Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 8, No. 3, 2020, pp. 597-609
DOI:10.22034/cmde.2020.31942.1489

Symbolic methods to construct a cusp, breathers, kink, rogue waves and some soliton waves solutions of nonlinear partial differential equations

Md. Nur Alam
School of Mathematical Sciences,
University of Science and Technology of China,
230026, Hefei, China.
E-mail: nuralam23@mail.ustc.edu.cn

Xin Li
School of Mathematical Sciences,
University of Science and Technology of China,
230026, Hefei, China.
E-mail: lixustc@ustc.edu.cn

Abstract A cusp, bright breathers, dark breathers, kink, bright rogue waves and some soliton waves solutions are obtained by using the $exp(-\phi(\xi))$-expansion method for the fourth order Benjamin-Ono equation and BBM equations. The obtained solutions might be indicated and meaningful for narrating the physical phenomena in the real-world. For compatible values of the arbitrary parameter included in the solution, we plot the 3D surface of the all obtained solutions which are shown in Figure 1 to Figure 10.

Keywords. The $exp(-\phi(\xi))$-expansion method, The fourth order Benjamin-Ono equation, BBM equation, Traveling wave solutions, Nonlinear evolution equation.

2010 Mathematics Subject Classification. 35C07, 35C08, 35P99.

1. Introduction

Finding the exact and analytical solutions to nonlinear partial differential equations (NLPDEs) play fundamental and imperative problems in mathematical physics, applied sciences and engineering. As a result, various groups of physicist and mathematicians have been working tirelessly to improve useful methods for providing different types soliton form solutions to NLPDEs. For this reason, several methods have been developed to search the analytical solutions, such as the novel (G'/G)-expansion method [2, 3, 4, 6, 8, 9, 10], the extended exp-function method [26, 28, 29], the modified exp-function method [27, 30], the sine-Gordon expansion method [13, 14], the homogeneous balance method [16, 25], the simplest equation method [23], the inverse scattering transform method [1, 33], the Hirota¡¯s bilinear method [38], the tanh-function method [37], Bell-polynomial method [41], the extended tanh-function method [17, 34, 40], the Exp-function method [20], the sine-cosine method [36], the

Received: 07 February 2019 ; Accepted: 28 April 2019.
∗ corresponding.

Backlund transformation method [22],the modified Exp-function method [32], the Jacobi elliptic function expansion method [21], the $exp(-\phi(\xi))$-expansion method [5, 7] and so on.

The objective of this article is to derive some fresh and wide-applicable a cusp, bright breathers, dark breathers, kink, bright rogue waves and some soliton waves solutions to the fourth order Benjamin-Ono equation and BBM equations through the $exp(-\phi(\xi))$-expansion method which are discussed in section 3. The new analytical solutions performed in this paper are characterized by exponent, trigonometric,rational, and hyperbolic functions. However, we observe that a cusp, bright bright breathers, dark breathers, kink, bright rogue waves and some soliton waves solutions are obtained to the fourth order Benjamin-Ono equation and BBM equations have not been reported previously.

The synopsis of this paper as follows: In Section 2, we give the algorithm of the $exp(-\phi(\xi))$-expansion method. In Section 3, new solutions of the fourth order Benjamin-Ono equation and BBM equations are formulated through the $exp(-\phi(\xi))$-expansion method. In Section 4, graphical representations and numerical experiments of the derived solutions are depicted. Finally, the conclusion of our study is given.

2. Algorithm of $exp(-\phi(\xi))$-expansion method

The following is given as the general nonlinear PDEs (the fourth order Benjamin-Ono equation and BBM equations) with two variables x and t as

$$\mathrm{P}(v, v_t, v_x, v_{tt}, v_{xx}, v_{xt}, ...) = 0, \tag{2.1}$$

where $v(x,t)$ is an unknown function and P is a polynomial in $v(x,t)$.

Step 1: The traveling wave variable

$$\mathrm{v}(x,t) = v(\xi), \xi = x + Vt, \tag{2.2}$$

where V is the speed of the traveling wave and the traveling wave variable (2.2) converts equation (2.1) into the ordinary differential equation

$$\mathrm{R}(v, -Vv\prime, v\prime, v\prime, v\prime, v\prime\prime, v\prime\prime, v\prime\prime, V^2 v\prime\prime, ...) = 0 \tag{2.3}$$

where R is a polynomial in v and its derivatives and the superscripts indicate the ordinary derivatives with respect to ξ.

Step 2: Suppose that the traveling wave solution of equation (2.3) can be expressed as

$$\mathrm{v}(\xi) = \sum_{i=0}^{N} A_i (exp(-\phi(\xi)))^i, \tag{2.4}$$

where $A_i (i = 0, ..., n)$ are constants to be determined, such that $A_N \neq 0$ and $\phi = \phi(\xi)$ satisfies the following ordinary differential equation:

$$(\phi(\xi))\prime = exp(-\phi(\xi)) + \mu exp(\phi(\xi)) + \lambda, \tag{2.5}$$

where $A_N ...., V, \lambda, \mu$ are constants to be determined latter.

Step 3: For a given ansatz equation (for example, the ansatz equation is $(\phi(\xi))\prime =$

$exp(-\phi(\xi)) + \mu exp(\phi(\xi)) + \lambda$ in this paper), the form of v is decided and the homogeneous balance method is used on equation (2.3) to find the coefficients of v.
Step 4: The homogeneous balance method is used to solve the ansatz equation.
Step 5: Finally, the solitary wave solutions of equation (2.1) are obtained by combining steps 3 and 4.

Next, we have five solutions including trigonometric, hyperbolic, exponent and rational function structure of equation (2.5).

When $\mu \neq 0$, $\lambda^2 - 4\mu > 0$,, the solution of equation (2.5) is

$$(\phi(\xi)) = ln(\frac{-\sqrt{\lambda^2-4\mu}tanh(\frac{\sqrt{\lambda^2-4\mu}}{2})(\xi+E)-\lambda}{2\mu}). \tag{2.6}$$

When $\mu \neq 0$, $\lambda^2 - 4\mu < 0$, the solution of equation (2.5) is

$$(\phi(\xi)) = ln(\frac{\sqrt{4\mu-\lambda^2}tan(\frac{\sqrt{4\mu-\lambda^2}}{2})(\xi+E)-\lambda}{2\mu}). \tag{2.7}$$

When $\mu = 0$, $\lambda \neq 0$, $\lambda^2 - 4\mu > 0$,, the solution of equation (2.5) is

$$(\phi(\xi)) = ln(\frac{\lambda}{exp(\lambda(\xi+E))-1}). \tag{2.8}$$

When $\mu \neq 0$, $\lambda \neq 0$, $\lambda^2 - 4\mu = 0$,, the solution of equation (2.5) is

$$(\phi(\xi)) = ln(-\frac{(2\lambda(\xi+E)+2)}{\lambda^2(\xi+E)}). \tag{2.9}$$

When $\mu = 0$, $\lambda = 0$, $\lambda^2 - 4\mu = 0$,, the solution of equation (2.5) is

$$(\phi(\xi)) = ln(\xi+E). \tag{2.10}$$

3. Formulation of solutions

In this section, the method is used to construct a cusp, bright breathers, dark breathers, kink, bright rogue waves and some soliton waves solutions for the fourth order Benjamin-Ono equation and BBM equations which are very important nonlinear evolution equations in the field of nonlinear dynamics.

3.1. **The fourth order Benjamin-Ono equation.** In this subsection, we will present the $exp(-\phi(\xi))$-expansion method to construct bright breathers, dark breathers, bright rogue waves and multiple soliton waves solutions of the fourth order Benjamin-Ono equation which are illustrated in Figure 1 to Figure 5. The fourth order Benjamin-Ono equation is of the form,

$$\mathrm{v}_{tt} + \alpha(v^2)_{xx} + \beta v_{xxxx} = 0, \tag{3.1}$$

where α and β are real non-zero parameters and $(v^2)_{xx}$ is the nonlinear term and v_{xxxx} is dispersion term. In mathematics, the fourth order Benjamin-Ono equation is an significant nonlinear partial integro differential equation that derives one-dimensional internal waves in deep water and it plays a important problem in many scientific applications, such as fluid dynamics, nonlinear optics, plasma physics [18, 35, 19, 15, 39].

By using traveling wave variable $v(\xi) = v(x,t)$, $\xi = x - Vt$, Equation (3.1) converts into a nonlinear ordinary differential equation

$$\mathrm{V}^2 v'' + \alpha(v^2)'' + \beta v^{iv} = 0. \tag{3.2}$$

Performing the first and second integrations of equation (3.2), we get

$$\mathrm{K} + V^2 v + \alpha(v^2) + \beta v'' = 0, \tag{3.3}$$

where K is constant. Making the homogeneous balance between v^2 and v'', we get $N = 2$. Putting the value of N in equation (3.3) our solution is of the form

$$\mathrm{v}(\xi) = A_0 + A_1(exp(-\phi(\xi)) + A_2(exp(-\phi(\xi))), \tag{3.4}$$

where the coefficients A_0, A_1 and A_2 are constants to be evaluated.

Putting the values of equation (3.4) into equation (3.3) and then equating each coefficients of $exp(-\phi(\xi)$ to zero, we get

$$6\beta A_2 + \alpha A_2^2 = 0, \tag{3.5}$$

$$10\beta A_2\lambda + 2\alpha A_1 A_2 + 2\beta A_1 = 0, \tag{3.6}$$

$$\mathrm{V}^2 A_2 + \alpha A_1^2 + 3\beta A_1\lambda + 2\alpha A_0 A_2 + 8\beta A_2\mu + 4\beta A_2\lambda^2 = 0, \tag{3.7}$$

$$\mathrm{V}^2 A_1 + 2\alpha A_1\mu + 6\beta A_2\mu\lambda + \beta A_1\lambda^2 + 2\alpha A_0 A_1 = 0, \tag{3.8}$$

$$\mathrm{V}^2 A_0 + 2\alpha A_0^2 + 2\beta A_2\mu^2 + K + \beta A_1\lambda\mu = 0, \tag{3.9}$$

Using algebraic software Maple, we solve the equation (3.5) to equation (3.9)

$$K = \frac{A_0^2\alpha^2 + 12\beta^2\mu^2 + 6\beta^2\lambda^2\mu + 8A_0\alpha\mu\beta + A_0\alpha\lambda^2\beta}{\alpha},$$
$$V = \sqrt{(}-(2\alpha A_0 + 8\beta\mu + \beta\lambda^2)),$$
$$A_0 = A_0, A_1 = -\frac{6\beta\lambda}{\alpha}, A_2 = -\frac{6\beta}{\alpha}.$$

where λ and μ are constants. Now setting the values of V, A_0, A_1, A_2 into equation (3.4), we have

$$\mathrm{v}(\xi) = A_0 - \frac{6\beta\lambda}{\alpha}(exp(-\phi(\xi))) - \frac{6\beta}{\alpha}(exp(-\phi(\xi)))^2, \tag{3.10}$$

where $\xi = x - (\sqrt{(}-(2\alpha A_0 + 8\beta\mu + \beta\lambda^2)))t$. A substitution of the equation (2.6) to equation (2.10) into equation (3.10), leads to the following five traveling wave solutions of the fourth order Benjamin-Ono equation.

When $\mu \neq 0$, $\lambda^2 - 4\mu > 0$, the solution of equation (3.1) is

$$v_1(\xi) = A_0 + \frac{6\beta\lambda}{\alpha}\Big(\frac{2\mu}{\sqrt{\lambda^2-4\mu}tanh(\frac{\sqrt{\lambda^2-4\mu}}{2})(\xi+E)+\lambda}\Big) - \frac{6\beta}{\alpha}\Big(\frac{2\mu}{\sqrt{\lambda^2-4\mu}tanh(\frac{\sqrt{\lambda^2-4\mu}}{2})(\xi+E)+\lambda}\Big)^2. \tag{3.11}$$

When $\mu \neq 0$, $\lambda^2 - 4\mu < 0$, the solution of equation (3.1) is

$$v_2(\xi) = A_0 - \frac{6\beta\lambda}{\alpha}\Big(\frac{2\mu}{\sqrt{4\mu-\lambda^2}tan(\frac{\sqrt{4\mu-\lambda^2}}{2})(\xi+E)-\lambda}\Big) - \frac{6\beta}{\alpha}\Big(\frac{2\mu}{\sqrt{4\mu-\lambda^2}tanh(\frac{\sqrt{4\mu-\lambda^2}}{2})(\xi+E)-\lambda}\Big)^2. \tag{3.12}$$

When $\mu = 0$, $\lambda \neq 0$, $\lambda^2 - 4\mu > 0$, the solution of equation (3.1) is

$$\mathrm{v}_3(\xi) = A_0 - \frac{6\beta\lambda}{\alpha}\Big(\frac{\lambda}{exp(\lambda(\xi+E))-1}\Big) - \frac{6\beta}{\alpha}\Big(\frac{\lambda}{exp(\lambda(\xi+E))-1}\Big)^2. \tag{3.13}$$

When $\mu \neq 0$, $\lambda \neq 0$, $\lambda^2 - 4\mu = 0$, the solution of equation (3.1) is

$$\mathrm{v}_4(\xi) = A_0 + \frac{6\beta\lambda}{\alpha}\Big(\frac{\lambda^2(\xi+E)}{2(\lambda(\xi+E))+2)}\Big) - \frac{6\beta}{\alpha}\Big(\frac{\lambda^2(\xi+E)}{2(\lambda(\xi+E))+2)}\Big)^2 \tag{3.14}$$

when $\mu = 0$, $\lambda = 0$, $\lambda^2 - 4\mu = 0$, the solution of equation (3.1) is

$$\mathrm{v}_5(\xi) = A_0 - \frac{6\beta\lambda}{\alpha}\Big(\frac{1}{(\xi+E)} - \frac{6\beta}{\alpha}\frac{1}{(\xi+E)^2} \tag{3.15}$$

3.2. **The BBM equation.** In this subsection, we will present the $exp(-\phi(\xi))$ -expansion method to construct a cusp, bright breathers, dark breathers, kink and multiple soliton waves solutions of the Benjamin-Bona-Mahony equation which are shown in Figure 6 to Figure 10. The Benjamin-Bona-Mahony equation is of the form [12, 24, 31]

$$\mathrm{v}_t + v_x + avv_x - bv_{xxt} = 0. \tag{3.16}$$

where a and b are constants and $v(x,t)$ is the function of the space variables x for the displacement of the water surface at location in the context of shallow water waves and time variable t. It was establised as a model for the unidirectional propagation of small amplitude long waves on the surface of water in a channel in nonlinear dispersive media [11]. According to the shallow water waves, it is applicable to the study of drift waves in the Rossby waves or plasma waves in rotating fluids and also the acoustic-gravity waves in compressible fluids, the acoustic waves in an harmonic crystals and the hydromagnetic waves in cold plasma [11].

By using traveling wave variable $\xi = x - \beta t$ carries equation (3.16) into a nonlinear ordinary differential equation

$$(1-\beta)v + \frac{a}{2}v^2 - b\beta v'' + K = 0. \tag{3.17}$$

Performing the first and second integrations of equation (3.17) and making the homogeneous balance between v^2 and $v\prime\prime$, we get $N = 2$. Putting the value of N in equation (3.17). Our solution is of the form

$$\text{v}(\xi) = A_0 + A_1(exp(-\phi(\xi)) + A_2(exp(-\phi(\xi))) \tag{3.18}$$

where the coefficients A_0, A_1 and A_2 are constants to be evaluated.

Putting the values of equation (3.18)into equation (3.17) and then equating each coefficients of $exp(-\phi(\xi)$ to zero, we get

$$\text{-}6\beta bA_2 + \frac{1}{2}aA_2^2 = 0, \tag{3.19}$$

$$\text{-}10\beta bA_2\lambda + aA_1A_2 - 2\beta bA_1 = 0, \tag{3.20}$$

$$\text{-}\beta A_2 + A_2 + \frac{1}{2}aA_1^2 - 3\beta bA_1\lambda + aA_0A_2 - 8\beta bA_2\mu - 4\beta bA_2\lambda^2 = 0, \tag{3.21}$$

$$\text{A}_1 - \beta A_1 - 2\beta bA_1\mu - 6\beta bA_2\mu\lambda - \beta bA_1\lambda^2 + aA_0A_1 = 0, \tag{3.22}$$

$$\text{A}_0 - \beta bA_1\lambda\mu - \beta A_0 + K + \frac{1}{2}aA_0^2 - 2\beta bA_2\mu^2 = 0, \tag{3.23}$$

Using algebraic software Maple, we solve the equation (3.19) to equation (3.23)

$$K = -\frac{2\beta - 1 - 8\beta^2b^2\lambda^2\mu - \beta^2 + \lambda^4b^2\beta^2 + 16\mu^2b^2\beta^2}{2a},$$
$$\beta = \beta, A_0 = \frac{\lambda^2b\beta + 8\mu b\beta + \beta - 1}{a},$$
$$A_1 = \frac{12b\beta\lambda}{a}, A_2 = \frac{12b\beta}{a},$$

where λ and μ are constants.

Now setting the values of V, A_0, A_1, A_2 into equation (3.18), we have

$$\begin{aligned}\text{v}(\xi) = {} & \frac{\lambda^2b\beta + 8\mu b\beta + \beta - 1}{a}\\ & + \frac{12b\beta\lambda}{a}(exp(-\phi(\xi))) + \frac{12b\beta}{a}(exp(-\phi(\xi)))^2,\end{aligned} \tag{3.24}$$

where $\xi = x - \beta t$.

A substitution of the equation (2.6) to equation (2.10) into equation (3.24), leads to the following five traveling wave solutions of the Benjamin-Bona-Mahony equation.

When $\mu \neq 0$, $\lambda^2 - 4\mu > 0$, the solution of equation (3.16) is

$$v_6(\xi) = \frac{\lambda^2 b\beta + 8\mu b\beta + \beta - 1}{a} - \frac{12b\beta\lambda}{a}\left(\frac{2\mu}{\sqrt{\lambda^2-4\mu}tanh(\frac{\sqrt{\lambda^2-4\mu}}{2})(\xi+E)+\lambda}\right) + \frac{12b\beta}{a}\left(\frac{2\mu}{\sqrt{\lambda^2-4\mu}tanh(\frac{\sqrt{\lambda^2-4\mu}}{2})(\xi+E)+\lambda}\right)^2. \tag{3.25}$$

When $\mu \neq 0$, $\lambda^2 - 4\mu < 0$, the solution of equation (3.16) is

$$v_7(\xi) = \frac{\lambda^2 b\beta + 8\mu b\beta + \beta - 1}{a} + \frac{12b\beta\lambda}{a}\left(\frac{2\mu}{\sqrt{4\mu-\lambda^2}tan(\frac{\sqrt{4\mu-\lambda^2}}{2})(\xi+E)-\lambda}\right) + \frac{12b\beta}{a}\left(\frac{2\mu}{\sqrt{4\mu-\lambda^2}tanh(\frac{\sqrt{4\mu-\lambda^2}}{2})(\xi+E)-\lambda}\right)^2. \tag{3.26}$$

When $\mu = 0$, $\lambda \neq 0$, $\lambda^2 - 4\mu > 0$, the solution of equation (3.16) is

$$\mathrm{v}_8(\xi) = \frac{\lambda^2 b\beta + 8\mu b\beta + \beta - 1}{a} + \frac{12b\beta\lambda}{a}\left(\frac{\lambda}{exp(\lambda(\xi+E))-1}\right) + \frac{12b\beta}{a}\left(\frac{\lambda}{exp(\lambda(\xi+E))-1}\right)^2. \tag{3.27}$$

When $\mu \neq 0$, $\lambda \neq 0$, $\lambda^2 - 4\mu = 0$, the solution of equation (3.16) is

$$\mathrm{v}_9(\xi) = \frac{\lambda^2 b\beta + 8\mu b\beta + \beta - 1}{a} - \frac{12b\beta\lambda}{a}\left(\frac{\lambda^2(\xi+E)}{2(\lambda(\xi+E))+2)}\right) + \frac{12b\beta}{a}\left(\frac{\lambda^2(\xi+E)}{2(\lambda(\xi+E))+2)}\right)^2. \tag{3.28}$$

When $\mu = 0$, $\lambda = 0$, $\lambda^2 - 4\mu = 0$, the solution of equation (3.16) is

$$\mathrm{v}_{10}(\xi) = \frac{\lambda^2 b\beta + 8\mu b\beta + \beta - 1}{a} + \frac{12b\beta\lambda}{a}\left(\frac{1}{(\xi+E)} + \frac{12b\beta}{a}\frac{1}{(\xi+E)^2}\right). \tag{3.29}$$

4. Graphical Representations and Numerical Experiments

In this section, we will provide the physical meaning of the obtained solutions of the equation (3.1) and equation (3.16) by using the mathematical software Maple, which are represented in Figure 1 to Figure 10. For further understanding the physical meaning of the solutions, we depicted 3D plot of visualize phenomena. We observe that in the numerical experimentations of the obtained solutions, Figure 1 is bright breather solutions, Figure 2 is the bark breather solutions, Figure 3 is bright rogue waves solutions, Figure 4 is also bright breather solutions, Figure 5 is multiple soliton solutions, Figure 6 is also bright breather solutions, Figure 7 is also multiple

FIGURE 1. Bright breathers solution $v_1(\xi)$ of equation (3.1) with $\alpha = -1$, $\beta = -1$, $\mu = 1$, $\lambda = 3$, $E = 1$ and $A_0 = 1$.

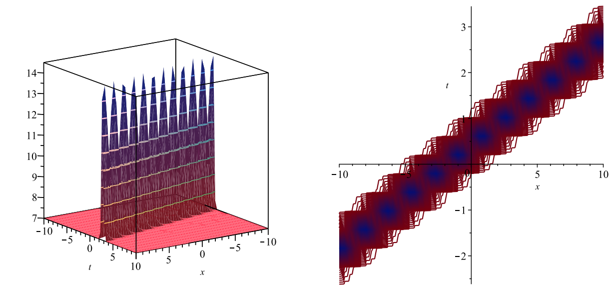


FIGURE 2. Dark breathers solution $v_2(\xi)$ of equation (3.1) with $\alpha = -5$, $\beta = -2$, $\mu = 3$, $\lambda = 1$, $E = 1$ and $A_0 = 5$.

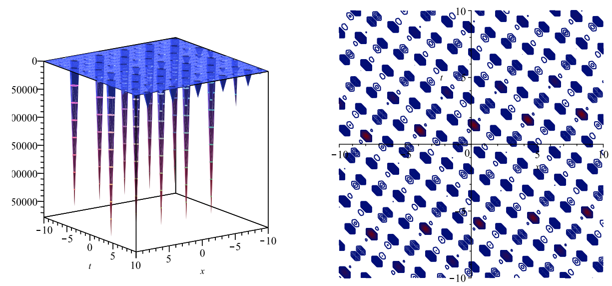
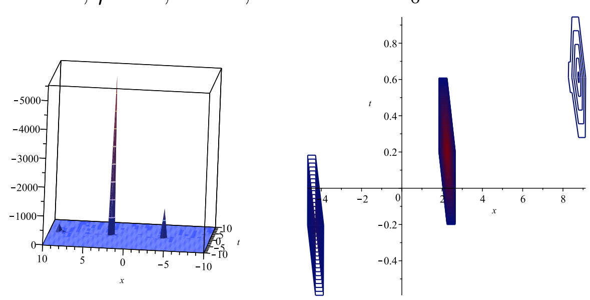


FIGURE 3. Bright rogue waves solution $v_3(\xi)$ of equation (3.1) with $\alpha = -9$, $\beta = -8$, $\mu = 0$, $\lambda = 2$, $E = 1$ and $A_0 = 12$.



soliton solutions, Figure 8 is cuspon waves solitons, Figure 9 is Singular kink waves solutions,and Figure 10 is soliton solutions.

FIGURE 4. Bright breathers solution $v_4(\xi)$ of equation (3.1) with $\alpha = -3$, $\beta = -3$, $\mu = 1$, $\lambda = 2$, $E = 5$ and $A_0 = 4$.

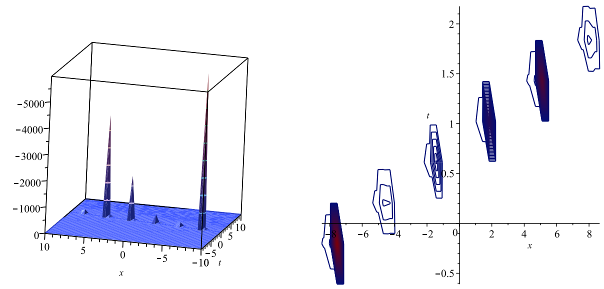


FIGURE 5. Multiple Soliton solution $v_5(\xi)$ of equation (3.1) with $\alpha = -8$, $\beta = -9$, $\mu = 0$, $\lambda = 0$, $E = 1$ and $A_0 = 4$.

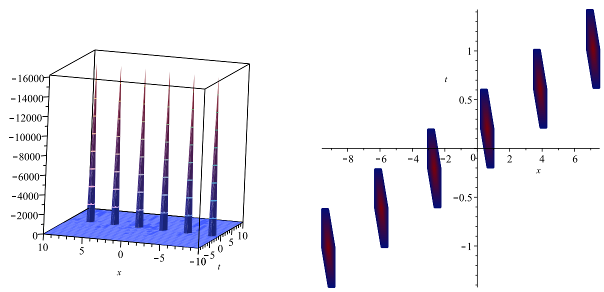
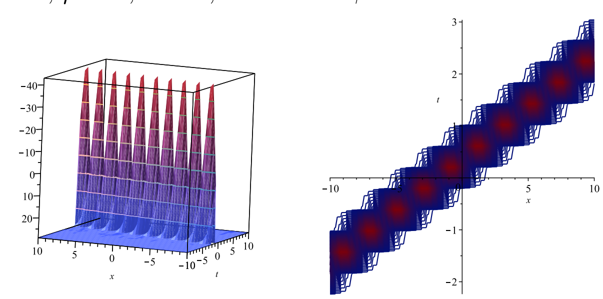


FIGURE 6. Bright breathers solution $v_6(\xi)$ of equation (3.16) with $a = 1$, $b = 1$, $\mu = 1$, $\lambda = 3$, $E = 1$ and $\beta = 5$.



5. Conclusion

In summary, applying the $exp(-\phi(\xi))$-expansion method to the fourth order Benjamin-Ono equation and BBM equation, we have successfully obtained a special kinds of

FIGURE 7. Multiple bright Soliton solition $v_7(\xi)$ of equation (3.16) with $a = 1$, $b = 1$, $\mu = 3$, $\lambda = 1$, $E = 1$ and $\beta = 1$.

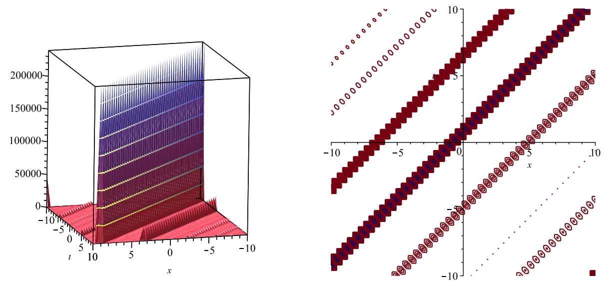


FIGURE 8. Cusp waves solution $v_8(\xi)$ of equation (3.16) with $a = 2$, $b = 2$, $\mu = 0$, $\lambda = 2$, $E = 1$ and $\beta = 1$.

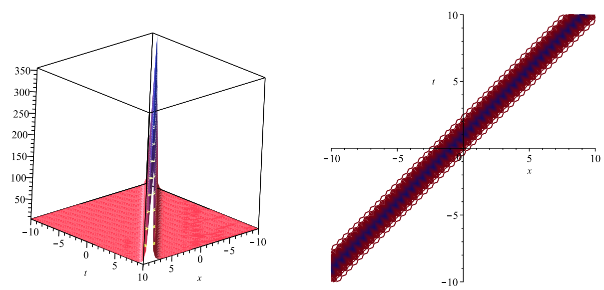
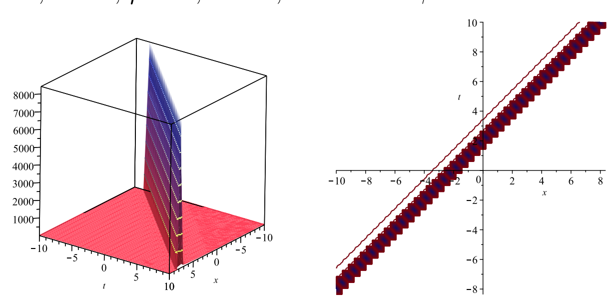
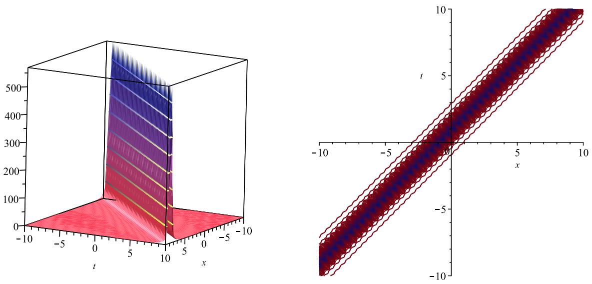


FIGURE 9. Singular kink waves solution $v_9(\xi)$ of equation (3.16) with $a = 1$, $b = 1$, $\mu = 1$, $\lambda = 2$, $E = 1$ and $\beta = 1$.



waves solutions such as bright bright breathers waves, dark breathers waves, a cusp wave, kink waves, bright rogue waves and some soliton solutions.

FIGURE 10. Soliton solution $v_{10}(\xi)$ of equation (3.16) with $a = 5$, $b = 8$, $\mu = 0$, $\lambda = 0$, $E = 1$ and $\beta = 1$.



6. ACKNOWLEDGMENT

The authors would like to acknowledge CAS-TWAS president¡¯s fellowship program.

REFERENCES

[1] M. J. Ablowitz and P. A. Clarkson, *Nonlinear evolution equations and inverse scattering*, Cambridge University Press, New York ,1991.

[2] M. N. Alam, M. A. Akbar, and S. T. Mohyud-Din, *A novel (G'/G)-expansion method and its application to the Boussinesq equation*, Chin. Phys. B, *23*(2) (2014), 020203.

[3] M. N. Alam and F. B. M. Belgacem, *New generalized (G/G)-expansion method applications to coupled Konno-Oono equation*, Advances in pure mathematics, *6*(3) (2016), 168-179.

[4] M. N. Alam and F. B. M. Belgacem, *Exact traveling wave solutions for the (1+1)-dimensional compound KdVB equation via novel (G'/G) -expansion method*, International Journal of Modern Nonlinear Theory and Application, *5*(1) (2016), 28-39.

[5] M. N. Alam and F. B. M. Belgacem, *Microtubules nonlinear models dynamics investigations through the $exp(-\phi(\xi))$-expansion method implementation*, Mathematics, *4* (2016), 6.

[6] M. N. Alam and F. B. M. Belgacem, *Application of the novel (G/G)-Expansion Method to the regularized long wave equation*, Waves Wavelets Fractals Adv. Anal., *1* (2015), 51-67.

[7] M. N. Alam, M. G. Hafez, M. A. Akbar, and H. O. Roshid, *Exact traveling wave solutions to the (3+1)-dimensional mKdV-ZK and the (2+1)-dimensional Burgers equations via $exp(-\phi(\xi))$-expansion method*, Alexandria Engineering Journal, *54* (2015), 635-644.

[8] M. N. Alam, M. G. Hafez, F. B. M. Belgacem, and M. A. Akbar, *Applications of the novel (G'/G)-expansion method to find new exact traveling wave solutions of the nonlinear coupled Higgs field equation*, Nonlinear Studies, *22*(4) (2015), 613-633.

[9] M. N. Alam, F. B. M. Belgacem, and M. A. Akbar, *Analytical treatment of the evolutionary (1+1)-dimensional combined KdV-mKdV equation via the novel (G'/G)-expansion method*, Journal of Applied Mathematics and Physics, *3*(12) (2015), 1571-1579.

[10] M. N. Alam, M. G. Hafez, M. A. Akbar, and F. B. M. Belgacem, *Application of new generalized (G'/G)-expansion method to the (3+1)-dimensional Kadomtsev-Petviashvili equation*, Italian Journal of Pure and Applied Mathematics, *36* (2016), 415-428.

[11] T. B. Benjamin, J. L. Bona and, J. J. Mahony, *Model equations for long waves in nonlinear dispersive systems*, Philos. Trans. Roy. Soc. London Ser. A, *272* (1972), 47-48.

[12] A. Biswas, S. Konar, *Soliton perturbation theory for the generalized Benjamin-Bona-Mahoney equation*, Commun. Nonlinear Sci. Numer. Simul, *13* (2008), 703-706.

[13] H. Bulut, T. A. Sulaiman, and H. M. Baskonus, *On the new soliton and optical wave structures to some nonlinear evolution equations*, Eur. Phys. J. Plus, *132*(11) (2017), 459.

[14] H. Bulut, T. A. Sulaiman, H. M. Baskonus, and T. Akturk, *Complex acoustic gravity wave behaviors to some mathematical models arising in fluid dynamics and nonlinear dispersive media*, Opt. Quant. Electron., *50*(1) (2018), 19.

[15] Z. Dai, J. Liu, and D. Li, *Applications of HTA and EHTA to YTSF equation*, Appl. Math. Comput. *207* (2009), 360-364.

[16] E. G. Fan, *Two new applications of the homogeneous balance method*, Phys. Lett. A., *265* (2000), 353-357.

[17] E. G. Fan, *Extended tanh-function method and its applications to nonlinear equations*, Phys. Lett. A, *277* (2000), 212-218.

[18] E. G. Fan, *Computer Algebra and Integrable System (Science Press, Beijing)*, 2004.

[19] Z. Fu, S. Liu, S. Liu, and Q. Zhao, *The JEFE method and periodic solutions of two kinds of nonlinear wave equations*, Commun. Nonlinear Sci. Numer. Simul. *8* (2003), 67-75.

[20] J. H. He and X. H. Wu, *Exp-function method for nonlinear wave equations*, Chaos, Solitons Fract., *30* (2006), 700-708.

[21] D. Liu, *Jacobi elliptic function solutions for two variant Boussinesq equations*, Chaos solitons Fract. *24* (2005), 1373-85.

[22] M. R. Miura, *Backlund transformation*, Springer, Berlin, 1978.

[23] S. T. Mohyud-Din, A. Yildirim, and S. Sariaydin, *Numerical soliton solution of the Kaup-Kupershmidt equation*, Int. J. Numer. Methods Heat Fluid Flow, *21*(3) (2011), 272-281.

[24] J. Nickel, *Elliptic solutions to a generalized BBM equation*, Phys. Lett. A, *364* (2007), 221-226.

[25] A. S. A. Rady, E. S. Osman, and M. Khalfallah, *The homogeneous balance method and its application to the Benjamin-Bona-Mahoney (BBM) equation*, Appl. Math. Comput., *217* (2010), 1385-1390.

[26] M. Shakeel, M. A. Iqbal, Q. Din, Q. M. Hassan, and K. Ayub, *New exact solutions for coupled nonlinear system of ion sound and Langmuir waves*, Indian Journal of Physics, *94* (2020), 885 894.

[27] M. Shakeel, M. A. Iqbal, and S. T. Mohyud-Din, *Closed Form Solutions for Nonlinear Biological Population Model*, Journal of Biological Systems, vol. *26*(1) (2018), 207-223.

[28] M. Shakeel, S. T. Mohyud-Din, and M. A. Iqbal, *Closed Form Solutions for Coupled Nonlinear Maccari System*, Computers and Mathematics with Applications, *76*(4) (2018), 799-809.

[29] M. Shakeel, S. T. Mohyud-Din, and M. A. Iqbal, *Complex Closed Form Solutions for Coupled Higgs System via Extended Exp-Function Method*, Nonlinear Science Letters A, *9*(3) (2018), 196-208.

[30] M. Shakeel, S. T. Mohyud-Din, and M. A. Iqbal, *Modified extended exp-function method for system of nonlinear partial differential equations defined by seismic sea waves*, Pramana-Journal of Physics, *91* (2018),28.

[31] H. Triki and M. S. Ismail, *Soliton solutions of a BBM(m,n) equation with generalized evolution*, Appl. Math. Comput., *217* (2010), 48-54.

[32] M. Usman, A. Nazir, T. Zubair, I. Rashid, Z. Naheed, and S. T. Mohyud-Din, *Solitary wave solutions of (3+1)-dimensional Jimbo Miwa and Pochhammer-Chree equations by modified Exp-function method*, Int. J. Modern Math. Sci., *5* (2013), 27-36.

[33] V. Vakhnenko, E. J. Parkes, and A.J. Morrison, *A Backlund transformation and the inverse scattering transform method for the generalised Vakhnenko equation*, Chaos, Solitons and Fract., *17*(4) (2003), 683-692.

[34] S. A. E. Wakil and M. A. Abdou, *New exact travelling wave solutions using modified extended tanh-function method*, Chaos Solitons Fract., *31*(4) (2007), 840-852.

[35] Z. Wang and D. Li, *An improved computing method for analysing the spatial resolved reflectance from biological tissues*, Chin. Phys. Lett. *14* (2005), 2158.

[36] A. M. Wazwaz, *A sine-cosine method for handle nonlinear wave equations*, Appl. Math. Compu. Model., *40* (2004), 499-508.

[37] A. M. Wazwaz, *The tanh method for generalized forms of nonlinear heat conduction and Burgers-Fisher equations*, Appl. Math.Comput., *169* (2005), 321-338.

[38] A. M. Wazwaz, *Multiple-soliton solutions for a (3+1)-dimensional generalized KP equation*, Commun. Nonlinear Sci. Numer. Simulat, *17* (2012), 491-495.
[39] Z. H. Xu, D. Q. Xian, and H. L. Chen, *New periodic solitary wave solutions for the Benjiamin Ono equation*, Appl. Math. Comput. *215* (2010), 4439-4442.
[40] E. Yusufoglu and A. Bekir, *On the extended tanh method applications of nonlinear equations*, Inter. J. of Non. Sci, *4*(1) (2007), 10-16.
[41] Z. Zhao, Y. Zhang, and W. Rui, *Backlund transformation and solutions of a (3 + 1)-dimensional nonlinear evolution equation*, Appl. Math. Comput, *248* (2014), 456-462.