infty, 0]$. Where $c > 0$, $w(t)$ is a scalar Brownian motion, while $D : C_r \to R$ is defined by:

$$D(\phi) = \frac{1}{2}\int_{-\infty}^{0} \phi(\theta)\mu(d\theta).$$

So that by Hölder inequality we have

$$|D(\varphi) - D(\phi)|^2 \leq \frac{1}{4}\int_{-\infty}^{0} |\varphi(\theta) - \phi(\theta)|^2\mu(d\theta), \quad \text{and} \quad k = 0.25$$

Define $b(\phi) = -c\phi, \sigma(\phi) = cos(\phi) + \int_{-\infty}^{0} \phi(\theta)\mu(d\theta)$. It is easy to show that:

$$\begin{aligned}\Big[\varphi(0) - \phi(0) - \big(D(\varphi) - D(\phi)\big)\Big]^T \Big[b(\varphi) - b(\phi)\Big] &= \Big[\varphi(0) - \phi(0) \\ &- \frac{1}{2}\int_{-\infty}^{0} \big(\varphi(\theta) - \phi(\theta)\big)\mu(d\theta)\Big]^T \Big[- c\varphi + c\phi\Big] \\ &\leq -c|\varphi(0) - \phi(0)|^2 + \frac{c}{2}\int_{-\infty}^{0} |\varphi(\theta) - \phi(\theta)|^2\mu(d\theta),\end{aligned} \tag{5.2}$$

and for $\varepsilon = k = 0.25$

$$\begin{aligned}\big|\sigma(\varphi) - \sigma(\phi)\big|^2 &= \big|cos(\varphi) - cos(\phi) + \int_{-\infty}^{0} (\varphi(\theta) - \phi(\theta))\mu(d\theta)\big|^2 \\ &\leq (1+\varepsilon)\big|cos(\varphi) - cos(\phi)\big|^2 + \frac{1+\varepsilon}{\varepsilon}\big|\int_{-\infty}^{0} (\varphi(\theta) - \phi(\theta))\mu(d\theta)\big|^2 \\ &\leq 1.25\big|\varphi(0) - \phi(0)\big|^2 + 5\int_{-\infty}^{0} \big|\varphi(\theta) - \phi(\theta)\big|^2\mu(d\theta).\end{aligned} \tag{5.3}$$

Thus, from (5.2) and (5.3) we get that: $\lambda_1 = c, \lambda_2 = \frac{c}{2}, \lambda_3 = 1.25$ and $\lambda_4 = 5$, hence for any $\mu^{(2r)} < 1 \in M_{2r}$, the solution map x_t of the equation (5.1) satisfies all the properties that we studied whenever $c > \dfrac{91.25 + 365\mu^{(2r)}}{2 - \mu^{(2r)}}$.

6. Conclusion

In this paper, under a sufficient strong monotone condition, we investigate the existence and uniqueness of the global solutions of NSFDEwID in the state space C_r and also address the stability of the solution map x_t.

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