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An efficient method to approximate eigenvalues and eigenfunctions of high order Sturm-Liouville problems

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Abstract determination of eigenvalues and eigenfunctions of a High-order Sturm-Liouville problem (HSLP) is considered. To this end, the Differential Transformation Method (DTM) is applied which is an efficient technique for solving differential equations. The results of the proposed approach are compared with those of some well-known methods reported in the literature. Four illustrative real life examples of mechanical engineering are provided to show the ability and the cost-effectiveness of this numerical approach.

Keywords. High order sturm-liouville problems, Differential transformation method, Eigenvalue.

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1. Introduction

The Sturm-Liouville problems play an important role in applied mathematics, physics and engineering. Although, the second order Sturm-Liouville equations describe many physical phenomena, mathematically, as classical and quantum mechanics (see [11, 12, 20]), and higher order Sturm-Liouville problems can be used in mathematical modeling of more important phenomena. For example, fourth order Sturm-Liouville problem is modeling the free vibration analysis of beam structures (see [5, 8, 16]), and a variety of fluid mechanics models are governed by high-order Sturm-Liouville problems. For instance, when the ordinary convection arises from heating the beneath layer of the fluid regarding the action of rotation, an eight order Sturm-Liouville problem can model the instability of moleculas. On the other hand, the

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marginal state is delineated by sixth order Sturm-Liouville problem (see [3,4,9,16]). The governed differential equation would be ten and twelfth order Sturm-Liouville boundary value problems provided that a uniform magnetic field was applied across the fluid in some direction as gravity (see [3,4,10,15]). However, not much work has been done for higher order Sturm-Liouville problems and it is still a challenging task for scientists.

Adomian decomposition method (ADM), variational iteration method (VIM), Chebyshev spectral collocation method (CSCM), and modified Adomian decomposition method, are numerical or semi-analytic schemes available on this subject (see [2,16,17]).

In this study, we will propose differential transformation method to solve a non-singular high order Sturm-Liouville problem in the following form

$$\begin{aligned}&(-1)^m(p_m(x)y^{(m)}(x))^{(m)}+(-1)^{(m-1)}(p_{m-1}(x)y^{(m-1)}(x))^{(m-1)}+\cdots\\&\quad+(p_2(x)y''(x))''-(p_1(x)y'(x))'+p_0(x)y(x)\\&=\lambda w(x)y(x),\ 0<x<b,\end{aligned}\tag{1.1}$$

subject to the separated boundary conditions as follows:

$$\begin{aligned}&y^{(i)}(0)=0,\\&y^{(i)}(b)=0,\end{aligned}\tag{1.2}$$

for $i\in S'\subset S:=\{0,1,2,\cdots,2m-1\}$, where S' has m elements. In equation (1.1), we assume that all cofficients $p_i(x),\ i=0,\cdots,m$, are real valued functions. For applying DTM, the following conditions are necessary:
The interval $(0,b)$ must be finite; the coefficient functions $p_i(x)$ $(0\le i\le m)$, the weight function $w(x)$, and $\dfrac{1}{p_m(x)}$ are in $L^1(0,b)$; $p_m(x)$ and $w(x)$ are both positive.

The eigenvalues $\lambda_k,\ k=1,2,3,\cdots$ can be ordered as an increasing sequence $\lambda_1,\lambda_2,\cdots$, where $\lim_{k\to\infty}\lambda_k=\infty$, and each eigenvalue has multiplicity at most m (see [10,15] for more details).

This paper is organized as follows: the differential transformation method and some important theorems pertaining to DTM will be addressed in section 2. Section 3 is devoted to showing the efficiency of the proposed method by presenting four examples. Some conclusions are drawn in section 4.

2. Differential transform method

Differential transform method has become a powerful and easy device for solving a bunch of differential equations arisen from physics and engineering (see [6]). Let

$$Y(k)=\frac{1}{k!}\left[\frac{d^ky(x)}{dx^k}\right]_{x=0},\tag{2.1}$$

where $y(x)$ is the solution of differential equation and $Y(k)$ is the differential transformation of y(x). Inverse differential transformation of $Y(k)$ is defined as follows

$$y(x)=\sum_{k=0}^{\infty} Y(k)x^k. \tag{2.2}$$

We consider the first N terms of series (2.2), as an approximation of the solution of Eq. (1.1), as follows

$$\hat{y}(x;\lambda)=\sum_{k=0}^{N} Y(k;\lambda)x^k. \tag{2.3}$$

Some essential properties of DTM that can be proved easily are listed as follows (see [18]).

(1) If $f(x)=g(x)\pm h(x)$, then $F(k)=G(k)\pm H(k)$,
(2) If $f(x)=\alpha g(x)$, then $F(k)=\alpha G(k)$,
(3) If $f(x)=g(x)h(x)$, then $F(k)=\sum_{l=0}^{k} G(l)H(k-l)$,
(4) If $f(x)=(x-x_0)^p$, then $F(k)=\delta(k-p)$
where,

$$\delta(k)=\begin{cases} 1 & if \quad k=0,\\ 0 & if \quad k\neq 0,\end{cases}$$

(5) If $f(x)=\dfrac{d^m g(x)}{dx^m}$, then $F(k)=(k+1)(k+2)\cdots(k+m)G(k+m)$.

3. DTM of high order SLP

Taking differential transformation of (1.1) and using all the properties mentioned in the previous section, we obtain

$$\begin{aligned}&(-1)^m(k+1)\cdots(k+m)\sum_{l=0}^{k+m}\left(P_m(l)(k+m-l+1)\cdots(k+2m-l)Y(k+2m-l)\right)\\ &+(-1)^{m-1}(k+1)\cdots(k+m-l)\sum_{l=0}^{k+m-l}(P_{m-1}(l)(k+m-l)\cdots(k+2m-l-1)\\ &\times Y(k+2m-l-1))+\cdots+(-1)^2(k+1)(k+2)\sum_{l=0}^{k+2}(P_2(l)(k+3-l)(k+4-l)\\ &\times Y(k+4-l))+(-1)(k+1)\sum_{l=0}^{k+1}P_1(l)(k-l+2)Y(k+2-l)+\sum_{l=0}^{k}P_0(l)Y(k-l)\\ &=\lambda\sum_{l=0}^{k} w(l)Y(k-l).\end{aligned}$$

So,

$$\begin{aligned} Y(k+2m) = \Bigg[\lambda \sum_{l=0}^{k} w(l)Y(k-l) - [(-1)^m (k+1)\cdots(k+m) \\ \times \sum_{l=1}^{k+m} P_m(l)(k+m-l+1)\cdots(k+2m-l)Y(k+2m-l) \\ + \cdots + \sum_{l=0}^{k} P_0(l)Y(k-l)]\Bigg] \Big/ \Bigg[(-1)^m(k+1)\cdots \\ (k+m)P_m(0)(k+m+1)\cdots(k+2m)\Bigg], \end{aligned} \tag{3.1}$$

where P_i is the differential transform of p_i, $i = 0, \cdots m$. According to (3.1), we need $2m$ starting values $Y(k), Y(k+1), \cdots, Y(k+2m-1)$ to apply DTM. It should be noted that two index subsets S' and $S \setminus S'$, $S := \{0, 1, \cdots, 2m-1\}$, have m elements. Starting values $Y(k)$ for $k \in S'$ can be easily obtained by (2.1). Let's put m other starting values $Y(k)$, $k \in S \setminus S'$, equal to unknown parameters c_k, $k = 1, \ldots, m$. According to the homogeneous boundary conditions (1.2) and the form of recursive relation (3.1), nonzero coefficients of differential transform, $Y(k;\lambda)$, include at least one parameter c_k, and also are linear with respect to these parameters. Therefore, the approximate solution (2.3) can be rewritten as follows:

$$\hat{y}(x;\lambda) = \sum_{k=1}^{m} c_k Q_k(x;\lambda), \tag{3.2}$$

where $Q_k \in \Pi_N$, Π_N is the set of polynomials of degree at most N with respect to x, which also include parameter λ. By imposing m boundary conditions (1.2) at point b, $y^{(i)}(b) = 0$, on (3.2), one obtains

$$\sum_{k=1}^{m} c_k Q_k^{(i)}(b;\lambda) = 0, \tag{3.3}$$

where $i \in S'$. The matrix form of the equations (3.3) is written as follows

$$\begin{pmatrix} Q_1^{(i_1)}(b,\lambda) & \cdots & Q_m^{(i_1)}(b,\lambda) \\ \vdots & & \vdots \\ Q_1^{(i_m)}(b,\lambda) & \cdots & Q_m^{(i_m)}(b,\lambda) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{3.4}$$

which is a homogeneous linear system of m equations with m unknown parameters $c_1, \ldots, c_m$, in where $i_1, \ldots, i_m \in S'$. In order to have a nontrivial solution for $c_1, \ldots, c_m$, the determinant of the coefficients matrix should be zero in (3.4) which is referred to as *characteristic equation*. This determinant is a polynomial of λ whose zeros correspond to the eigenvalues of original problem (1.1)-(1.2). In this paper, we use Mathematica to solve the characteristic equation, numerically by applying NSolve command.

4. Numerical Results

In this section, we apply the differential transform method to find the eigenvalues of some high order Sturm-Liouville problems, numerically. These examples have already been brought in the literature (see [1, 2, 9, 16, 19]). Our results will be compared with the results mentioned in some papers.

Example 4.1. *Consider the following fourth order Sturm-Liouville problem related to mechanicals non-linear systems identification [2, 14, 16].*

$$\begin{cases} y^{(4)}(x) - 2\alpha x^2 y'' - 4\alpha x y' + (\alpha^2 x^4 - 2\alpha)y = \lambda y, \quad x \in (0,5) \\ y(0) = y''(0) = 0, \\ y(5) = y''(5) = 0. \end{cases} \tag{4.1}$$

Applying DTM in this example, according to (3.1) we have

$$\begin{aligned} Y(k+4) = \Bigg[&\lambda Y(k) + 2\alpha Y(k) - \alpha^2 \sum_{l=0}^{k} \delta(l-4) Y(k-l) \\ &+ 4\alpha \sum_{l=0}^{k} \delta(l-1)(k-l+1) Y(k-l+1) \\ &+ 2\alpha \sum_{l=0}^{k} \delta(l-2)(k-l+1)(k-l+2) Y(k-l+2) \Bigg] \\ &\Big/ \Big[(k+1)(k+2)(k+3)(k+4) \Big], \end{aligned}$$

with starting values

$$Y(0) = 0; \quad Y(1) = c_1; \quad Y(2) = 0; \quad Y(3) = c_2.$$

Then, the approximate solution (3.2) will be

$$\hat{y}(x,\lambda) = c_1 Q_1(x;\lambda) + c_2 Q_2(x;\lambda), \tag{4.2}$$

in which

$$\begin{aligned} Q_1(x;\lambda) = x &+ \frac{1}{120}(0.06+\lambda)x^5 \\ &+ \frac{-0.0001 + 0.00166667(0.06+\lambda) + 1/120(0.06+\lambda)(0.42+\lambda)}{3024} x^9 \end{aligned}$$

and

$$Q_2(x;\lambda) = x^3 + \frac{1}{840}(0.26+\lambda)x^7$$

when $N = 9$ in (2.3). The approximate solution (4.2) just satisfies the initial conditions $y(0) = y''(0) = 0$ and by imposing boundary conditions $y(5) = y''(5) = 0$, one can readily obtain the following matrix form of equations

$$\begin{pmatrix} Q_1(5;\lambda) & Q_2(5;\lambda) \\ Q_1''(5;\lambda) & Q_2''(5;\lambda) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.3}$$

TABLE 1. Estimated eigenvalues of Example 4.1

k	λ_k			
	current work	ADM [2]	CSCM [16]	PDQ [19]
1	0.2150508643697155	0.21505086436971596	0.2150508643160	0.21505086437
2	2.7548099346830	2.754809934682985	2.7548099336169	2.75480993468
3	13.2153515405582	13.215351540558824	13.215351540581	13.2153515406
4	40.950819759160	40.95081975913761	40.95081978144	40.9508197591
5	99.05347806671	99.05347813813881	99.0534780383535	99.0534780633
6	204.35573364024	204.35449348957832	204.355735479344	204.355732256

We find the eigenvalues of example 4.1 by solving the characteristic equation of (4.3), i.e.,

$$Q_1(5;\lambda)Q_2''(5;\lambda) - Q_2(5;\lambda)Q_1''(5;\lambda) = 0.$$

Table 1 shows the first six calculated eigenvalues of problem (4.1) for $\alpha = 0.01$ by proposed approach when $N = 65$, Adomian decomposition method (ADM), Chebyshev spectral collocation method (CSCM), and polynomial-based differential quadrature (PDQ) (see [2, 16, 19]). Moreover, we gain one more extra eigenvalue $\lambda_7 = 377.40980259317$, to show the efficiency of our proposed method.

Example 4.2. *The following differential equation*

$$\left(\frac{d^2}{dx^2} - a^2\right)\left(\frac{d^2}{dx^2} - a^2 - v\right)\left(\frac{d^2}{dx^2} - a^2 - \frac{v}{p}\right)y = a^2Ry \tag{4.4}$$

arises in the analysis of a Benard problem [9] where parameters a, v, and R are constants, and v is regarded as the eigen parameter. We consider this problem with the following self-adjoint boundary conditions

$$y(0) = y''(0) = y^{(4)}(0) = 0,$$

$$y(1) = y''(1) = y^{(4)}(1) = 0.$$

Since the dependence on v is obviousely nonlinear, the equation (4.4) can be converted to the following linear eigenvalue problem [9].

$$Ly := \left(\frac{d^2}{dx^2} - a^2\right)\left(\frac{d^2}{dx^2} - a^2 - v\right)\left(\frac{d^2}{dx^2} - a^2 - \frac{v}{p}\right)y - a^2Ry = \lambda y, \quad x \in (0,1),$$

TABLE 2. Two first eigenvalues of Example 4.2

v	λ_1 [9]	λ_2 [9]	λ_1 [1]	λ_2 [1]	λ_1 present	λ_2 present
v_1^*	-1.0000050	106275.180	-1	2.15755×10^{-6}	-1	-35487.692788
v_2^*	-1.0001000	009528.170	-1	5.43301×10^{5}	-9530.184561	-1
v_3^*	-0.0000001	034762.361	-0.9934	8.74092×10^{7}	-2	-34764.361139
v_4^*	-0.0000300	009629.620	-0.9001	3.44567×10^{6}	-9630.467937	-13.051237

where the eigenvalues $\lambda_k,\ k = 1, 2, \cdots$ are functions of v and v-eigenvalues are given by

$$v_k^{(\pm)} = -\frac{(1+p)(k^2\pi^2 + a^2)}{2} \pm \sqrt{\frac{(1-p)^2(k^2\pi^2+a^2)^2}{4} + \frac{a^2Rp}{(k^2\pi^2+a^2)}}, \quad k = 1, 2, \cdots .$$

In the proposed method we have

$$\begin{aligned} Y(k+6) = \Big[&\big((v+1)^2 + 1 + \lambda\big)Y(k) - (v^2+4v+3)(k+1)(k+2)Y(k+2) \\ &+ (3+2v)(k+1)(k+2)(k+3)(k+4)Y(k+4)\Big] \\ &\Big/ \Big[(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)\Big], \end{aligned}$$

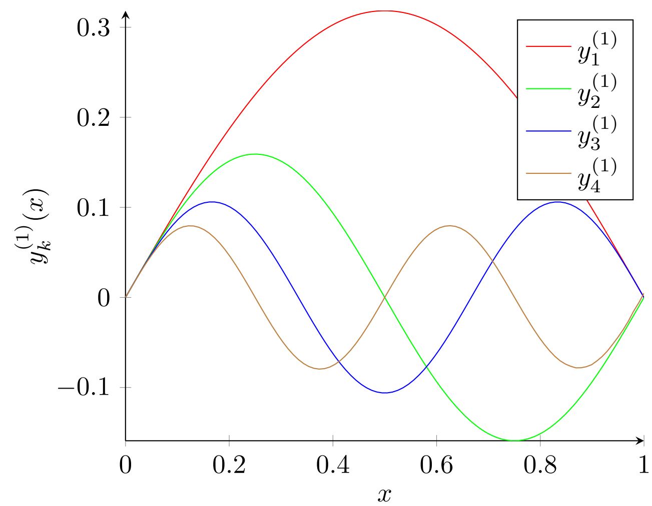
with starting values $Y(0) = Y(2) = Y(4) = 0,\ Y(1) = c_1,\ Y(3) = c_2,\ Y(5) = c_3$.

We compare the first two eigenvalues computed by DTM with those obtained by Greenberg and Marletta [9], and by Al-Mdallal and Syam [1], for $a = P = R = 1$, and several values of v; $v_1^* = (v_1^{(-)} + v_1^{(+)})/2$, $v_2^* = (v_2^{(-)} + v_2^{(+)})/2$, $v_3^* = v_1^{(-)}$ and $v_4^* = v_2^{(-)}$ (See table 2). Table 3 shows three more approximated eigenvalues of this problems of various nonlinear eigenvalues v_s^*; $s = 1, 2, 3, 4$. The eigenvalues λ_k, and related eigenfunctions $y_k^{(s)} := y^{(s)}(x; \lambda_k)$, $s = 1, 2, 3, 4$, can be calculated. The *kth* eigenfunction of v_1^*, i.e., $y_k^{(1)} := y^{(1)}(x; \lambda_k)$, $k = 1, 2, 3, 4$, is plotted in figure 1.

TABLE 3. Three more eigenvalues of Example 4.2

λ_k	v_1^*	v_2^*	v_3^*	v_4^*
λ_2	-559995.346453	-218748.79583021	-555701.148433753	-217313.268087
λ_3	-3.48468623×10^2	-2.229068748×10^6	-3.470429911×10^6	-2.2254295$\times 10^6$
λ_4	-1.047985334×10^7	-1.063340640×10^7	-1.044762528$\times 10^7$	-7.5469537$\times 10^6$

Figure 4.2. Illustration of some eigenfunctions in Example 4.2 in mentioned interval



It is worth noting that the kth eigenfunction has exactly $k-1$ zeros in the interval mentioned. To show the high accuracy of our results we define the following error function

$$e_k^{(s)} := L(y_k^{(s)}(x)) - \lambda_k y_k^{(s)}(x).$$

We illustrated the error functions for two first computed eigenfunctions, i.e. $e_1^{(s)}(x)$ and $e_2^{(s)}(x)$ in Figure 2. One can also see the property of orthogonality of eigenfunctions, e.g.,

$$\langle y_1^{(1)}(x), y_2^{(1)}(x)\rangle = 8.02 \times 10^{-16}$$
$$\langle y_1^{(2)}(x), y_2^{(2)}(x)\rangle = 6.34 \times 10^{-7}$$

Example 4.3. *Consider the following tenth order Sturm-Liouville problem*

$$\begin{cases} -y^{(10)}(x) = \lambda y(x), & x \in (0,\pi) \\ y(0) = y''(0) = y^{(4)}(0) = y^{(6)}(0) = 0, \\ y(\pi) = y''(\pi) = y^{(4)}(\pi) = y^{(6)}(\pi) = 0. \end{cases}$$

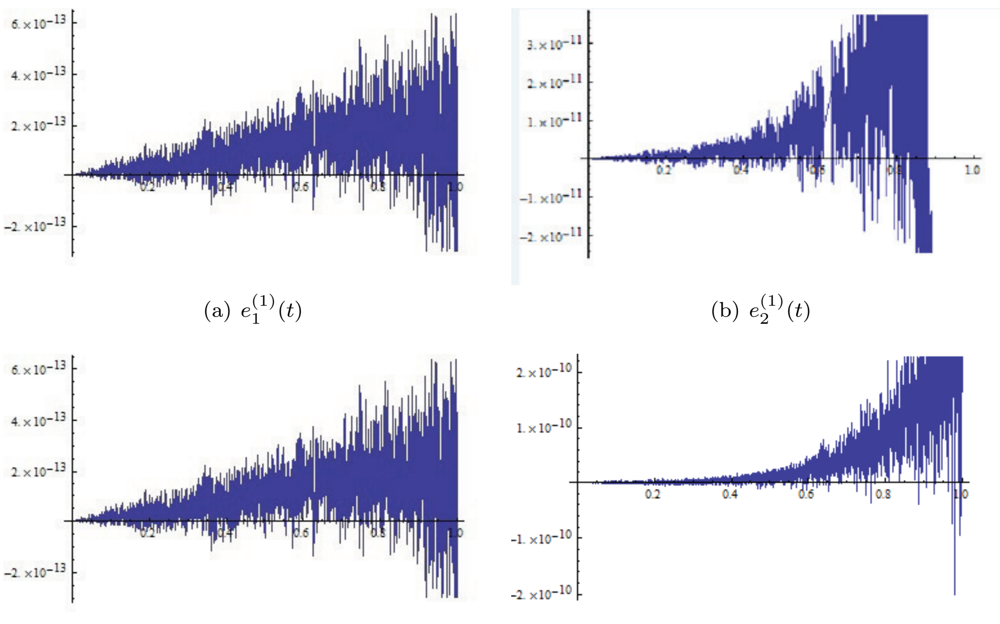
By applying the DTM we get

$$Y(k+10) = \frac{-\lambda Y(k)}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)(k+9)Y(k+9)}$$

with starting values

$$Y(0) = Y(2) = Y(4) = Y(6) = Y(8) = 0;$$
$$Y(1) = c_1;\ Y(3) = c_2;\ Y(5) = c_3;\ Y(7) = c_4;\ Y(9) = c_5.$$

Figure 1. Illustration of some error functions in examples 4.2 at the mentioned interval.



(a) $e_1^{(1)}(t)$ (b) $e_2^{(1)}(t)$

(c) $e_1^{(2)}(t)$ (d) $e_2^{(2)}(t)$

The kth exact eigenvalue of this problem is known to be k^{10}. The first six calculated eigenvalues are presented in table 4.

Example 4.4. *The following eigenproblem arises in the study of vibration in a turbine blade (see [13], p.141).*

$$(EIy'')'' - ((F - \omega^2 I\rho)y')' - \omega^2 \rho y = 0, \qquad 0 < x < l, \tag{4.5}$$

subject to the boundary conditions

$$y(0) = y'(0) = 0,$$
$$EIy''(l) = (EIy'')'(l) - ((F - \omega^2 I\rho)y')(l) = 0,$$

TABLE 4. Estimated relative error for Example 4.3

λ_k		relative error
λ_1	1	0
λ_2	1024	0
λ_3	59049.0000001	1.693×10^{-12}
λ_4	1048576.00000	4.163×10^{-10}
λ_5	9765625.00000	4.601×10^{-8}
λ_6	6046176.0000	2.267×10^{-7}

where ω is the vibrational frequency; y is the displacement perpendicular to the blade; E is the Young 's modulus; I is the moment of inertia of a cross-section of the blade; ρ is the linear density of the blade; and F is the (variable) centrifugal force

$$F(x) = \Omega^2 \int_x^l \rho(x)A(s)(r+s)ds,$$

in which Ω is the angular velocity, $A(.)$ is the cross-sectional area of the blade, and r is the radius of the turbine.

To compare our results with [10], we took $E = I = A(x) = \Omega = l = 1$, $r = \frac{2}{3}$ and $\rho(x) = x$. So, the eigenproblem (4.5) is converted to a standard eigenproblem by introducing a new eigen parameter λ as follows

$$y^{(iv)} - (((\frac{1}{3})(2+2x+x^2)(1-x) - \omega^2 xy = \lambda y, \quad 0 < x < 1, \tag{4.6}$$

subject to the boundary condition

$$y(0) = y'(0) = 0,$$
$$y''(1) = y'''(1) + \omega^2 y'(1) = 0.$$

λ-eigenvalues are functions of ω-eigenvalues for which the problem (4.6) has an infinite sequence of eigenvalues

$$\lambda_0(\omega) \leqslant \lambda_1(\omega) \leqslant \lambda_2(\omega) \leqslant \cdots .$$

The results in [7,10] imply that $\lambda_k(\omega)$ is a strictly decreasing function of ω. We obtain the kth eigenvalue of problem (4.5) by using simple rootfinding of equation $\lambda_k(\omega) = 0$.

ω	[10]	$(current\ work)_{N=30}$	$(current\ work)_{N=37}$
0	1.8115460	1.7790252978909484	1.7790253
1	5.9067512	5.651298414864634	5.6512999
2	10.8786209	10.807637812419266	10.5599679

5. Conclusion

In this paper, differential transformation method (DTM) was applied for high order Sturm-Liouville problems successfully. This method was implemented to calculate eigenvalues and related eigenfunctions of this kind of problems. Generally speaking, our proposed method is more convenient than other existed methods and it is easy to use for even complicated high order Sturm-Liouville problems. The proposed procedure leads to a system of equation and there are many Mathematical Packages for solving such systems. On the other hand, the obtained results in tables and the illustrated plots in figures showed the high accuracy of DTM which was applied on high order Sturm-Liouville problem.

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