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Numerical analysis of fractional differential equation by TSI-wavelet method

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Abstract In this paper, we propose a new numerical algorithm for the approximate solution of non-homogeneous fractional differential equation. Using this algorithm the fractional differential equations are transformed into a system of algebraic linear equations by operational matrices of block-pulse and hybrid functions. Based on our new algorithm, this system of algebraic linear equations can be solved by a proposed (TSI) method. Further, some numerical examples are given to illustrate and establish the accuracy and reliability of the proposed algorithm.

Keywords. Fractional differential equation, Block-pulse wavelet, Hybrid function, Operational matrices, Two stage iterative method.

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1. Introduction

Fractional differential equations (FDE)s are generalized from integer order ones, which are achieved by replacing the integer order derivatives by fractional ones. Recently, fractional derivatives have been used to new applications in differential equation [5, 14, 15, 16, 17]; also the interested readers may refer to other sources [3, 4, 6, 14, 23] for the underlying theory and applications of fractional calculus. In this paper we consider a FDE in the following form,

$$\sum_{i=0}^{n} a_i({}_aD_t^{\alpha_i} y(t)) = u(t), \tag{1.1}$$

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∗ corresponding.

subject to the initial conditions

$$y^{(i)}(a) = d_i, i = 0, ..., n, \tag{1.2}$$

where $a_i \in R, n < \alpha \leq n+1, 0 < \alpha_1 < \alpha_2 < ... < \alpha_n < \alpha$ and ${}_aD_t^{\alpha_n} y(t)$ denotes the Caputo fractional derivative of order α. We can see the conditions of existence and uniqueness of solutions to the FDE in [14]. However, many of the existing numerical methods transform the FDE into an algebraic equation and then solve it [1, 2, 11, 18].

In this study, first, we use the orthogonal functions such as block-pulse wavelet to transform a non-homogeneous fractional differential equation into an algebraic linear equation and then we investigate the two-stage iterative method (TSI) for solving this linear systems which is an iterative method for solving a linear algebraic equation based on two inner and outer splitting of a matrix. According to this process, which denoted by TSI-Wavelet, we can create an extremely effective and practical algorithm. This paper is organized as follows. In Section 2, we present a number of definitions about fractional calculus, block-pulse wavelets, hybrid functions and their properties. After reviewing the two-stage iterative (TSI) method and introducing some related essential concepts and results in Section 3, we further set up our new numerical algorithm for the approximate solution of non-homogeneous fractional differential equation in this section. In Section 4, we examine the advantages of our results by carrying out numerical computations. The conclusions are presented in Section 5.

2. Elementary Definitions and Operational Matrices

Here and in this part of the study, we present some basic definitions and properties of fractional calculus, wavelets and operational matrices [8, 14, 19, 24].

Definition 2.1. The Riemann-Liouville fractional integral of order α is

$$I^\alpha(f(x)) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau) d\tau, \quad alpha > 0. \tag{2.1}$$

A real function $f(x), \quad x \geq 0$ is said to be in space $C_\mu, \mu \in R$ if there exists a real number $p > \mu$, such that $f(x) = x^p f_1(x)$ where $f_1(x) \in [0, \infty)$, and it is said to be in the space C_μ^m if $f^m \in C_\mu, m \in N$.

Definition 2.2. The fractional derivative of $f(x)$ in the Caputo sense is defined as

$${}_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \tag{2.2}$$

where $n-1 < \alpha \leq n, n \in N, x > 0, f \in C_{-1}^n$.

For the Caputo's derivative, we have $D_t^\alpha C = 0$, which C is a constant and

$$D_x^\alpha x^n = \begin{cases} 0 & n \in N, n < \lceil \alpha \rceil, \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha} & n \in N, n > \lfloor \alpha \rfloor. \end{cases} \tag{2.3}$$

The relation between the Riemann-Liouville operator and the Caputo operator is given by the following expressions [26]:

$${}_aD_x^\alpha I^\alpha f(x) = f(x), \tag{2.4}$$

$$I^{\alpha}{}_aD_x^{\alpha}f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+)\frac{(x-a)^k}{k!}, \quad x > 0. \tag{2.5}$$

Definition 2.3. The m-set of Block-Pulse functions on $[0, \eta)$ is defined as:

$$b_i(t) = \begin{cases} 1 & \frac{\eta i}{m} \leq t < \frac{\eta(i+1)}{m}, \\ 0 & otherwise. \end{cases} \tag{2.6}$$

where $i = 0, 1, 2, \cdots, m-1$.

The functions b_i are disjoint and orthogonal [8], that is

$$\int_0^{\eta} b_i(t)b_j(t)dt = \delta_{i,j},$$

where $\delta_{i,j} = 0$ for $i \neq j$ and $\delta_{i,j}$ is the constant for $i = j$.

Definition 2.4. The shifted Legendre polynomials are defined on the interval $[0, 1]$ and can be determined with the aid of the following recurrence formulae [18]:

$$P_{i+1}(x) = \frac{(2i+1)(2x-1)}{i+1}P_i(x) - \frac{i}{i+1}P_{i-1}(x), \; i = 1, 2, ...,$$

where $P_0(x) = 1$ and $P_1(x) = 2x - 1$. The analytic form of the shifted Legendre polynomial $P_i(x)$ of degree i given by

$$P_i(x) = \sum_{k=0}^{i} (-1)^{i+k}\frac{(i+k)!x^k}{(i-k)(k!)^2},$$

where $P_i(0) = (-1)^i$ and $P_i(1) = 1$.

Theorem 2.5. *A function $f(x) \in L^2([0, T))$ may be expanded by the block-puls functions as*

$$f(x) \simeq \sum_{i=1}^{m_1} f_i b_i(t) = F^T B_m(x), \tag{2.7}$$

where

$$F = \begin{pmatrix} f_1 & \cdots & f_m \end{pmatrix}, \; B_m(x) = \begin{pmatrix} b_1(x) & \cdots & b_m(x) \end{pmatrix}.$$

The block-pulse coefficients f_i are obtained as

$$f_i = \frac{T}{h}\int_{(i-1)h}^{ih} f(x)dx. \tag{2.8}$$

Proof. See [8]. □

Definition 2.6. Hybrid functions $hy_{i,j}(x)$, $i = 0, ...m-1$ *and* $j = 0, ..., n-1$ are defined on the interval $[0, T)$ as

$$hy_{i,j}(x) = \begin{cases} P_j(\frac{m}{T}x - i) & \frac{iT}{m} \leq x < \frac{(i+1)T}{m}, \\ 0 & otherwise, \end{cases} \tag{2.9}$$

where $P_j(t)$ is the j^{th} shifted Legendre polynomials on $[0, 1)$.

Now for approximating the function $f(x)$, we can set [13, 25]

$$f(x) \simeq C^T Hy_{n,m}(x), \tag{2.10}$$

where

$$C^T = \begin{pmatrix} c_{0,0} & \cdots & c_{0,n-1} & c_{(m-1),(n-1)} \end{pmatrix},$$

$$Hy_{n,m}(x) = \begin{pmatrix} hy_{0,0}(x) & \cdots & hy_{0,n-1}(x) & \cdots & hy_{(m-1),(n-1)}(x) \end{pmatrix}, \tag{2.11}$$

and

$$c_{i,j} = \frac{< f(x), hy_{i,j} >}{< hy_{i,j}, hy_{i,j} >},$$

where

$$< u(x), v(x) >= \int_0^T u(x)v(x)dx.$$

Now, we introduce the operational matrix methods based on Block-Pulse and hybrid functions of Block-Pulse and shifted Legendre polynomials.

Fractional integration of the Block-Pulse function vector is given as

$$(I^\alpha B_m)(t) = F^{(\alpha)} B_m(t), \tag{2.12}$$

where $F^{(\alpha)}$ is the block-pulse operational matrix of the fractional order integration [26]

$$F^{(\alpha)} = (\tfrac{T}{m})^\alpha \tfrac{1}{\Gamma(\alpha+2)} \begin{pmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{m-2} \\ 0 & 0 & 1 & \cdots & \xi_{m-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \tag{2.13}$$

where $\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$.

Now let $Hy_{n,m} \simeq \Phi B_{mn}(x)$ and $I^\alpha Hy_{n,m}(x) = Q^{(\alpha)} Hy_{n,m}(x)$, then we can construct the operational matrix of fractional order integration for hybrid functions as:

$$Q^{(\alpha)} = \Phi F^{(\alpha)} \Phi^{-1}. \tag{2.14}$$

According to the results of the operational matrices in this section we can transport (1.1) under the conditions (1.2) to an linear algebraic equations which drawn in Section 5. Since, buy increasing m and n the size of such linear algebraic equations to be large, we need a good numerical method for solving an algebraic liner equation with large size. So in the next section we will propose two stage iterative method for solving large sparse linear algebraic equations.

3. TSI method

In this section, we are going to present a new numerical algorithm for solving an algebraic linear equations.
The spectral radius $\rho(A)$ of a matrix A with eigenvalues λ_1, λ_2, ... ,λ_n is defined to be the number

$$\rho(A) = \max\{|\lambda_1|, |\lambda_2|, ..., |\lambda_n|\}.$$

Thus $\rho(A)$ is the largest absolute value of the eigenvalues of matrix A.
There are many numerical method for solving linear equations [20, 21, 22]. Consider the linear system

$$Ax = b. \tag{3.1}$$

For any splitting $A = M - N$ with $det(M) \neq 0$, the basic iterative methods for solving (3.1) is

$$x_i = M^{-1}Nx_{i-1} + M^{-1}b, \ i = 1, 2, ... \tag{3.2}$$

with outer splitting $A = M - N$ and inner splitting $M = F - G$. Then the algorithm of TSI method for solving (3.1) is as follows.

Algorithm 1.

Step 1. Choose an initial vector x_0, a prescribed tolerance, number of outer iteration m and a sequence of number of inner iterations, $s(k), k = 1, ..., m$.
Step 2. For $i = 1, ..., m$ do
$y_0 = x_{i-1}$,
For $j = 1, ..., s(k)$ do
$Fy_j = Gy_{j-1} + Ny_{i-1} + b$
$x_i = y_{s(k)}$.
Step 3. If $||b - Ax_i|| \leq tol$, then stop.

When the number of inner iterations is fixed in each outer step, i.e., $s(k) = s$, $s \geq 1$, it is said that the method is stationary, while a non-stationary two-stage method is such that the number of inner iterations may change with the outer iterations. Throughout the paper, it is assumed that $s(k) = s$, $s \geq 1$. By replacing the loop over j and by (3.2), TSI methods for solving the system of linear equations (3.1) have the following form

$$x_i = (F^{-1}G)^s x_{i-1} + \sum_{j=0}^{s-1}(F^{-1}G)^j F^{-1}(Nx_{i-1} + b), i = 1, 2, ... \ . \tag{3.3}$$

Clearly, the iteration matrix corresponding to the relation (3.3) is

$$T_s = (F^{-1}G)^s + \sum_{j=0}^{s-1}(F^{-1}G)^j F^{-1}1N = I - (I - (F^{-1}G)^s)(I - M^{-1}N), \tag{3.4}$$

where I denotes the $n \times n$ identity matrix. If $\rho(F^{-1}G) < 1$, then $I - (F^{-1}G)^s$ is nonsingular. Then there exists a unique pair of matrices [9], B_s and C_s, such that $M = B_s - C_s$ and $R = (F^{-1}G)^s = B_s^{-1}C_s$ where

$$B_s = M(I - R)^{-1}, \tag{3.5}$$

$$C_s = M(I - R)^{-1}R, \tag{3.6}$$

so

$$T_s = B_s^{-1}(C_s + N). \tag{3.7}$$

Now we present a new theorem under suitable conditions. Based on this theorem, we can find which double splitting is more efficient compared to the other ones. We should begin with some basic notations and preliminary results first.

Definition 3.1. Let A be a real matrix. The splitting $A = M - N$ is called
(a) convergent if $\rho(M^{-1}N) < 1$,
(b) regular if $M^{-1} \geq 0$ and $N \geq 0$, and
(c) weak regular if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$.

Lemma 3.2. *Let $A = M - N$ be a convergent regular splitting, and let $R \geq 0$, $\rho(R) < 1$. If the unique splitting be as $M = B_s - C_s$ such that $R = B_s^{-1}C_s$ be a weak regular splitting, then the two-stage iterative method for any nonnegative s of inner iterations will be convergent.*

Proof. See [9]. □

Lemma 3.3. *Let $A = M - N$ be regular or a weak regular splitting of A. Then $\rho(M^{-1}N) < 1$ if and only if $A^{-1} \geq 0$.*

Proof. See [7]. □

Lemma 3.4. *Let $A = M - N$ Let $A = M_1 - N_1 = M_2 - N_2$ be two weak regular splitting of A, where $A^{-1} \geq 0$. If $M_1^{-1} \geq M_2^{-1}$, then $\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2)$.*

Proof. See [12]. □

Now, we establish new results in the following theorem.

Theorem 3.5. *Let $A^{-1} \geq 0$, $A = M_1 - N_1 = M_2 - N_2$ be regular splitting and let $M_1 = F_1 - G_1$, $M_2 = F_2 - G_2$ be weak regular splitting. If $M_2^{-1} \geq \alpha M_1^{-1}$ then $\rho(T_s(M_2 - N_2)) \leq \rho(T_s(M_1 - N_1)) < 1$ where $\alpha = \frac{1-\rho_1}{1-\rho_2}$ with $\rho_i = \rho(F_i^{-1}G_i)$ for $i = 1, 2$.*

Proof. By Lemma 3.2, it is easy to show that $\rho(T_s(M_1 - N_1)) < 1$. But by two-stage iterative method (Algorithm1), for $i = 1, 2$ we have $A = M_i - N_i$, $M_i = F_i - G_i = B_{i,s} - C_{i,s}$ and $R_i = (F_i^{-1}G_i)^s = B_{i,s}^{-1}C_{i,s}$, so we get $A = M_{i,T} - N_{i,T}$, where $M_{i,T} = B_{i,s} = M_i(I - R_i)^{-1}$ and $N_{i,T} = (C_{i,s} + N_i) = M_i(I - R_i)^{-1}R_i + N_i$.

Since $A = M_i - N_i$ are regular splitting, $M_i = F_i - G_i$ is the weak regular splitting. By Lemma 3.3 we have $\rho_1, \rho_2 < 1$. Furthermore, it can be shown that $A = M_{i,T} - N_{i,T}$

is the weak regular splitting. Therefore, to apply Lemma 3.4, it is only necessary to show that $M_{2,T}^{-1} \geq M_{1,T}^{-1}$. Since $M_2^{-1} \geq \alpha M_1^{-1}$, we have

$$(1 - \rho(F_2^{-1}G_2))M_1 \geq M_2(1 - \rho(F_1^{-1}G_1)).$$

Since $GF^{-1} = F(F^{-1}G)F^{-1}$, we have $\rho(GF^{-1}) = \rho(F^{-1}G)$, so

$$(I - (G_2F_2^{-1}))M_1 \geq M_2(I - (F_1^{-1}G_1))$$

$$\Rightarrow M_2^{-1} - M_2^{-1}(G_2F_2^{-1})^s \geq (I - R_1)M_1^{-1}.$$

Since $M(F^{-1}G)^s = (GF^{-1})^s M$, we have $(I - (F_2^{-1}G_2)^s)M_2^{-1} \geq (I - R_1)M_1^{-1}$, and the proof is completed. □

According to the operational matrices described in Section 2, by integrating from (1.1) of order α_n, we can change (1.1) to a linear algebraic equations and then by using Algorithm 1 solve it. So we have the following algorithm.

Algorithm 2(TSI-wavelets).

Step 1. Compute $Ax = b$ related to the relation (1.1), choose initial vectors x_0, tol, number of outer iteration m and sequence of number of inner iterations, $s(k), k = 1, ..., m$.
Step 2. For $i = 1, ..., m$ do
$y_0 = x_{i-1}$,
For $j = 1, ..., s(k)$ do
$Fy_j = Gy_{j-1} + Ny_{i-1} + b$
$x_i = y_{s(k)}$.
Step 3. If $||b - Ax_i|| \leq tol$, then stop.

In order to show the efficiency of our new algorithm for solving initial value problem (1.1), we apply it to solve different types of FDEs whose exact solutions are known.

4. Illustrative Examples

In this section, we use $E_{Ma}W$ for the absolute error generated by the Matlab command $linsolve(A, b)$ and $E_{Ts}W$ for the absolute error generated by the Algorithm 2, which W is the block-pulse wavelet B_m or the hybrid functions $Hy_{m,n}$.

Example 4.1. Consider the Bagley-Torvik equation with initial value as

$${}_0D_x^2 y(x) +_0 D_x^{\frac{3}{2}} y(x) + y(x) = 1 + x \; ; \; y(0) = 1, \; y'(0) = 1. \tag{4.1}$$

The exact solution is $y(x) = 1 + x$, [18]. The integral representation of (4.1) is given by

$$y(x) - x - 1 + I^{\frac{1}{2}}(y(x)) + I^{\frac{1}{2}}(-x - 1) + I^2(y(x)) = I^2(1 + x). \tag{4.2}$$

In this example, we use the operational matrix of fractional order integration with respect to the Block-Pulse wavelet. By applying Theorem 2.5 , we can approximate solution y(x) and $1+x$ as follows

$$y(x) = C_b^T B_m(x),\ 1+x = C_1^T B_m(x), \tag{4.3}$$

by substituting (4.3) into (4.2) and using operational matrices we have

$$C_b^T(I_m + F^{(\frac{1}{2})} + F^{(2)}) = C_1^T(I_m + F^{(\frac{1}{2})} + F^{(2)}). \tag{4.4}$$

From (2.13), we can see that the entries of principal diagonal of upper triangular matrix $I_m + F^{(\frac{1}{2})} + F^{(2)}$ are positive and, thus the matrix $I_m + F^{(\frac{1}{2})} + F^{(2)}$ is nonsingular. This shows that (4.4) has a unique solution as $C_b = C_1$. Since $y(x) = C_b^T B_m(x) = C_1^T B_m(x) \simeq 1+x$, Theorem 2.5 shows that $C_1^T B_m(x) \to 1+x$ as $m \to \infty$. Therefore, the numerical solution can be regarded as $1+x$, which is the exact solution here.

TABLE 1. Absolute error for $\alpha = 1.5$, $a = 12$ for example 4.2, case 1.

x	0.2	0.5	0.8	1.1	1.4
$E_{Ma}B_{32}$	9.1×10^{-3}	8.1×10^{-4}	8.6×10^{-4}	6.0×10^{-4}	5.2×10^{-4}
$E_{Ma}Hy_{8,3}$	1.4×10^{-3}	7.6×10^{-4}	1.2×10^{-3}	1.4×10^{-4}	5.0×10^{-4}
$E_{Ts}B_{32}$	5.6×10^{-6}	4.9×10^{-7}	1.3×10^{-7}	2.1×10^{-8}	1.3×10^{-7}
$E_{Ts}Hy_{8,3}$	3.2×10^{-6}	2.5×10^{-7}	9.1×10^{-6}	7.2×10^{-7}	5.2×10^{-8}

Example 4.2. Consider

$$\begin{aligned} &{}_0D_x^{\alpha} y(x) + a y(x) = f(x),\ (t > 0), \\ &y^k(0) = a_k,\ (k = 0, 1, \cdots, n-1), \end{aligned} \tag{4.5}$$

where $n-1 < \alpha \leq n$. For $0 < \alpha \leq 2$ and $a_k = 0$, this equation is called the *relaxation-oscillation* equation [14].

TABLE 2. Absolute error for $\alpha = 1$, by $B_{32}(x)$ and $Hy_{8,3}(x)$, for example 4.2, case 2.

x	0.2	0.5	0.8	1.1	1.4
$E_{Ma}B_{32}$	2.6×10^{-2}	9.0×10^{-3}	1.6×10^{-3}	7.5×10^{-3}	1.0×10^{-2}
$E_{Ma}Hy_{8,3}$	8.5×10^{-4}	3.9×10^{-4}	1.1×10^{-4}	4.3×10^{-5}	1.2×10^{-4}
$E_{Ts}B_{32}$	1.3×10^{-5}	6.2×10^{-6}	5.1×10^{-7}	5.6×10^{-6}	1.1×10^{-5}
$E_{Ts}Hy_{8,3}$	7.6×10^{-8}	1.3×10^{-7}	5.8×10^{-7}	6.7×10^{-8}	7.2×10^{-9}

Case 1. Consider $a_k = 0$ and $f(x) \equiv H(x)$, where $H(x)$ is the Heaviside function. In this case the analytical solution [14] is $y(x) = \int_0^x G(x-\tau)f(\tau)$, $G(x) = x^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})$ and

$$E_{\beta,\gamma}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\beta n + \gamma) n!} z^n,$$

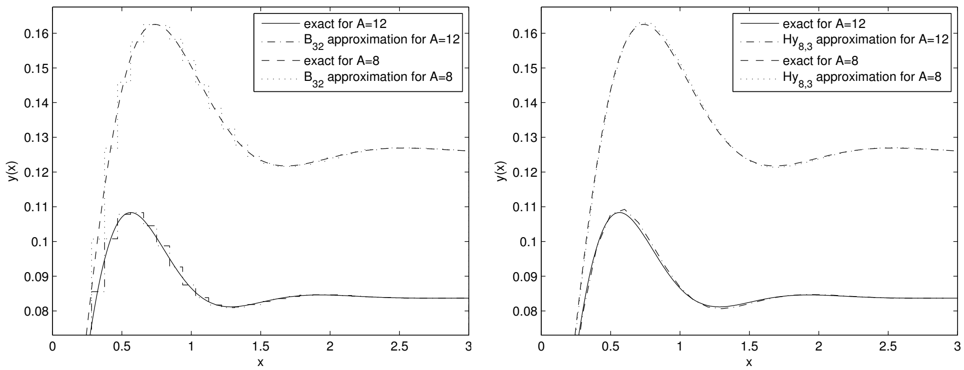

FIGURE 1. Comparison of the exact solution with numerical solution for $a = 8, 12$ and $\alpha = 1.5$, by $B_{32}(x)$ and $Hy_{8,3}(x)$, for example 4.2, case 1.

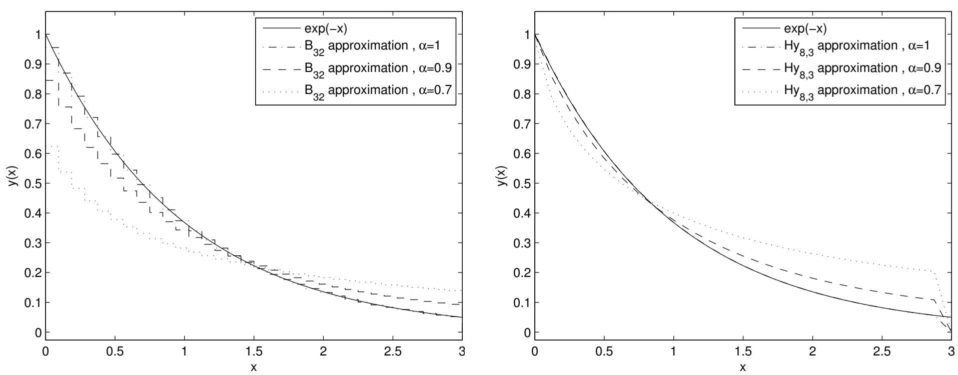

FIGURE 2. Numerical solution of example 4.2, case 2, by $B_{32}(x)$ and $Hy_{8,3}(x)$ with $\alpha = 1, 0.9, 0.7$.

where $\beta > 0$, $\gamma > 0$ and z is in complex number. The integral representation of (4.5) for $1 < \alpha < 2$ is

$$y(x) + aI^{\alpha}(y(x)) = I^{\alpha}(f(x)), \tag{4.6}$$

we, by applying Algorithm 2 described in the previous sections on $[0, 3)$ for $\alpha = 1.5$, solve the problem. The algebraic equations corresponding to (4.6) are of the form

$$C(I + aO^{\alpha}) = C_f O^{\alpha}, \tag{4.7}$$

Figure. 1 shows the numerical results generated by the block-pulse (B_{32}) and hybrid functions ($Hy_{8,3}$) for $a = 12$ and 8 with $\alpha = 1.5$. The absolute error for $a = 12$ are shown in Table 1. From Table 1, we see that the operational matrix methods achieve a good approximation with the exact solution. Moreover from Table 1, we found that if we use Algorithm 2, the errors are less than the errors generated by Matlab command $linsolve(A, b)$.

Case 2. In this case, we consider $f(x) = 0$, $0 \leq \alpha \leq 1$ and $a_0 = 1$, $a_1 = 0$, the analytical solution is [18], $y(x) = \sum_{k=0}^{\infty} \frac{(-x^{\alpha})^k}{\Gamma(\alpha k+1)}$. For $\alpha = 1$ we have, $y(x) = \exp(-x)$. The exact solution $exp(-x)$, with $\alpha = 1$ and numerical solution by block-pulse wavelets and hybrid function for $\alpha = 1$, 0.9 and 0.7 are shown in Figure. 2 . From Figure. 2 we can see that the numerical solution converges to exact solution as $\alpha \longrightarrow 1$. Also, the absolute errors generated buy the Matlab command $linsolve(A, b)$ and two stage iterative method for $\alpha = 1$ are shown in Table 2. From Table 2, we can see that if we use Algorithm 2 then we found more efficient numerical solutions. Moreover from Table 2 we can see that the hybrid operational matrix method gives an efficient numerical solution for $\alpha = 1$.

5. Conclusion

In this paper, a fractional differential equation has been solved by a new numerical algorithm based on operational matrices and two stage iterative method. This algorithm transforms a fractional differential equation with initial conditions into a system of algebraic linear equations and then solve it by TSI method. Figures and tables show that this method is extremely effective and practical. Moreover, the absolute error shows that the two stage iterative method for solving system of algebraic equations, is more efficient than the Matlab command $linsolve(A, b)$.

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