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Solving Itô integral equations with time delay via basis functions

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Abstract In this paper, a direct method for solving Volterra-Fredholm integral equations with time delay by using orthogonal functions and their stochastic operational matrix of integration is proposed. Stochastic integral equations can be reduced to a sparse system which can be directly solved. Numerical examples show that the proposed scheme has a suitable degree of accuracy.

Keywords. Delay integral equations; Stochastic operational matrix; Stochastic Volterra-Fredholm integral equations; Itô integral.

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1. Introduction

Basis functions were used to derive solutions of dynamic systems [1, 2]. Integral and integro-differential equations of convolution type and system of the first kind integral equations that is an ill-posed problem is solved by basis functions [3, 4]. Stochastic integral equations arise in many applications such as mechanics, finance, biomathematics, engineering, etc. Dynamic systems are mostly dependent on a noise source followed by well-defined probability laws, so that modeling such problems requires the use of different stochastic differential equations [5, 6, 7, 8]. Also linear and nonlinear stochastic integral equations are proposed in recent years [9, 10]. Integral equations with time delay frequently encountered in physical and biological modeling processes. Delays occur frequently in chemical, transportation, electronic, communication, manufacturing and power systems. Delay integral equations (DIEs) and delay integro-differential equations (DIDEs) are solved by different methods [11, 12, 13]. Because in most problems the accurate solution can not be solved exactly, we constrain to obtain approximate solution by numerical schemes.
We intend the following linear stochastic Volterra-Fredholm integral equation,

$$\begin{aligned} X(t) =& f(t) + \int_a^b k_1(s,t)X(s)ds + \int_0^t k_2(s,t)X(s-\tau)ds \\ &+ \int_0^t k_3(s,t)X(s)dB(s), \qquad t\in[0,T), \tau\in(0,T) \end{aligned} \tag{1.1}$$

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where $X(t)$, $f(t)$, $k_1(s,t)$, $k_2(s,t)$ and $k_3(s,t)$, for $s,t \in [0,T)$, are the stochastic processes defined on the same probability space (Ω, F, P), and $X(t)$ is unknown. Also $B(t)$ is a Brownian motion process and $\int_0^t k_3(s,t)X(s)dB(s)$ is the Itô integral.

The present paper is organized as follows. After Introduction Section, Section 2 reviews block pulse functions and integration operational matrix and functions containing time delay $f(t-\tau)$. In Section 3, we introduce concept of the stochastic integration operational matrix. In Section 4, we solve stochastic Volterra-Fredholm integral equations by using stochastic integration operational matrix. Section 5 is allocated to error in block pulse functions approximation and in Section 6, we achieve numerical examples to show the accuracy of the method and the culmination of paper in Section 7 is the conclusion.

2. Block Pulse Functions (BPFs)

The aim of this section is to interprate notations and definition of the block pulse functions that have been expressed entirely in [2].

2.1. **Definition.** We define the m-set of BPFs as,

$$\phi_i^{(m)}(t) = \begin{cases} 1 & (i-1)h \leq t < ih, \\ 0 & otherwise, \end{cases} \tag{2.1}$$

with $t \in [0,T)$, $i = 1,2,...,m$ and $h = \frac{T}{m}$.

The primary properties of BPFs are disjointness and orthogonality that can be expressed as follows

$$\phi_i^{(m)}(t)\phi_j^{(m)}(t) = \delta_{ij}\phi_i^{(m)}(t), \tag{2.2}$$

$$\int_0^T \phi_i^{(m)}(t)\phi_j^{(m)}(t)dt = h\delta_{ij}, \tag{2.3}$$

where $i,j = 1,2,...,m$ and δ_{ij} is Kronecker delta.

Also if $m \to \infty$, then the BPFs set is complete; i.e. for every $f \in L^2([0,T))$, Parseval's identity holds,

$$\int_0^T f^2(t)dt = \sum_{i=1}^{\infty} f_i^2 \|\phi_i^{(m)}(t)\|^2, \tag{2.4}$$

where

$$f_i = \frac{1}{h}\int_0^T f(t)\phi_i^{(m)}(t)dt, \tag{2.5}$$

by considering first m terms of BPFs, we can write them brevity as m-vector form

$$\Phi(t) = \left(\phi_1(t), \phi_2(t), \ldots, \phi_m(t)\right)^T, \quad t \in [0, T).$$

2.2. **Functions approximation.** A real bounded function $f(t)$, which $f(t) \in L^2[0, T)$, can be expanded into a block pulse series as

$$f(t) \simeq \hat{f}_m(t) = \sum_{i=1}^{m} f_i \phi_i^{(m)}(t), \tag{2.6}$$

where f_i is the block pulse coefficient with respect to the ith BPF $\phi_i^{(m)}(t)$. In the vector form we have,

$$f(t) \simeq \hat{f}_m(t) = F^T \Phi(t) = \Phi^T(t) F, \tag{2.7}$$

where

$$F = \left(f_1, f_2, \ldots, f_m\right)^T.$$

Let $k(s,t) \in L^2\left([0, T_1) \times [0, T_2)\right)$. It can be expanded as

$$k(s,t) = \Psi^T(s) K \Phi(t) = \Phi^T(t) K^T \Psi(s), \tag{2.8}$$

where $\Psi(s)$ and $\Phi(t)$ are m_1 and m_2 dimensional BPFs vectors respectively, and $K = \left(k_{ij}\right), i = 1, 2, ..., m_1, j = 1, 2, ..., m_2$ is the $m_1 \times m_2$ block pulse coefficient matrix with

$$k_{ij} = \frac{1}{h_1 h_2} \int_0^{T_1} \int_0^{T_2} k(s,t) \Psi_i^{(m_1)}(s) \Phi_j^{(m_2)}(t) dt ds,$$

where $h_1 = \frac{T_1}{m_1}$, $h_2 = \frac{T_2}{m_2}$. For convenience, we put $m_1 = m_2 = m$.

2.3. **Integration operational matrix.** Computing $\int_0^t \phi_i^{(m)}(s) ds$ follows

$$\int_0^t \phi_i^{(m)}(s) ds = \begin{cases} 0 & 0 \le t < (i-1)h, \\ t - (i-1)h & (i-1)h \le t < ih, \\ h & ih \le t < T. \end{cases} \tag{2.9}$$

From [2], We will have:

$$\int_0^t \Phi(s) ds \simeq P \Phi(t), \tag{2.10}$$

where operational matrix of integration is given by

$$P = \frac{h}{2} \begin{pmatrix} 1 & 2 & 2 & \ldots & 2 \\ 0 & 1 & 2 & \ldots & 2 \\ 0 & 0 & 1 & \ldots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}_{m \times m}. \tag{2.11}$$

So, the integral of every function $f(t)$ can be approximated as follows

$$\int_0^t f(s)ds \simeq \int_0^t F^T\Phi(s)ds \simeq F^T P\Phi(t). \tag{2.12}$$

2.4. **Functions containing time delay $f(t-\tau)$.** In order to approximate a function containing time delay, we consider the whole block pulse function containing time delay $\tau = (q+\lambda)h$ with a nonnegative integer q and $0 \leq \lambda < 1$ that can be expressed into its block pulse series in a vector form:

$$\Phi(t-\tau) = \big((1-\lambda)H^q + \lambda H^{q+1}\big)\Phi(t). \tag{2.13}$$

In the above relation, the matrix $(1-\lambda)H^q + \lambda H^{q+1}$ is usually called the block pulse operational matrix for time delay, or simply the delay operational matrix. Expressing concretely, it is:

$$(1-\lambda)H^q + \lambda H^{q+1} = \begin{pmatrix} 0 & \cdots & 0 & \overset{\substack{(q+1)\text{th-column}\\ \downarrow}}{1-\lambda} & \lambda & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1-\lambda & \lambda & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \lambda \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1-\lambda \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{m\times m}. \tag{2.14}$$

Therefore, the block pulse series of a function containing time delay $\tau = (q+\lambda)h$ can easily be obtained as :

$$f(t-\tau) \simeq F^T\Phi(t-\tau) = F^T\big((1-\lambda)H^q + \lambda H^{q+1}\big)\Phi(t). \tag{2.15}$$

3. Stochastic integration operational matrix

From [14], the Itô integral of every function $f(t)$ can be approximated as follows

$$\int_0^t f(s)dB(s) \simeq \int_0^t F^T\Phi(s)dB(s) \simeq F^T P_S\Phi(t). \tag{3.1}$$

Where stochastic operational matrix of integration is given by

$$P_S = \begin{pmatrix} B(\frac{h}{2}) & B(h) & B(h) & \dots & B(h) \\ 0 & B(\frac{3h}{2})-B(h) & B(2h)-B(h) & \dots & B(2h)-B(h) \\ 0 & 0 & B(\frac{5h}{2})-B(2h) & \dots & B(3h)-B(2h) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & B(\frac{(2m-1)h}{2})-B((m-1)h) \end{pmatrix}. \tag{3.2}$$

4. Solving stochastic Volterra-Fredholm integral equations with time delay by using stochastic operational matrix

We intend following linear stochastic Volterra-Fredholm integral equations with time delay,

$$X(t) = f(t) + \int_a^b k_1(s,t)X(s)ds + \int_0^t k_2(s,t)X(s-\tau)ds + \int_0^t k_3(s,t)X(s)dB(s), \qquad t \in [0,T), \tau \in (0,T). \tag{4.1}$$

Usually we set $[0, mh]$ instead of $[a, b]$ in relation (4.1) to convenience the use of block pulse functions.

We approximate $X(t)$, $f(t)$, $k_1(s,t)$, $k_2(s,t)$ and $k_3(s,t)$ by relations (2.7), (2.8) as follows

$$X(t) \simeq X^T\Phi(t) = \Phi^T(t)X,$$
$$f(t) \simeq F^T\Phi(t) = \Phi^T(t)F,$$
$$k_1(s,t) \simeq \Psi^T(s)K_1\Phi(t) = \Phi^T(t)K_1^T\Psi(s),$$
$$k_2(s,t) \simeq \Psi^T(s)K_2\Phi(t) = \Phi^T(t)K_2^T\Psi(s).$$
$$k_3(s,t) \simeq \Psi^T(s)K_3\Phi(t) = \Phi^T(t)K_3^T\Psi(s).$$

In the above approximates, X and F are stochastic block pulse coefficient vector, and K_1, K_2 and K_3 are stochastic block pulse coefficient matrix. We approximate $X(s-\tau)$ as follows,

$$X(s-\tau) \simeq X^T\Psi(s-\tau) \simeq X^T\big((1-\lambda)H^q + \lambda H^{q+1}\big)\Psi(s),$$

and by letting $A = (1-\lambda)H^q + \lambda H^{q+1}$,we can write,

$$X(s-\tau) \simeq X^T A\Psi(s).$$

With substituting above approximations in equation (4.1), we get

$$\begin{aligned} X^T\Phi(t) \simeq F^T\Phi(t) + X^T\Big(\int_0^{mh} \Psi(s)\Psi^T(s)ds\Big)K_1\Phi(t) \\ + X^T A\Big(\int_0^{t} \Psi(s)\Psi^T(s)ds\Big)K_2\Phi(t) \\ + X^T\Big(\int_0^{t} \Psi(s)\Psi^T(s)dB(s)\Big)K_3\Phi(t). \end{aligned} \tag{4.2}$$

Let K_2^i be the ith row of the constant matrix K_2, and K_3^i be the ith row of the constant matrix K_3, and R^i be the ith row of the integration operational matrix P, R_S^i be the ith row of the stochastic integration operational matrix P_S, $D_{K_2^i}$ be a diagonal matrix with K_2^i as its diagonal entries, and $D_{K_3^i}$ be a diagonal matrix with K_3^i as its diagonal entries. By the previous relations and assuming $m_1 = m_2$, we have,

$$\begin{aligned} \Big(\int_0^{mh} \Psi(s)\Psi^T(s)ds\Big)K_1\Phi(t) &= \Big(\int_0^{mh} \Phi(s)\Phi^T(s)ds\Big)K_1\Phi(t) \\ &= (hI)K_1\Phi(t) = B_1\Phi(t). \end{aligned} \tag{4.3}$$

where

$$B_1 = h\begin{pmatrix} k_{11}^1 & k_{12}^1 & k_{13}^1 & \cdots & k_{1m}^1 \\ k_{21}^1 & k_{22}^1 & k_{23}^1 & \cdots & k_{2m}^1 \\ k_{31}^1 & k_{32}^1 & k_{33}^1 & \cdots & k_{3m}^1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_{m1}^1 & k_{m2}^1 & k_{m3}^1 & \cdots & k_{mm}^1 \end{pmatrix}_{m\times m}, \tag{4.4}$$

$$\begin{aligned} \Big(\int_0^{t} \Psi(s)\Psi^T(s)ds\Big)K_2\Phi(t) &= \Big(\int_0^{t} \Phi(s)\Phi^T(s)ds\Big)K_2\Phi(t) \\ &= \begin{pmatrix} R^1\Phi(t)K_2^1\Phi(t) \\ R^2\Phi(t)K_2^2\Phi(t) \\ \vdots \\ R^m\Phi(t)K_2^m\Phi(t) \end{pmatrix} = \begin{pmatrix} R^1 D_{K_2^1} \\ R^2 D_{K_2^2} \\ \vdots \\ R^m D_{K_2^m} \end{pmatrix}\Phi(t) = B_2\Phi(t), \end{aligned} \tag{4.5}$$

where

$$B_2 = \frac{h}{2}\begin{pmatrix} k^2_{11} & 2k^2_{12} & 2k^2_{13} & \cdots & 2k^2_{1m} \\ 0 & k^2_{22} & 2k^2_{23} & \cdots & 2k^2_{2m} \\ 0 & 0 & k^2_{33} & \cdots & 2k^2_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & k^2_{mm} \end{pmatrix}_{m\times m}, \tag{4.6}$$

also,

$$\begin{aligned} \Big(\int_0^t \Psi(s)\Psi^T(s)dB(s)\Big)K_3\Phi(t) &= \Big(\int_0^t \Phi(s)\Phi^T(s)dB(s)\Big)K_3\Phi(t) \\ &= \begin{pmatrix} R^1_S\Phi(t)K^1_3\Phi(t) \\ R^2_S\Phi(t)K^2_3\Phi(t) \\ \vdots \\ R^m_S\Phi(t)K^m_3\Phi(t) \end{pmatrix} = \begin{pmatrix} R^1_S D_{K^1_3} \\ R^2_S D_{K^2_3} \\ \vdots \\ R^m_S D_{K^m_3} \end{pmatrix}\Phi(t) = B_3\Phi(t), \end{aligned} \tag{4.7}$$

where

$$B_3 = \begin{pmatrix} k^3_{11}B(\frac{h}{2}) & k^3_{12}B(h) & k^3_{13}B(h) & \cdots & k^3_{1m}B(h) \\ 0 & k^3_{22}\Big(B(\frac{3h}{2}) - B(h)\Big) & k^3_{23}\Big(B(2h) - B(h)\Big) & \cdots & k^3_{2m}\Big(B(2h) - B(h)\Big) \\ 0 & 0 & k^3_{33}\Big(B(\frac{5h}{2}) - B(2h)\Big) & \cdots & k^3_{3m}\Big(B(3h) - B(2h)\Big) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & k^3_{mm}\Big(B(\frac{(2m-1)h}{2}) - B((m-1)h)\Big) \end{pmatrix}. \tag{4.8}$$

With substituting relations (4.3), (4.5) and (4.7) in (4.2), we get

$$X^T\Phi(t) \simeq F^T\Phi(t) + X^TB_1\Phi(t) + X^TAB_2\Phi(t) + X^TB_3\Phi(t).$$

Then,

$$X^T(I - B_1 - AB_2 - B_3) \simeq F^T. \tag{4.9}$$

So, by setting $M = (I - B_1 - AB_2 - B_3)^T$ and replacing $\simeq$ by $=$, we will have,

$$MX = F. \tag{4.10}$$

Which is a linear system of equations with lower triangular coefficients matrix that gives the approximate block pulse coefficient of the unknown stochastic processes $X(t)$.

5. Error estimation and rate of convergence

In this section, we will show that the rate of convergence of the proposed method for solving stochastic Volterra-Fredholm integral equations with time delay is $O(h)$.

Theorem 5.1. *Suppose that* $f(t) \in C^1\big([0,1)\big)$ *and* $e(t) = f(t) - \hat{f}_m(t)$, $t \in I = [0,1)$, *which* $\hat{f}_m(t) = \sum_{i=1}^{m} f_i \phi_i^{(m)}(t)$ *is the block pulse series of* $f(t)$. *Then,*

$$\|e(t)\| \leq \frac{h}{2\sqrt{3}} \sup_{t\in I} |f'(t)|. \tag{5.1}$$

Proof. Let,

$$e_i(t) = \begin{cases} f(t) - f_i & t \in D_i, \\ 0 & t \in I - D_i. \end{cases} \tag{5.2}$$

where $D_i = \{t : (i-1)h \leq t < ih, h = \frac{1}{m}\}$ and $i = 1, 2, ..., m$.
We have,

$$e_i(t) = f(t) - \frac{1}{h}\int_{(i-1)h}^{ih} f(s)ds = \frac{1}{h}\int_{(i-1)h}^{ih} \Big(f(t) - f(s)\Big)ds,$$

now by mean value theorem, we get,

$$e_i(t) = \frac{f'(\eta_i)}{h}\int_{(i-1)h}^{ih} (t-s)ds = f'(\eta_i)\Big(t + (-i + \frac{1}{2})h\Big),$$

where $t, \eta_i \in D_i$, for $i = 1, 2, ..., m$. Then,

$$\|e_i(t)\|^2 = \int_{(i-1)h}^{ih} |e_i(t)|^2 dt = \big(f'(\eta_i)\big)^2 \int_{(i-1)h}^{ih} \Big(t + (-i + \frac{1}{2})h\Big)^2 dt = \frac{h^3}{12}\big(f'(\eta_i)\big)^2,$$

Consequently

$$\begin{aligned} \|e(t)\|^2 &= \int_0^1 |e(t)|^2 dt = \int_0^1 \Big(\sum_{i=1}^{m} e_i(t)\Big)^2 dt \\ &= \int_0^1 \Big[\sum_{i=1}^{m} e_i^2(t) + 2\sum_{i<j} e_i(t)e_j(t)\Big]dt = \sum_{i=1}^{m}\int_0^1 e_i^2(t)dt \\ &= \sum_{i=1}^{m} \|e_i(t)\|^2 = \frac{h^3}{12}\sum_{i=1}^{m}\big(f'(\eta_i)\big)^2 \leq \frac{h^2}{12}\sup_{t\in I}|f'(t)|^2, \end{aligned} \tag{5.3}$$

or,

$$\|e(t)\| \leq \frac{h}{2\sqrt{3}} \sup_{t\in I} |f'(t)|.$$

Hence, $\|e(t)\| = O(h)$. □

Theorem 5.2. *Suppose that* $f(s,t)$ *is a function in* $L^2\big([0,1)\times[0,1)\big)$ *and* $e(s,t) = f(s,t) - \hat{f}_m(s,t)$*, which* $\hat{f}_m(s,t) = \sum_{i=1}^m \sum_{j=1}^m f_{ij}\psi_i^{(m)}(s)\phi_j^{(m)}(t)$ *is the block pulse series of* $f(s,t)$*. Then,*

$$\|e(s,t)\| \leq \frac{h}{2\sqrt{3}}\Big(\sup_{(x,y)\in D}|f_s'(x,y)|^2 + \sup_{(x,y)\in D}|f_t'(x,y)|^2\Big)^{\frac{1}{2}}. \tag{5.4}$$

Proof. Let,

$$e_{ij}(s,t) = \begin{cases} f(s,t) - f_{ij} & (s,t)\in D_{ij}, \\ 0 & (s,t)\in D - D_{ij}. \end{cases} \tag{5.5}$$

where $D_{ij} = \{(s,t) : (i-1)h \leq s < ih, (j-1)h \leq t < jh, h = \frac{1}{m}\}$ and $i,j = 1,2,...,m$, we have,

$$\begin{aligned} e_{ij}(s,t) &= f(s,t) - \frac{1}{h^2}\int_{(i-1)h}^{ih}\int_{(j-1)h}^{jh} f(x,y)dydx \\ &= \frac{1}{h^2}\int_{(i-1)h}^{ih}\int_{(j-1)h}^{jh}\Big(f(s,t) - f(x,y)\Big)dydx, \end{aligned} \tag{5.6}$$

now by mean value theorem, we get,

$$\begin{aligned} e_{ij}(s,t) &= \frac{1}{h^2}\int_{(i-1)h}^{ih}\int_{(j-1)h}^{jh}\Big((s-x)f_s' + (t-y)f_t'\Big)dydx \\ &= f_s'\Big(s + (-i+\frac{1}{2})h\Big) + f_t'\Big(t + (-j+\frac{1}{2})h\Big). \end{aligned} \tag{5.7}$$

Then,

$$\|e_{ij}(s,t)\|^2 = \int_{(i-1)h}^{ih}\int_{(j-1)h}^{jh}|e_{ij}(s,t)|^2 dtds = \frac{h^4}{12}\Big(f_s'^2 + f_t'^2\Big), \quad (\eta_i,\eta_j)\in D_{ij}.$$

Consequently

$$\begin{aligned} \|e(s,t)\|^2 &= \int_0^1\int_0^1 |e(s,t)^2 dtds \\ &= \int_0^1\int_0^1\Big(\sum_{i=1}^m\sum_{j=1}^m e_{ij}(s,t)\Big)^2 dtds \\ &= \sum_{i=1}^m\sum_{j=1}^m\int_0^1\int_0^1 e_{ij}^2(s,t)dtds = \sum_{i=1}^m\sum_{j=1}^m \|e_{ij}(s,t)\|^2 \\ &= \frac{h^4}{12}\sum_{i=1}^m\sum_{j=1}^m\Big(f_s'^2(\eta_i,\eta_j) + f_t'^2(\eta_i,\eta_j)\Big) \\ &\leq \frac{h^2}{12}\Big(\sup_{(x,y)\in D}|f_s'(x,y)|^2 + \sup_{(x,y)\in D}|f_t'(x,y)|^2\Big), \end{aligned} \tag{5.8}$$

or,

$$\|e(s,t)\| \leq \frac{h}{2\sqrt{3}}\Big(\sup_{(x,y)\in D}|f_s'(x,y)|^2 + \sup_{(x,y)\in D}|f_t'(x,y)|^2\Big)^{\frac{1}{2}}.$$

Hence, $\|e(s,t)\| = O(h)$. □

Theorem 5.3. *Suppose $X(t)$ is exact solution of (4.1) and $\hat{Y}_m(t)$ is the proposed method solution and so $X(t) \simeq \hat{X}_m(t)$, $Y(t) \simeq \hat{Y}_m(t)$. Then*

$$\begin{aligned}\|X(t) - \hat{Y}_m(t)\|_\infty \leq & \frac{h}{2\sqrt{3}}\Big(\|M^{-1}\|\Big(1 + (b-a)M_1 \\ & + M_2 + M_3 \sup_{t\in I}|B(t)|\Big)\|X'\|_\infty + \|Y'\|_\infty\Big), \end{aligned} \tag{5.9}$$

where M is defined in (4.10) and

$$\|X(t) - \hat{Y}_m(t)\|_\infty = \sup_{t\in I}|X(t) - \hat{Y}_m(t)|,$$

$$M_j = \sup_{s,t\in I}|k_j(s,t)|, \quad j = 1,2,3.$$

Proof. We can write

$$\sup_{t\in I}|X(t) - \hat{Y}_m(t)| \leq \sup_{t\in I}|X(t) - Y(t)| + \sup_{t\in I}|Y(t) - \hat{Y}_m(t)|.$$

Furthermore from (5.4) we have

$$\sup_{t\in I}|Y(t) - \hat{Y}_m(t)| \leq \frac{h}{2\sqrt{3}}\|Y'\|_\infty,$$

then it is enough to find a bound for $\sup_{t\in I}|X(t) - Y(t)|$. We know that for any arbitrary real bounded function $f(t)$, which is square integrable in the interval $t \in [0,1)$, and for any $\epsilon > 0$, there exists m such that the inequality $\|f - \hat{f}_m\| < \epsilon$, holds. So we can write Stochastic Itô integral equation (4.1) as

$$\begin{aligned} f(t) =& \hat{X}_m(t) - \int_a^b k_1(s,t)\hat{X}_m(s)ds - \int_0^t k_2(s,t)\hat{X}_m(s-\tau)ds \\ & + \int_0^t k_3(s,t)\hat{X}_m(s)dB(s) \qquad t\in[0,1). \end{aligned} \tag{5.10}$$

If we substitute $Y(t)$ instead of $X(t)$ in above equation then the write hand side of integral equation is exchanged by a new function that we denote it by $\hat{f}_m(t)$. So we have,

$$\begin{aligned} \hat{f}(t) =& \hat{Y}_m(t) - \int_a^b k_1(s,t)\hat{Y}_m(s)ds - \int_0^t k_2(s,t)\hat{Y}_m(s-\tau)ds \\ & + \int_0^t k_3(s,t)\hat{Y}_m(s)dB(s) \qquad t\in[0,1). \end{aligned} \tag{5.11}$$

From relation (4.10) we have

$$X(t) = M^{-1}f(t), \quad Y(t) = M^{-1}\hat{f}(t).$$

Consequently we have

$$\sup_{t\in I}|X(t) - Y(t)| \leq \|M^{-1}\| \sup_{t\in I}|f(t) - \hat{f}(t)|. \tag{5.12}$$

For finding a bound for $\sup_{t\in I}|f(t)-\hat{f}(t)|$, we write,

$$\begin{aligned} f(t) =& X(t) - \int_a^b k_1(s,t)X(s)ds - \int_0^t k_2(s,t)X(s-\tau)ds \\ &+ \int_0^t k_3(s,t)X(s)dB(s) \qquad t\in[0,1). \end{aligned}$$

and

$$\begin{aligned} \hat{f}(t) =& \hat{X}_m(t) - \int_a^b k_1(s,t)\hat{X}_m(s)ds - \int_0^t k_2(s,t)\hat{X}_m(s-\tau)ds \\ &+ \int_0^t k_3(s,t)\hat{X}_m(s)dB(s) \qquad t\in[0,1). \end{aligned}$$

So that,

$$\begin{aligned} f(t)-\hat{f}(t) =& X(t)-\hat{X}_m(t) - \int_a^b k_1(s,t)\Big(X(s)-\hat{X}_m(s)\Big)ds \\ &- \int_0^t k_2(s,t)\Big(X(s-\tau)-\hat{X}_m(s-\tau)\Big)ds \\ &- \int_0^t k_3(s,t)\Big(X(s)-\hat{X}_m(s)\Big)dB(s). \end{aligned}$$

Therefore,

$$\begin{aligned} \sup|f(t)-\hat{f}(t)| \le& \sup|X(t)-\hat{X}_m(t)| + \sup\Big|\int_a^b k_1(s,t)\Big(X(s)-\hat{X}_m(s)\Big)ds\Big| \\ &+ \sup\Big|\int_0^t k_2(s,t)\Big(X(s-\tau)-\hat{X}_m(s-\tau)\Big)ds\Big| \\ &+ \sup\Big|\int_0^t k_3(s,t)\Big(X(s)-\hat{X}_m(s)\Big)dB(s)\Big| \\ &\le \sup|X(t)-\hat{X}_m(t)| + \sup\int_a^b |k_1(s,t)||X(s)-\hat{X}_m(s)|ds \\ &+ \sup\int_0^t |k_2(s,t)||X(s-\tau)-\hat{X}_m(s-\tau)|ds \\ &+ \sup\int_0^t |k_3(s,t)||X(s)-\hat{X}_m(s)|dB(s) \\ &\le \sup|X(t)-\hat{X}_m(t)| + (b-a)M_1\sup|X(t)-\hat{X}_m(t)| \\ &+ M_2\sup|X(t-\tau)-\hat{X}_m(t-\tau)| + M_3\sup|B(t)|\sup|X(t)-\hat{X}_m(t)| \\ &\le \Big(1+(b-a)M_1+M_2+M_3\sup|B(t)|\Big)\frac{h}{2\sqrt{3}}\|X'\|_\infty, \end{aligned}$$

so by substituting this bound in the inequality (5.12) we get,

$$\sup_{t\in I}|X(t)-Y(t)| \le \|M^{-1}\|\Big(1+(b-a)M_1+M_2+M_3\sup|B(t)|\Big)\frac{h}{2\sqrt{3}}\|X'\|_\infty, \quad (5.13)$$

then from relation (5.12) and (5.13) we have

$$\|X(t)-\hat{Y}_m(t)\|_\infty \leq \frac{h}{2\sqrt{3}}\Big(\|M^{-1}\|\Big(1+(b-a)M_1+M_2+M_3\sup_{t\in I}|B(t)|\Big)\|X'\|_\infty+\|Y'\|_\infty\Big), \tag{5.14}$$

and the proof is complete. □

6. Numerical examples

To show the efficiency of the proposed numerical method, we consider the following examples. Because we don't have examples with exact solutions, for given m, we compute and plot the values of $|X_{2m}(t) - X_m(t)|$. We do this with $m = 32$, $m = 64$ for $\tau = 0.001$, $\tau = 0.005$, $\tau = 0.010$.

Example 6.1. Consider the following linear stochastic Volterra-Fredholm integral equation with time delay,

$$X(t) = 1 + \int_0^T s^2 X(s)ds \int_0^t s^2 X(s-\tau)ds + \int_0^t sX(s)dB(s), \quad s,t \in [0,1), \tau \in (0,0.5), \tag{6.1}$$

where $X(t)$ is an unknown stochastic processes. The numerical results of $|X_{2m}(t) - X_m(t)|$ with $m = 32$, $m = 64$ in three constant τ are shown in Table 1. The curves in Figure 1 reresent a trajectory of the approximate solution computed by the presented method.

Example 6.2. Consider the following linear stochastic Volterra-Fredholm integral equation with time delay,

$$X(t) = \frac{1}{16} + \int_0^T (1+\cos(s))X(s)ds + \int_0^t (1-\cos(s))X(s-\tau)ds + \int_0^t \sin(s)X(s)dB(s), \quad s,t \in [0,1), \tau \in (0,0.5), \tag{6.2}$$

where $X(t)$ is an unknown stochastic processes. The numerical results of $|X_{2m}(t) - X_m(t)|$ with $m = 32$, $m = 64$ in three constant τ are shown in Table 2. The curves in Figure 2 reresent a trajectory of the approximate solution computed by the presented method.

Table 1: The values of $|X_{2m}(t) - X_m(t)|$ for Example 1 with $m = 32$, $m = 64$ in three constant τ.

t	$\tau = 0.001$		$\tau = 0.005$		$\tau = 0.010$	
	$m = 32$	$m = 64$	$m = 32$	$m = 64$	$m = 32$	$m = 64$
0	$2.127752E-4$	$1.217680E-3$	$2.316497E-5$	$3.125784E-5$	$6.215425E-5$	$3.225236E-5$
0.1	$3.692284E-3$	$3.672124E-4$	$1.021215E-5$	$6.215423E-5$	$1.212252E-5$	$7.215147E-5$
0.2	$1.29763E-4$	$2.684928E-5$	$7.548799E-5$	$5.215424E-5$	$3.262421E-5$	$2.363696E-5$
0.3	$2.561221E-4$	$6.124578E-4$	$2.132355E-4$	$1.121114E-4$	$3.000124E-4$	$6.363535E-4$
0.4	$7.754215E-4$	$3.303101E-4$	$9.001542E-4$	$6.215425E-4$	$2.045102E-4$	$7.171717E-4$
0.5	$1.246795E-4$	$3.167965E-4$	$2.012154E-4$	$1.021542E-4$	$6.325331E-4$	$5.124512E-4$
0.6	$2.454565E-4$	$2.281297E-4$	$6.215421E-4$	$9.326523E-4$	$3.215241E-4$	$7.124974E-4$
0.7	$2.111452E-3$	$3.265178E-3$	$1.374961E-3$	$2.124512E-3$	$7.112421E-3$	$7.284756E-3$
0.8	$3.746512E-3$	$2.179435E-3$	$7.578771E-3$	$4.553333E-3$	$2.575022E-3$	$1.756863E-3$
0.9	$5.377912E-3$	$8.369636E-3$	$3.278866E-3$	$2.467333E-3$	$2.011122E-3$	$2.776888E-3$

Table 2: The values of $|X_{2m}(t) - X_m(t)|$ for Example 2 with $m = 32$, $m = 64$ in three constant τ.

t	$\tau = 0.001$		$\tau = 0.005$		$\tau = 0.010$	
	$m = 32$	$m = 64$	$m = 32$	$m = 64$	$m = 32$	$m = 64$
0	$2.215175E-4$	$2.151151E-4$	$1.266544E-4$	$3.126546E-4$	$3.012256E-4$	$2.756894E-4$
0.1	$5.217551E-4$	$6.025646E-4$	$2.134656E-4$	$5.23215E-4$	$2.457561E-4$	$2.756894E-4$
0.2	$2.386974E-4$	$3.265487E-4$	$5.262622E-4$	$5.124567E-4$	$2.101275E-4$	$5.437698E-4$
0.3	$3.679411E-3$	$6.232232E-3$	$6.323265E-3$	$2.011227E-3$	$5.346734E-3$	$2.357677E-3$
0.4	$4.215478E-3$	$6.200021E-3$	$4.154816E-3$	$8.267777E-3$	$1.278398E-3$	$2.467888E-3$
0.5	$3.265855E-3$	$2.352633E-3$	$5.187694E-3$	$5.623367E-3$	$2.376866E-3$	$1.276811E-3$
0.6	$9.384625E-3$	$4.124512E-3$	$6.130512E-3$	$2.021810E-3$	$2.575689E-3$	$7.353531E-3$
0.7	$4.271564E-3$	$3.268965E-3$	$5.167899E-3$	$8.055579E-3$	$2.437857E-3$	$2.018001E-3$
0.8	$2.356258E-3$	$5.242415E-3$	$5.021219E-3$	$2.011003E-3$	$8.246877E-3$	$3.045780E-3$
0.9	$6.245105E-3$	$3.022211E-3$	$4.001215E-3$	$3.021554E-3$	$2.768988E-3$	$6.487301E-3$

FIGURE 1. Trajectory of the approximate solution of Example 1 for $m = 32$, $m = 64$ with $\tau = 0.005$.

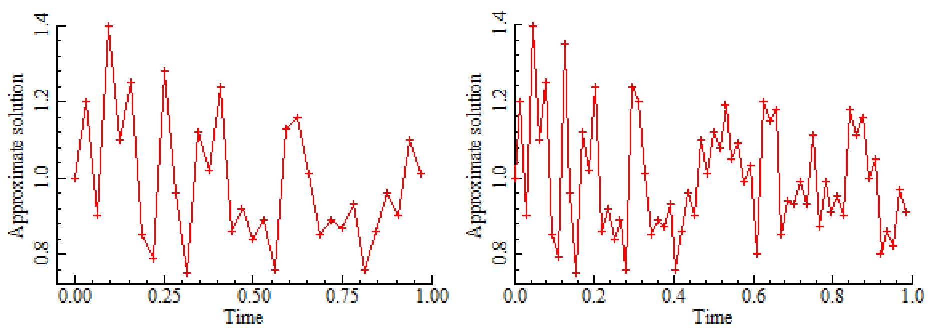
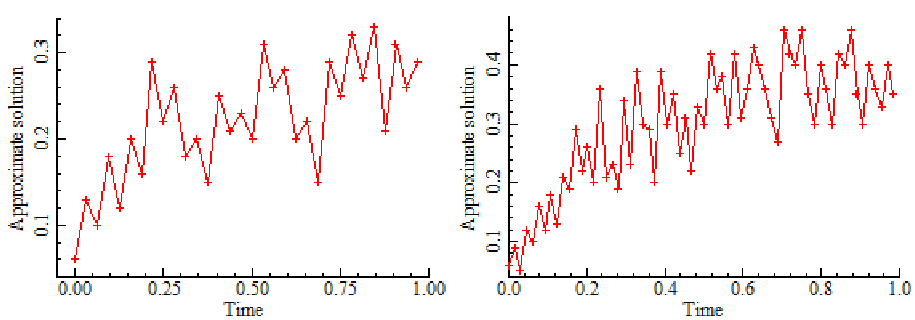


FIGURE 2. Trajectory of the approximate solution of Example 2 for $m = 32$, $m = 64$ with $\tau = 0.005$.



Conclusion

Because of the structure of stochastic equations, closed-form solutions of many important stochastic functional equations are virtually impossible to obtain. Thus, numerical solutions are a viable alternative. Using piecewise constant orthogonal functions as basis functions to solve the stochastic Volterra-Fredholm integral equations was very simple and effective in comparison with other methods. The main advantage of the proposed method was to transform the main problem into linear systems of algebraic equations which can be simply solved. Its applicability and accuracy is checked on some examples. Moreover, the Ito-Taylor expansion described by Kloeden and Platen [5], or generalized hat basis functions together with their stochastic operational matrix of Ito-integration [10], has a lot of steps and complicated computing.

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