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Applying new wavelet transform method on the generalized-FKPP equation

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Abstract The numerous methods for solving differential equations exist, every method have benefits and drawbacks, in this field, the combined methods are very useful, one of them is the wavelet transform method (WTM). This method based on the wavelets and corresponding wavelet transform, that dependent on the differential invariants obtained by the Lie symmetry method. In this paper, we apply the WTM on the generalized version of FKPP equation (GFKPP) with non-constant coefficient

$$fu_{tt}(x,t) + u_t(x,t) = u_{xx}(x,t) + u(x,t) - u^2(x,t),$$

where f is a smooth function of either x or t. We will see for suitable wavelets, this method proposes the interesting solutions.

Keywords. Wavelet, Quasi-wavelet, Mother wavelet, The wavelet transform, Differential invariants, The GFKPP equation.

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1. Introduction

The theory of Lie symmetry groups of differential equations was developed by Sophus Lie [3]. Such Lie groups are invertible point transforms of both the dependent and independent variables of the differential equations. The symmetry group methods provide an ultimate arsenal for analysis of differential equations and is of great importance to understand and to construct solutions of differential equations. Several applications of Lie groups in the theory of differential equations were discussed in the literature, the most important ones are: reduction of order of ordinary differential equations, construction of invariant solutions, mapping solutions to other solutions and the detection of linearizing transforms (for many other applications of Lie symmetries see [7, 8]).

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In mathematics, Fisher's equation, also known as the Fisher–Kolmogorov equation and the Fisher–KPP equation, named after R. A. Fisher and A. N. Kolmogorov, is the partial differential equation $u_t = u_{xx} + u - u^2$. Fisher proposed this equation to describe the spatial spread of an advantageous allele and explored its travelling wave solutions and nowadays, this equation proposed as a model of diffusion in biomathematics [15].

The experimental observation of an initially flat liquid-film interface that evolves with time to a propagating diffusion front with a constant front velocity are specific characteristics of the Fisher- Kolmogorov- Petrovskii- Piskounov (FKPP) equation obeying a traveling wave solution [2].

The FKPP equation occurs, e.g., in ecology, physiology, combustion, crystallization, plasma physics, and in general phase transition problems, this equation is a well known and widely applied nonlinear reaction-diffusion equations and is traditionally applied to model the spread of genes in the population genetics [16].

The generalized version of FKPP equation with function coefficient is

$$fu_{tt} + u_t = u_{xx} + u - u^2, \tag{1.1}$$

where f is a smooth function of either x or t. So far, this version of equation was solved with numerical methods and any explicit solution was not found. By the Lie symmetry method, the generalized FKPP equation will be converted to ODEs & all symmetries and generalized vector fields will be determined.

The wavelets are important functions in the functional and harmonic analysis. First wavelet was introduced by Alfred Haar (the Hungarian mathematician) in 1909 [6]. Nowadays, the wavelets have numerous applications in the some fields of science and technology: seismology, image processing, signal processing, coding theory, biosciences, financial mathematics, fractals and so on [1]. The application of wavelets for solving differential equations limited to ODEs or PDEs with the numerical solutions in the special conditions [5]. The famous wavelets such as Haar, Daubechie, Coiflet, Symlet, CDF, Mexican hat and Gaussian are extendible to two or more variables by tensor product. The wavelets with two or more variables (that in connection with PDEs are useful) is very important [14]. In this paper, we apply the Wavelet Transform Method (WTM) on the generalized version of FKPP (GFKPP) and obtain solutions.

The remainder of the paper is organized as follows. In section 2, we recall some needed results to construct differential invariants, the mother wavelets and the wavelet transforms. In section 3, the wavelet transform method is proposed. In sections 4, the proposed method will be applied on the GFKPP equation. Finally, the conclusions & future works are presented.

2. Preliminaries

2.1. **The Lie symmetry method.** In this section, we recall the general procedure for determining symmetries for any system of partial differential equations (see [3, 7, 11]). To begin, let us consider the general case of a nonlinear system of partial differential equations of order nth in p independent and q dependent variables is given as a system of equations:

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \cdots, l, \tag{2.1}$$

involving $x = (x^1, \cdots, x^p)$, $u = (u^1, \cdots, u^q)$ and the derivatives of u with respect to x up to n, where $u^{(n)}$ represents all the derivatives of u of all orders from 0 to n. We consider a one-parameter Lie group of infinitesimal transforms acting on the independent and dependent variables of the system (2.1):

$$(\tilde{x}^i, \tilde{u}^j) = (x^i, u^i) + s(\xi^i, \eta^j) + O(s^2), \quad i = 1 \cdots, p,\ j = 1 \cdots, q, \tag{2.2}$$

where s is the parameter of the transform and ξ^i, η^j are the infinitesimals of the transforms for the independent and dependent variables, respectively. The infinitesimal generator $\mathbf{v}$ associated with the above group of transforms can be written as $\mathbf{v} = \sum_{i=1}^{p} \xi^i \partial_{x^i} + \sum_{j=1}^{q} \eta^j \partial_{u^j}$, A symmetry of a differential equation is a transform which maps solutions of the equation to other solutions. The invariance of the system (2.1) under the infinitesimal transforms leads to the invariance conditions (Theorem 2.36 of [7]):

$$\mathrm{Pr}^{(n)}\mathbf{v}\big[\Delta_\nu(x, u^{(n)})\big] = 0, \quad \Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \cdots, l, \tag{2.3}$$

where $\mathrm{Pr}^{(n)}$ is called the n^{th} order prolongation of the infinitesimal generator given by $\mathrm{Pr}^{(n)}\mathbf{v} = \mathbf{v} + \sum_{\alpha=1}^{q} \sum_J \phi_J^\alpha(x, u^{(n)})\partial_{u_J^\alpha}$, where $J = (j_1, \cdots, j_k)$, $1 \leq j_k \leq p$, $1 \leq k \leq n$ and the sum is over all J's of order $0 < \#J \leq n$. If $\#J = k$, the coefficient ϕ_J^α of $\partial_{u_J^\alpha}$ will only depend on k-th and lower order derivatives of u, and $\phi_\alpha^J(x, u^{(n)}) = D_J(\phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha) + \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha$, where $u_i^\alpha := \partial u^\alpha / \partial x^i$ and $u_{J,i}^\alpha := \partial u_J^\alpha / \partial x^i$.

One of the most important properties of these infinitesimal symmetries is that they form a Lie algebra under the usual Lie bracket. The first advantage of symmetry group methods is to construct new solutions from known solutions. The second is when a nonlinear system of differential equations admits infinite symmetries, so it is possible to transform it to a linear system. Neither the first advantage nor the second will be investigated here, but symmetry group method will be applied to the PDE to be connected directly to some order differential equations. To do this, a particular linear combinations of infinitesimals are considered and their corresponding invariants are determined.

For every vector field, by establishing the characteristics system as follows

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\phi}, \tag{2.4}$$

and solving this system, we can obtain differential invariants corresponding to vector fields. By expressing the PDE in the coordinates (x, t, u), this equation was reduced, and the final solution was obtained. These coordinates will be constructed by searching for independent invariants (y, v) corresponding to the infinitesimal generator. Thus by using the chain rule, the expression of equation in the new coordinate, allows us to the reduced equation. For more informations and examples, see [7].

2.2. **The wavelets.** The wavelets are important functions in the mathematics and other scientific fields. In this section, we introduce wavelets as functions belong to $L^2(\mathbb{R}^2)$ (The space of squared integrable functions equipped with integral norm).

Remark 2.1. The function ψ belong to $L^2(\mathbb{R}^2)$ is a wavelet, if it satisfies in the following admissible condition

$$C_\psi = \int_{\mathbb{R}^2} \frac{|F(\psi)(\omega)|^2 d\omega}{|\omega|^2} > 0, \tag{2.5}$$

where $F(\psi)(\omega)$ is the Fourier transform of wavelet ψ and defined as follows

$$F(\psi)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^2} \exp(-ix.\omega)\psi(x)d\omega, \tag{2.6}$$

C_ψ is called the wavelet coefficient of ψ. Here, $\omega = (\omega_1, \omega_2)$ and $x = (x_1, x_2)$ belong to $\mathbb{R}^2$. For further informations and examples, see [2].

Remark 2.2. The wavelet ψ is called mother wavelet, if it satisfies in the following properties

$$\int_{\mathbb{R}^2} \psi(x)dx = 0, \tag{2.7}$$

$$\int_{\mathbb{R}^2} |\psi(x)|^2 dx < \infty, \tag{2.8}$$

$$\lim_{|\omega|\to\infty} F(\psi(\omega)) = 0, \tag{2.9}$$

Note that, the first property equivalents to $C_\psi > 0$ (the admissible condition) for mother wavelet ψ.

For more details see [6, 9].

Indeed, the mother wavelets have the admissible condition, n-zero moments and exponential decay properties. The mother wavelet have two parameters: the translation parameter $b = (b_1, b_2)$ and scaling parameter $a > 0$. The mother wavelet corresponding to (a, b) is

$$\psi_{a,b}(x) = \psi(\frac{x-b}{a}) = \psi(\frac{x_1 - b_1}{a}, \frac{x_2 - b_2}{a}) \tag{2.10}$$

If the function ψ don't satisfy in the some properties of the mother wavelets, or approximately satisfies, ψ is called quasi-wavelet. The quasi-wavelets have numerous applications in the applied mathematics and other scientific fields for solving PDEs, for more details and examples see [9, 13]. In this paper, we provide the quasi-wavelets based on the differential invariants of PDEs, and by using them, we will analyze PDEs.

Remark 2.3. The wavelet transform corresponding to the mother wavelet ψ for the function $f \in L^2(\mathbb{R}^2)$ with parameters (a, b) (that $a > 0$) defined as follows

$$W_\psi(f)(a,b) = \frac{1}{\sqrt{a.C_\psi}} \int_{\mathbb{R}^2} \psi_{a,b}(x).f(x)dx, \tag{2.11}$$

Thus, the wavelet transform depends on the wavelet ψ, the function f, and the parameters (a, b).

Theorem 2.4. *The wavelet transform is an operator from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^3)$ that satisfies in the following properties:*

1 . *Linearity:* $W_\psi[\alpha f(x) + \beta g(x)] = \alpha W_\psi[f(x)] + \beta W_\psi[g(x)]$,
2 . *Translation:* $W_\psi[f(x-k)] = W(a, b-k), \quad \forall k \in \mathbb{R}^2$,
3 . *Scaling:* $W_\psi[\frac{1}{\sqrt{s}} f(\frac{x}{s})] = W(\frac{a}{s}, \frac{b}{s})$,
4 . *Wavelet shifting:* $W_{\psi(x-k)}[f(x)] = W(a, b + ak)$,
5 . *Linear combination:* $W_{\alpha\psi_1+\beta\psi_2}[f(x)] = \alpha W_{\psi_1}[f(x)] + \beta W_{\psi_2}[f(x)]$,
6 . *Wavelet scaling:* $W_{\frac{\psi(x/s)}{\sqrt{(|s|)}}}[f(x)] = W(as, b)$.

Proof. For proof and more details see [6]. □

Actually, the wavelet transforms corresponding mother wavelets are isometries [10]. Therefore, In the smooth manifold M, the collection of wavelet tranforms of M denoted by W(M), is a Lie subgroup of I(M) (the isometry group of M) [2].
The admissible condition implies that the wavelet transform is invertible, on the other hand, because of the wavelet transform is isometry, it is invertible, the inversion formula for the wavelet transform $W_\psi(f)$ is

$$f(x) = f(x_1, x_2) = \frac{1}{C_\psi} \int_{\mathbb{R}^+ \times \mathbb{R}^n} W_\psi f(a, b)\, \psi_{a,b}(x) \frac{da\, db_1\, db_2}{a^3} \tag{2.12}$$

In fact, by the inversion formula (also called the synthesis formula), the function $f(x)$ corresponds to the wavelet transform $W_\psi(f)$ will be obtained [12].

3. The wavelet transform method

The wavelet tranform method (WTM) have 4 following steps:

1 . Apply equivalence algorithms (for example, the Lie symmetry method) on DE, and obtain differential invariants.
2 . Build the suitable quasi-wavelet based on the differential invariants.
3 . Multiply the quasi-wavelet in the both sides of the equation and take the wavelet transform. Solve the reduced DE, and obtain the wavelet transform.
4 . According to obtained wavelet transform and By the inversion formula, calculate the analytic solution.

In the following, some WTM formula are proposed.

Theorem 3.1. *Assume $\Delta_\nu(x, t, u^{(m)}) = 0$ is m-th order DE with two independent variables (x, t), ψ is a mother wavelet based on differential invariants, t is constant and x is variable. Then, we have:*

1) $W_\psi(\partial_t u)(x, t) = \frac{d}{dt} W_\psi(u)(x, t)$,
2) $W_\psi(\partial_t^n u)(x, t) = \frac{d^n}{dt^n} W_\psi(u)(x, t)$,
3) $W_\psi(\partial_x u)(x, t) = -W_{\partial_x \psi}(u)(x, t)$,
4) $W_\psi(\partial_x^n u)(x, t) = (-1)^n W_{\partial_x^n \psi}(u)(x, t)$,

TABLE 1. The Lie Symmetry Method: exact symmetries, differential invariants and invariant solutions

f	V.F.	dim(g)	Invariants	Invariantsolutions
0	$\partial_t,\ \partial_x,$ $x\partial_x + 2t\partial_t + 2u\partial_u$	3	$(\frac{x^2}{t}, \frac{u}{t}),$ $(x,u),(t,u)$	$\int \frac{\sqrt{3}\,du}{\sqrt{2u^3-3u^2+3C_1}} = \lvert t+C_2\rvert,$ $u=(1+F(x)\exp(-t))^{-1},$
c	∂_x, ∂_t	2	$(t,u),(x,u)$	$cu''(t)+u'(t)+u(t)^2=u(t),$
$f(x)$	$\partial_t, F(x,t)\partial_u$	∞	(x,u)	$\int \frac{\sqrt{3}\,du}{\sqrt{2u^3-3u^2+3C_1}} = \lvert t+C_2\rvert,$
$f(t)$	$\partial_x, x\partial_u, \partial_u,$ $tx+\frac{x^3}{6}\partial_u, t+\frac{x^2}{2}\partial_u$	5	$(t,u),(x,t)$	$f(t)u''(t)+u'(t)=u(t)-u(t)^2,$

Proof. For proof and more details see [14]. □

In fact, we take the wavelet transform from both side of $\Delta_\nu(x,t,u^{(m)}) = 0$ by assuming that $t = cte, x = variable, a = 1, b = 0$, solve the reduced equation according to $\tilde{u}(x,t)$ and its t-derivations, and obtain $\tilde{u}(x,t)$, here after, for the given mother wavelet $\psi(x,t)$ and obtained wavelet transform $\tilde{u}(x,t)$, calculate $u(x,t)$ from the following formula ($1D$-inversion formula)

$$u(x,t) = \int_{\mathbb{R}} \tilde{u}(x,t)\psi(x,t), dx \tag{3.1}$$

where, $u(x,t)$ is a desired analytic solution, in this way, the PDE is solved by WTM based on ψ (according to the differential invariants). In the following section, we apply WTM on the GFKPP equation.

4. Apply WTM on the GFKPP

In this section, we implement WTM on the GFKPP equation and obtain solutions, finally, the WTM results will be proposed. First, apply the Lie symmetry method on the GFKPP equation $fu_{tt} + u_t = u_{xx} + u - u^2$, and obtain the symmetry groups, vector fields and differential invariants, for more detailed calculations & results of the Lie symmetry method implementation on the GFKPP equation, see [15]. The Lie symmetry method results for the GFKPP equation proposed in Table 1 (for more details and computation, see [15, 16]):

In table 1, the symmetry groups are translation and scaling. For every differential invariant and symmetry group, we offer adequate quasi-wavelets as below:

Here, first we apply WTM by quasi-wavelets ψ_1, ψ_2 on the $cu_{tt} + u_t = u_{xx}$ with $f = c$. At this way, consider quasi-wavelet ψ_1 as follows

$$\psi_1 := \exp(\frac{-t^2}{2})\sin(\frac{\pi(x-2t)}{2}) \tag{4.1}$$

and, the quasi-wavelet ψ_2 as follows

$$\psi_2 := \exp(-t^2/2)\cos(\pi(x-ct)/2) \tag{4.2}$$

TABLE 2. The Quasi-wavelets: Symmetry groups, differential invariants and quasi-wavelets

Symmetry groups	Differential invariants	Quasi-wavelets
Translation	$x - ct, u$	$\exp(-t^2/2)\sin(\pi(x-ct)/2)$, $\exp(-t^2/2)\cos(\pi(x-ct)/2)$,
Scaling	$(x/t), (x/\sqrt{t}), (u/t^a)$	$\exp(-t^2/2)\sin(x/t)$, $\exp(-t^2/2)\cos(x/t)$,

then, by multiplying the both sides of FKPP in ψ_1 and taking the wavelet transform, we get

$$c\frac{d^2}{dt^2}\tilde{u} + \frac{d}{dt}\tilde{u} = -\frac{\pi^2}{4}\tilde{u} \tag{4.3}$$

by solving this equation with characteristics method (for more details about the solving methods of PDEs, please see [4]), we see that the solution depends on the coefficient c, indeed,

1 . If $c = \frac{1}{\pi^2}$, then we have double root, and the wavelet transform as follows

$$\tilde{u}(x,t) = \{\tilde{F}(x) + \tilde{G}(x)t)\}\exp(-5t), \tag{4.4}$$

where $\tilde{F}(x)$ and $\tilde{G}(x)$ (respectively) are the wavelet transform related to the functions F, G of x. Thus, the analytic solution from (3.1) is

$$u(x,t) = \{F(x) + G(x)t)\}\exp(-5t), \tag{4.5}$$

2 . If $c > \frac{1}{\pi^2}$, we have the complex roots $D = \alpha \pm i\beta$ and the wavelet transform is

$$\tilde{u}(x,t) = \exp(\alpha t)\{\tilde{F}(x)\cos(\beta t) + \tilde{G}(x)\sin(\beta t))\}, \tag{4.6}$$

and the analytic solution from (3.1) obtained as below

$$u(x,t) = \exp(\alpha t)\{F(x)\cos(\beta t) + G(x)\sin(\beta t))\}, \tag{4.7}$$

3 . If $c > \frac{1}{\pi^2}$, we have two distinct real roots $D_{1,2}$, therefore

$$\tilde{u}(x,t) = \tilde{F}(x)\exp(D_1 t) + \tilde{G}(x)\exp(D_2 t), \tag{4.8}$$

So, the analytic solution from (3.1) is

$$u(x,t) = F(x)\exp(D_1 t) + G(x)\exp(D_2 t), \tag{4.9}$$

For instance, with $c = 1$, the complex roots $D = -0.5 \pm 0.75i$ are obtained, thus

$$\tilde{u}(x,t) = \exp(-0.5t)\{\tilde{F}(x)\cos(0.75t) + \tilde{G}(x)\sin(0.75t))\}, \tag{4.10}$$

and the analytic solution as follows

$$u(x,t) = \exp(-0.5t)\{F(x)\cos(0.75t) + G(x)\sin(0.75t))\}, \tag{4.11}$$

TABLE 3. WTM on the FKPP equation: The parameter c, The wavelet transform, The analytic solution

c	The wavelet tranform	The analytic solution
$\frac{1}{\pi^2}$	$\{\tilde{F}(x)+\tilde{G}(x)t)\}\exp(-5t),$	$\{F(x)+G(x)t)\}\exp(-5t),$
$>\frac{1}{\pi^2}$	$\tilde{F}(x)\exp(D_1t)+\tilde{G}(x)\exp(D_2t),$	$F(x)\exp(D_1t)+G(x)\exp(D_2t),$
$<\frac{1}{\pi^2}$	$\exp(\alpha t)\{\tilde{F}(x)\cos(\beta t)+\tilde{G}(x)\sin(\beta t))\},$	$\exp(\alpha t)\{F(x)\cos(\beta t)+G(x)\sin(\beta t))\},$

The computations and results for ψ_2 are similar. Note that, for the generalized version of FKPP equation, with known $f(x)$ or $f(t)$, by taking the wavelet transform from function f, the final results will be obtained.

The following table, shows the results of implementing the wavelet transform method on the FKPP equation:

5. Conclusions and future works

In this paper, we applied the novel method based on the wavelets; wavelet transforms method (WTM) based on the quasi-wavelets on the GFKPP equation. We proposed the suitable quasi-wavelets based on the Lie symmetry method results; symmetry groups, differential invariants and invariant solutions. In fact, we used result obtained by the equivalence methods like the Lie symmetry method for constructing the quasi-wavelets and applying WTM with them on the GFKPP equation. Finally, we proposed results; the analytic solutions. This research shows the power and performance of WTM for analyzing and solving different PDEs, in the future works, we will try to apply WTM on the PDEs at every order and degree and generalize this method for solving PDEs at any condition. We will hope that can propose a universal method for analyzing & solving differential equations.

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