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On the numerical approximation of Volterra integro-differential equation using Laplace transform

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Abstract In this work a numerical scheme is constructed to approximate the Volterra integro-differential equations of convolution type. The proposed numerical scheme is based on Laplace transform, inverse Laplace transform and integration. The solution of the problem is represented in the form of contour integral in the complex plane. This integral is approximated along optimal contour using trapezoidal rule with equal step size. The solution accuracy depends on optimal contour which is needed for accurate approximation of the inverse Laplace transform. For better accuracy two types of contours, parabolic and hyperbolic, are used which are available in the literature. The performance of the numerical scheme is tested for different examples. The actual error well agree with the corresponding error estimates of the proposed numerical scheme for both parabolic as well as hyperbolic contours.

Keywords. Volterra Integro-differential equation, Hyperbolic and parabolic contours, Laplace Transforms, Trapezoidal rule.

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1. Introduction

The Volterra integro-differential equation was developed by Volterra while studying the population growth model [25]. The Volterra integro-differential equations have applications in fluid dynamics, electrostatics [6], diffusion problems [2], and some special problem of mechanics etc. As the analytical solutions of VIDEs are difficult in many cases, so numerical methods are used to find the solution of such types of VIDEs. Some of the effective numerical methods which are available in the literature for the solution of VIDEs include operational matrix approach for integration with block pulse functions [3, 4], Lagrange interpolation for integrals and integro-differential equation [18], the homotopy perturbation and finite difference methods [8, 10], the Elzaki transform method [7], and the power series method [20] etc. A

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stability and convergence analysis for solving VIDEs are discussed for some numerical methods in [5, 19]. Some real life problems using VIDEs are solved using various analytical and numerical solutions [1, 9, 12]. In the present work we used Laplace transform to approximate Volterra integro-differential equations and construct a numerical algorithm for the solution of inverse Laplace transform which is the extension of some earlier work [22, 23, 24] for VIDEs.

2. Description of the method

The Laplace transform of an arbitrary function $f(t)$ of a real variable $t > 0$ [15, p.449] is defined by

$$f(z) = \mathcal{L}[f(t), z] = \int_0^{\infty} e^{-zt} f(t) dt, \tag{2.1}$$

where $z = s + ia$ is a complex variable. The Laplace transform is valid for any continuous or piecewise-continuous functions satisfying the condition $|f(t)| < Me^{a_0 t}$ with some $M > 0$ and $a_0 \geq 0$. The inverse Laplace transform is given by the following formula [15, p.450]

$$f(t) = \frac{1}{2\pi\ i} \int_{C-i\infty}^{C+i\infty} f(z) e^{zt} dz, \tag{2.2}$$

here the integration path is parallel to the imaginary axis and lies to the right of all singularities of $f(z)$, which corresponds to $C > a_0$. The general form of Volterra integro-differential equation is given by[25, p.162]

$$u^{(n)}(t) = f(t) + \int_0^t k(t, s) u(s) ds, \tag{2.3}$$

where $u^{(m)}(0) = b_m$, $0 \leq m \leq n-1$, and $u^{(n)}(t)$ indicates the nth derivative of $u(t)$, and b_m are constant which denote the initial conditions. The Volterra integro-differential equation of first kind is given by [25, p.181]

$$\int_0^t k_1(t, s) u(s) ds + \int_0^t k(t, s) u^{(n)}(s) ds = f(t). \tag{2.4}$$

In the present work, we consider the case in which $k_1(t, s) = 0$, the above equation becomes

$$\int_0^t k(t, s) u^{(n)}(s) ds = f(t). \tag{2.5}$$

Since we are concerned only with convolution type of equations, so the kernel will be of the form $k(t, s) = k(t - s)$, and the equations (2.3)-(2.5) become

$$u^{(n)}(t) = f(t) + \int_0^t k(t - s) u(s) ds, \tag{2.6}$$

and

$$\int_0^t k(t-s)u^{(n)}(s)ds = f(t). \tag{2.7}$$

Applying the Laplace transform to equation (2.7) we have

$$\mathcal{L}\left[\int_0^t k(t-s)u^{(n)}(s)ds\right] = \mathcal{L}\{f(t)\}, \tag{2.8}$$

and using the convolution theorem we have

$$\mathcal{L}\{k(t)\}\mathcal{L}\{u^{(n)}(t)\} = \mathcal{L}\{f(t)\}. \tag{2.9}$$

If we denote

$$\mathcal{L}\{k(t)\} = K(z), \mathcal{L}\{u(t)\} = U(z), \text{ and } \mathcal{L}\{f(t)\} = F(z),$$

and

$$\mathcal{L}\{u^{(n)}(t)\} = z^n U(z) - z^{n-1}u(0) - z^{n-2}u'(0) - ... - u^{(n-1)}(0), \tag{2.10}$$

using these values in equation (2.9), we get

$$U(z) = \frac{F(z)}{z^n K(z)} + \frac{z^{n-1}u(0) + z^{n-2}u'(0) + ... + u^{(n-1)}(0)}{z^n}. \tag{2.11}$$

Apply Laplace transform to (2.6) we have

$$U(z) = \frac{F(z)}{z^n - K(z)} + \frac{z^{n-1}u(0) + z^{n-2}u'(0) + ... + u^{(n-1)}(0)}{z^n - K(z)}. \tag{2.12}$$

Hence the solution of the problem is represented as an integral in the complex plane

$$u(t) = \frac{1}{2\pi\ i}\int_{C-i\infty}^{C+i\infty} U(z)e^{zt}dz. \tag{2.13}$$

Now we need to select the contour of integration to approximate the path from $C-i\infty$ to $C+i\infty$, here we consider two such contours namely the parabolic contour [22] and the hyperbolic contour [14] respectively.

The parametric equation of parabolic contour is given by

$$z = \mu((1-c)^2 - \eta^2) + 2i\mu\eta(1-c), -\infty < \eta < \infty,\ (C_1) \tag{2.14}$$

while that of hyperbolic contour is given by

$$z = \mu(1-\sin(\alpha+c)\cosh(\eta)) + i\mu\cos(\alpha+c)\sinh(\eta),\ -\infty < \eta < \infty,\ (C_2) \tag{2.15}$$

where μ, c, and α are parameters and need to be optimized for better accuracy.

$$u(t) = \frac{1}{2\pi i}\int_\Gamma U(z)e^{zt}dz. \tag{2.16}$$

The numerical solution can be represented in the following form incorporating either of the two contours C_1 or C_2

$$u(t) = \frac{1}{2\pi\ i} \int_{\Gamma} U(z(\eta)) e^{z(\eta)t} z'(\eta) d\eta. \tag{2.17}$$

If we use equal weight quadrature rule, i.e trapezoidal rule with step size h, then $z_j = z(\eta_j), z'_j = z'(\eta_j)$ the equation (2.17) can be approximated as

$$u_N(t) = \frac{h}{2\pi\ i} \sum_{j=-N}^{N} U(z_j) e^{z_j t} z'_j. \tag{2.18}$$

3. Application of the method to numerical experiments

Here we apply the present numerical scheme for solving some problems, where the following optimal parameters for better accuracy are used (see for example [14, 22]). We used the following parameters for all our numerical experiments $t = 0.1$, $T = 1$, $t_0 = 0.01$, for the path C_1 we used $c = 0.3$, $h = 3/N$, $\mu = \pi N/(12t)$, while for the path C_2, the parameters $c = 0.3$, $\alpha = 1.1721$, $A_\alpha = \cosh^{-1}(\frac{2\alpha}{(4\alpha-\pi)\sin(\alpha))}$, $h = A_\alpha/N$, $\mu = ((4\pi\alpha - \pi^2)N)/(A_\alpha t)$ are used. Here the error estimate of the method is $l(\rho N)e^{-\nu N}$.

3.1. **Problem 1.** We apply the present numerical scheme to solve the following problem [18]

$$\int_0^t \cos(t-s) u''(s) ds = 2\sin(t), \tag{3.1}$$

with $u(0) = 0$, and $u'(0) = 0$, application of the Laplace transform to equation (3.1) and using convolution theorem we have

$$\mathcal{L}\{\cos(t) * u''(t)\} = \mathcal{L}\{2\sin(t)\}, \tag{3.2}$$

and hence we have

$$\mathcal{L}\{\cos(t)\} = \frac{z}{z^2+1}, \mathcal{L}\{\sin(t)\} = \frac{1}{z^2+1}, \mathcal{L}\{u''(t)\} = z^2 U(z) - zu(0) - u'(0). \tag{3.3}$$

Using the above values we have

$$U(z) = \frac{2}{z^3}. \tag{3.4}$$

The approximate solution can be obtained from (2.18) using (3.4), where the exact solution of the above problem is $u(t) = t^2$. We obtained the results of the above problem using both contours C_1 and C_2 in form of absolute error, and compared with the error estimate of the numerical method. The results are compared with other methods and are shown in the Table 1.

TABLE 1. Problem 1: Solution of Volterra integro-differential using our numerical scheme and both contours C_1 and C_2.

N	Absolute error (C_1)	Absolute error (C_2)	$l(\rho N)e^{-\nu N}$
32	$5.2493e-018$	$6.4591e-008$	$2.8175e-004$
48	$1.0500e-017$	$1.4780e-012$	$1.8711e-006$
64	$3.4993e-017$	$2.1086e-016$	$1.2658e-08$
66	$6.6180e-015$	$1.0432e-016$	$6.7847e-009$
67	$4.7624e-015$	$6.8084e-017$	$4.9674e-009$
80	$1.2457e-014$	$1.1312e-015$	$8.6551e-011$
88	$6.8617e-014$	$5.7250e-017$	$7.1766e-012$
89	$9.0819e-014$	$1.4521e-015$	$5.2577e-012$
96	$3.4870e-016$	$1.8844e-016$	$5.9593e-013$
109	$8.5166e-013$	$4.7709e-017$	$1.0474e-014$
112	$1.4810e-012$	$2.0766e-016$	$4.1235e-015$
122	$1.7977e-012$	$1.0531e-015$	$1.8457e-016$
128	$3.1323e-014$	$1.4592e-015$	$2.8639e-017$
158	$2.7697e-010$	$3.1509e-015$	$2.5902e-021$
179	$4.2485e-009$	$4.4829e-017$	$3.8429e-024$
Method 1 [18]	$9.9000e-016$		
Method 2 [18]	$5.3000e-016$		
[4]	$1.7000e-014$		

3.2. **Problem 2.** Here we used the present numerical scheme to solve the following problem (see [3])

$$u'(t) + u(t) = \int_0^t e^{(s-t)}u(s)ds. \tag{3.5}$$

Applying the Laplace transform to equation (3.5) and using the convolution theorem we get

$$\mathcal{L}\{u'(t)\} + \mathcal{L}\{u(t)\} = \mathcal{L}\{e^{-t} * u(t)\}. \tag{3.6}$$

Simplify we get

$$\mathcal{L}\{u'(t)\} + \mathcal{L}\{u(t)\} = \mathcal{L}\{e^{-t}\}\mathcal{L}\{u(t)\}. \tag{3.7}$$

These values may be calculated as

$$\mathcal{L}\{e^{-t}\} = \frac{1}{z+1}, \mathcal{L}\{u'(t)\} = zU(z) - u(0),\ \mathcal{L}\{u(t)\} = U(z). \tag{3.8}$$

Using the above result we have

$$U(z) = \frac{z+1}{z^2+2z}. \tag{3.9}$$

The approximate solution can be obtained from (2.18) using (3.9), where the exact solution is $u(t) = e^{-t}\cos ht$. The results of the present method in term of absolute error and error estimate and comparison with an other method for the same model are shown in Table 2.

TABLE 2. Problem 2: Solution of Volterra integro-differential by the present method with two contours C_1 and C_2.

N	Absolute error (C_1)	Absolute error (C_2)	$l(\rho N)e^{-\nu N}$
12	$3.5213e-008$	$2.2844E-004$	$1.5920e-001$
16	$4.8442e-011$	$4.4124e-006$	$4.4300e-002$
32	$1.5462e-015$	$1.8282e-010$	$2.8175e-004$
48	$6.6553e-015$	$2.0334e-014$	$1.8711e-006$
64	$2.2519e-014$	$2.3120e-014$	$1.2658e-08$
80	$1.7172e-010$	$2.6291e-013$	$8.6551e-011$
96	$5.6308e-012$	$5.8978e-014$	$5.9593e-013$
112	$1.0731e-008$	$1.5174e-014$	$4.123e-015$
128	$3.0271e-010$	$8.0386e-013$	$2.8639e-017$
144	$1.2774e-006$	$3.0109e-012$	$1.9948e-019$
160	$2.1435e-005$	$6.8549e-012$	$1.3928e-021$
176	$8.1798e-005$	$1.0315e-011$	$9.7926e-024$
192	$2.9887e-006$	$8.4420e-012$	$6.8262e-026$
[16]	$3.2717e-003$		

3.3. **Problem 3.** We solve the following problem by the present method

$$u''(t) = t + \int_0^t (t-s)u(s)ds, \tag{3.10}$$

as discussed in [3], where $u(0) = 0$ and $u'(0) = 1$, apply the Laplace transform to equation (3.10), and using convolution theorem we have

$$\mathcal{L}\{u''(t)\} = \mathcal{L}\{t\} + \mathcal{L}\{t * u(t)\}, \tag{3.11}$$

or

$$\mathcal{L}\{u''(t)\} = \mathcal{L}\{t\} + \mathcal{L}\{t\}.\mathcal{L}\{u(t)\}, \tag{3.12}$$

where

$$\mathcal{L}\{t\} = \frac{1}{z^2},\ \mathcal{L}\{u(t)\} = U(z),\ \mathcal{L}\{u''(t)\} = z^2U(z) - zu(0) - u'(0). \tag{3.13}$$

Using the above result we have

$$U(z) = \frac{1}{z^2-1}. \tag{3.14}$$

The use of equation (3.14) in equation (2.18), we obtained the approximate solution, where the exact solution of the problem is given by $u(t) = \sinh t$. The results are shown in Table 3.

Table 3. Problem 3: Solution of Volterra integral-equation by the present method using C_1 and C_2, $t = \frac{1}{8}$.

N	Absolute error C_1	Absolute error C_2	$l(\rho N)e^{-\nu N}$
16	$1.1971e-008$	$2.7000e-003$	$4.43e-002$
32	$2.8905e-017$	$3.4800e-007$	$2.8175e-004$
48	$6.5608e-017$	$2.5026e-011$	$1.8711e-006$
64	$9.7942e-016$	$1.4754e-015$	$2.5757e-008$
80	$1.3746e-012$	$1.5375e-014$	$8.6551e-011$
96	$2.5609e-014$	$3.4867e-015$	$5.9593e-013$
112	$1.1470e-010$	$2.0595e-015$	$4.123e-015$
128	$2.0707e-012$	$3.0714e-014$	$2.8639e-017$
144	$8.3449e-009$	$9.7298e-014$	$1.9948e-019$
160	$1.2514e-007$	$2.0166e-013$	$9.7926e-024$
176	$3.7030e-007$	$2.3259e-013$	$9.7926e-024$
[3]	$1.6000e-004$		

3.4. **Problem 4.** Lastly we consider the following problem

$$u(t) = a\sin(t) + 2\int_0^t u'(s)\sin(t-s)ds. \tag{3.15}$$

Applying the Laplace transform to equation (3.15), and using convolution theorem we have

$$\mathcal{L}\{u(t)\} = \mathcal{L}\{a\sin(t)\} + 2\mathcal{L}\{\sin(t) * u'(t)\}, \tag{3.16}$$

Performing the following computations

$$\mathcal{L}\{\sin(t)\} = \frac{1}{z^2+1}, \mathcal{L}\{u'(t)\} = zU(z) - u(0),\ \mathcal{L}\{u(t)\} = U(z),\ u(0) = 0 \tag{3.17}$$

and using the above results we have

$$U(z) = \frac{a}{(z-1)^2}. \tag{3.18}$$

Use this value in equation (2.18) we get the numerical solution by the present method, where the exact solution of the current problem is $u(t) = ate^t$ with $a = 1$. The results are given in Table 4

4. Conclusion

In this paper we constructed a numerical scheme for approximating Volterra integro-differential equations. The proposed numerical scheme is based on Laplace transform and quadrature rule with high order accuracy. The proposed numerical scheme recovered the results with much better accuracy as compared to the available methods, e.g the Lagrange interpolation method, the integral expansion method, the operational matrix method, the optimal homotopy asymptotic method for solving the Volterra

Table 4. Problem 4: Solution of Problem 4 by the current method using C_1 and C_2 at $t = \frac{1}{8}$.

N	Absolute error C_1	Absolute error C_2	$l(\rho N)e^{-\nu N}$
15	$2.5372e-007$	$2.7600e-002$	$6.0900e-002$
25	$7.1277e-013$	$1.5338e-004$	$2.600e-003$
35	$1.0496e-014$	$5.5997e-007$	$1.0981e-004$
45	$1.2823e-014$	$1.6229e-009$	$4.7825e-006$
55	$8.7341e-016$	$4.0392e-012$	$2.0995e-007$
65	$4.5596e-013$	$1.2181e-014$	$9.2672e-009$
75	$1.2082e-012$	$1.4281e-015$	$4.1067e-010$
85	$3.0601e-012$	$4.2221e-014$	$1.8254e-011$
95	$1.1794e-011$	$1.4797e-014$	$8.1331e-013$
105	$1.2802e-011$	$3.5085e-014$	$3.6309e-014$

integro-differential equations of convolution types. The proposed method is a very well alternative for solving such types of problems in science and engineering.

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