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Interval extensions of the Halley method and its modified method for finding enclosures of roots of nonlinear equations

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Abstract In this paper, interval extensions of the Halley method and its modified method for finding enclosures of roots of nonlinear equations are produced. Error analysis and convergence will be discussed. Also, these methods are compared together with the interval Newton method.

Keywords. Interval analysis, Nonlinear equations, Halley method, Guaranteed convergence.

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1. Introduction

Interval analysis was formally introduced by Moore in the 1960's, see [9, 10]. It provides a natural framework for self-verified numerical computing with its ability to correctly and automatically account for errors from many sources, including rounding errors due to limited precision of the floating point representation of real numbers, approximation errors due to algebraic manipulation of formulas, and measurement error in the initial data. A real interval is of the form $\boldsymbol{x} = [\underline{x}, \overline{x}]$, where $\underline{x}$ and $\overline{x}$ are the lower and upper bounds of the interval number $\boldsymbol{x}$, respectively. The set of compact real intervals is denoted by

$$\mathbb{IR} = \{\boldsymbol{x} = [\underline{x}, \overline{x}] \mid \underline{x}, \overline{x} \in \mathbb{R}, \ \underline{x} \leq \overline{x}\}.$$

In this Section, we review some of the fundamental definitions and properties of interval analysis that will be used throughout this paper. For a more in-depth discussion of topics related to interval analysis, see [1, 7, 8, 11, 12].

The width, radius, mid-point, and absolute value of an interval $\boldsymbol{x} = [\underline{x}, \overline{x}]$ are defined by $w(\boldsymbol{x}) = \overline{x} - \underline{x}$, $rad(\boldsymbol{x}) = \frac{1}{2}(\overline{x} - \underline{x})$, $m(\boldsymbol{x}) = \frac{1}{2}(\underline{x} + \overline{x})$ and $|\boldsymbol{x}| = \max\{|\underline{x}|, |\overline{x}|\}$, respectively. Let $\boldsymbol{x} = [\underline{x}, \overline{x}]$ and $\boldsymbol{y} = [\underline{y}, \overline{y}]$ be two intervals. Moore [10] defined the

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interval arithmetic as follows:

$$\begin{aligned}
&\boldsymbol{x}+\boldsymbol{y}=[\underline{x}+\underline{y},\overline{x}+\overline{y}],\\
&\boldsymbol{x}-\boldsymbol{y}=[\underline{x}-\overline{y},\overline{x}-\underline{y}],\\
&\boldsymbol{x}\times\boldsymbol{y}=[\min\{\underline{xy},\underline{x}\overline{y},\overline{x}\underline{y},\overline{xy}\},\max\{\underline{xy},\underline{x}\overline{y},\overline{x}\underline{y},\overline{xy}\}],\\
&\boldsymbol{x}/\boldsymbol{y}=[\underline{x},\overline{x}]\times[1/\overline{y},1/\underline{y}] \qquad if \qquad 0\notin\boldsymbol{y}.
\end{aligned}$$

Note that subtraction and division are not the inverse operations of addition and multiplication, respectively.

Definition 1.1. We say that $\boldsymbol{f}$ is an interval extension of f on the interval $\boldsymbol{x}=[\underline{x},\overline{x}]$, if

$$\begin{aligned}
&\boldsymbol{f}([x,x])=f(x), \quad (restriction),\\
&\boldsymbol{f}(\boldsymbol{x})\supseteq\{f(x)\,|\,x\in\boldsymbol{x}\}, \quad (inclusion).
\end{aligned}$$

Definition 1.2. An interval extension $\boldsymbol{f}$ is said to be Lipschitz in $\boldsymbol{x}^{(0)}$ if there is a constant L such that $rad(\boldsymbol{f}(\boldsymbol{x}))\leq L\,rad(\boldsymbol{x})$ for every $\boldsymbol{x}\subseteq\boldsymbol{x}^{(0)}$.

Lemma 1.3 (see [11])**.** *If $\boldsymbol{f}$ is a natural interval extension of a real rational function with $\boldsymbol{f}(\boldsymbol{x})$ defined for $\boldsymbol{x}\subseteq\boldsymbol{x}^{(0)}$, where $\boldsymbol{x}$ and $\boldsymbol{x}^{(0)}$ are intervals or n-dimensional interval vectors, then $\boldsymbol{f}$ is Lipschitz in $\boldsymbol{x}^{(0)}$.*

Definition 1.4. An interval valued function $\boldsymbol{f}$ is inclusion monotonic if $\boldsymbol{x}\subseteq\boldsymbol{y}$ implies $\boldsymbol{f}(\boldsymbol{x})\subseteq\boldsymbol{f}(\boldsymbol{y})$.

Definition 1.5. An interval sequence $\{\boldsymbol{x}^{(k)}\}$ is nested if $\boldsymbol{x}^{(k+1)}\subseteq\boldsymbol{x}^{(k)}$ for all k.

Lemma 1.6 (see [11])**.** *For any real numbers a and b and any intervals $\boldsymbol{x}$ and $\boldsymbol{y}$, we have the following relations:*

$$\begin{aligned}
&w(a\boldsymbol{x}+b\boldsymbol{y})=|a|w(\boldsymbol{x})+|b|w(\boldsymbol{y}),\\
&w(\boldsymbol{xy})\leq|\boldsymbol{x}|w(\boldsymbol{y})+|\boldsymbol{y}|w(\boldsymbol{x}),\\
&w(1/\boldsymbol{y})\leq|1/\boldsymbol{y}|^2w(\boldsymbol{y}) \quad if \quad 0\notin\boldsymbol{y}.
\end{aligned}$$

2. Background and methodology

2.1. **Interval Newton method.** Newton's method is the well-known iterative method for finding a simple zero of function. To obtain the root enclosures of a real-valued function f of a real variable x, an interval version of Newton method [11] is well-known as follows:

$$\boldsymbol{x}^{(k+1)}=\left\{m(\boldsymbol{x}^{(k)})-\frac{f(m(\boldsymbol{x}^{(k)}))}{\boldsymbol{f}'(\boldsymbol{x}^{(k)})}\right\}\cap\boldsymbol{x}^{(k)}, \qquad k=0,1,2,\ldots, \tag{2.1}$$

where $\boldsymbol{f}'(\boldsymbol{x}^{(k)})$ is an inclusion monotonic interval extension of $f'(x)$ for all $x\in\boldsymbol{x}^{(k)}$, and $0\notin\boldsymbol{f}'(\boldsymbol{x}^{(k)})$. The interval Newton method has the following properties:

- (i) If $\{m(\boldsymbol{x}^{(k)})-f(m(\boldsymbol{x}^{(k)}))/\boldsymbol{f}'(\boldsymbol{x}^{(k)})\}\cap\boldsymbol{x}^{(k)}=\emptyset$, then $\boldsymbol{x}^{(k)}$ does not contain any zero of f.
- (ii) If $x^*\in\boldsymbol{x}^{(0)}$ and $\{m(\boldsymbol{x}^{(k)})-f(m(\boldsymbol{x}^{(k)}))/\boldsymbol{f}'(\boldsymbol{x}^{(k)})\}\subseteq\boldsymbol{x}^{(k)}$, then $\boldsymbol{x}^{(k)}$ contains exactly one zero of f.

Theorem 2.1 (see [11])**.** *Given a real rational function f of a single real variable x with rational extensions $\boldsymbol{f}$, $\boldsymbol{f}'$ of f, f', respectively, such that f has a simple zero x^* in an interval $\boldsymbol{x}^{(0)}$ for which $\boldsymbol{f}(\boldsymbol{x}^{(0)})$ is defined and $\boldsymbol{f}'(\boldsymbol{x}^{(0)})$ is defined and does not contain zero i.e. $0 \notin \boldsymbol{f}'(\boldsymbol{x}^{(0)})$, then there is a positive real number C such that*

$$rad(\boldsymbol{x}^{(k+1)}) \leq C\, rad(\boldsymbol{x}^{(k)})^2. \tag{2.2}$$

Based on the interval extension of the Newton method, some interval methods have been produced for computing the enclosure solutions of nonlinear equations in [2–4, 14].

2.2. **Halley method.** The Halley method is an important method for finding a simple root of a nonlinear equation and is given by

$$x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{(f'(x_n))^2 - \frac{1}{2}f(x_n)f''(x_n)}.$$

This method was first considered by the astronomer Halley [6] and is an improvement of the Newton method with the order of convergence equal to 3.

2.3. **Modified Halley method.** The modified Halley method [13] is given by

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{2f(x_n)f(y_n)f'(y_n)}{2f(x_n)(f'(y_n))^2 - (f'(x_n))^2 f(y_n) + f'(x_n)f'(y_n)f(y_n)}. \end{cases}$$

Order of convergence of this method is equal to 5.

3. Main results

In this section two new interval methods,interval extensions of the Halley method and its modified method, are produced for computing enclosures of roots of nonlinear equations. Note that we use the symbol O in Theorems 3.2, 3.4 according to the following convention [16]: If $\frac{f}{g} \to C$, where C is a nonzero constant, we write $f = O(g)$ or $f \sim Cg$.

3.1. **Interval extension of the Halley method.** Let $x^* \in \boldsymbol{x}^{(0)} = [x_1^{(0)}, x_2^{(0)}]$ be a simple root of $f \in C^2(\boldsymbol{x}^{(0)})$ and $f'(x), f''(x) \neq 0$ for all $x \in \boldsymbol{x}^{(0)}$. By considering the first order Taylor series expansion of $f(x^*)$ around the point x, we have

$$0 = f(x^*) = f(x) + (x^* - x)f'(\xi),$$

for some ξ between x and x^*. Since $f(x^*) = 0$ and $f'(\xi) \neq 0$, we obtain

$$x^* = x - \frac{f(x)}{f'(\xi)}. \tag{3.1}$$

Now we write the second order Taylor series expansion of $f(x^*)$ around the point x as follows:

$$0 = f(x^*) = f(x) + (x^* - x)f'(x) + \frac{(x^* - x)^2}{2}f''(\eta),$$

for some η between x and x^*. Therefore

$$0 = f(x) + (x^* - x)\left(f'(x) + \frac{(x^* - x)}{2} f''(\eta)\right). \tag{3.2}$$

From (3.1) and (3.2), we obtain

$$0 = f(x) + (x^* - x)\left(f'(x) - \frac{f(x)}{2f'(\xi)} f''(\eta)\right),$$

hence

$$x^* = x - \frac{f(x)f'(\xi)}{f'(\xi)f'(x) - \frac{1}{2}f(x)f''(\eta)}.$$

If $\boldsymbol{f}'(\boldsymbol{x}^{(0)})$ denotes the interval extension of $f'(\xi), f'(x)$ and $\boldsymbol{f}''(\boldsymbol{x}^{(0)})$ denotes the interval extension of $f''(\eta)$, then

$$x^* \in x - \frac{f(x)\boldsymbol{f}'(\boldsymbol{x}^{(0)})}{\left(\boldsymbol{f}'(\boldsymbol{x}^{(0)})\right)^2 - \frac{1}{2}f(x)\boldsymbol{f}''(\boldsymbol{x}^{(0)})},$$

for any $x \in \boldsymbol{x}^{(0)}$, in particular for $x = m(\boldsymbol{x}^{(0)}) = (x_1^{(0)} + x_2^{(0)})/2$. Hence

$$x^* \in m(\boldsymbol{x}^{(0)}) - \frac{f(m(\boldsymbol{x}^{(0)}))\boldsymbol{f}'(\boldsymbol{x}^{(0)})}{\left(\boldsymbol{f}'(\boldsymbol{x}^{(0)})\right)^2 - \frac{1}{2}f(m(\boldsymbol{x}^{(0)}))\boldsymbol{f}''(\boldsymbol{x}^{(0)})}.$$

Let

$$\boldsymbol{H}(\boldsymbol{x}^{(0)}) = m(\boldsymbol{x}^{(0)}) - \frac{f(m(\boldsymbol{x}^{(0)}))\boldsymbol{f}'(\boldsymbol{x}^{(0)})}{\left(\boldsymbol{f}'(\boldsymbol{x}^{(0)})\right)^2 - \frac{1}{2}f(m(\boldsymbol{x}^{(0)}))\boldsymbol{f}''(\boldsymbol{x}^{(0)})},$$

therefore, $x^* \in \boldsymbol{H}(\boldsymbol{x}^{(0)})$. Since $x^* \in \boldsymbol{x}^{(0)}$, it is clear that $x^* \in \boldsymbol{H}(\boldsymbol{x}^{(0)}) \cap \boldsymbol{x}^{(0)}$. Define $\boldsymbol{x}^{(1)} = \boldsymbol{H}(\boldsymbol{x}^{(0)}) \cap \boldsymbol{x}^{(0)}$. By continuing this process, we build the sequence

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{H}(\boldsymbol{x}^{(k)}) \cap \boldsymbol{x}^{(k)}, \tag{3.3}$$

where

$$\boldsymbol{H}(\boldsymbol{x}^{(k)}) = m(\boldsymbol{x}^{(k)}) - \frac{f(m(\boldsymbol{x}^{(k)}))\boldsymbol{f}'(\boldsymbol{x}^{(k)})}{\left(\boldsymbol{f}'(\boldsymbol{x}^{(k)})\right)^2 - \frac{1}{2}f(m(\boldsymbol{x}^{(k)}))\boldsymbol{f}''(\boldsymbol{x}^{(k)})}.$$

Thus, an interval extension of the Halley method is produced.

Theorem 3.1. *Let $f \in C^2(\boldsymbol{x}^{(0)})$ and suppose that*

$$0 \notin \boldsymbol{f}'(\boldsymbol{x}^{(k)}), \ \left\{\left(\boldsymbol{f}'(\boldsymbol{x}^{(k)})\right)^2 - \frac{1}{2}f(m(\boldsymbol{x}^{(k)}))\boldsymbol{f}''(\boldsymbol{x}^{(k)})\right\}, \quad k = 0, 1, 2, \ldots.$$

If an interval $\boldsymbol{x}^{(0)}$ contains a root x^* of f, then so do intervals $\boldsymbol{x}^{(k)}, k = 1, 2, \ldots$. Besides, the nested interval sequence $\{\boldsymbol{x}^{(k)}\}$ of the form (3.3) converges to x^*.

Proof. By induction, if $x^* \in \boldsymbol{x}^{(0)}$, then $x^* \in \boldsymbol{x}^{(k)}$ for $k = 1, 2, \ldots$. Also, if there is a k such that $x^* = m(\boldsymbol{x}^{(k)})$, then we have $w(\boldsymbol{x}^{(k+1)}) = 0$, and, therefore, the convergence is proved. Now let $x^* \neq m(\boldsymbol{x}^{(k)})$ for $k = 0, 1, 2, \ldots$. Since

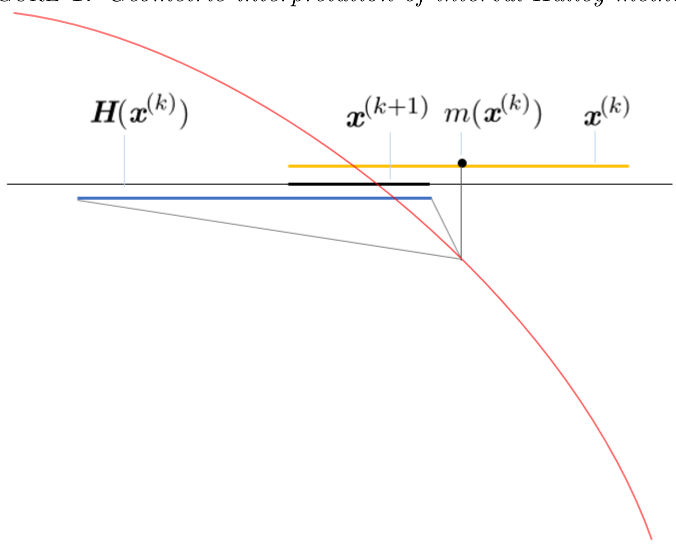
$$0 \notin \boldsymbol{f}'(\boldsymbol{x}^{(k)}),\ \left\{\left(\boldsymbol{f}'(\boldsymbol{x}^{(k)})\right)^2 - \frac{1}{2}f(m(\boldsymbol{x}^{(k)}))\boldsymbol{f}''(\boldsymbol{x}^{(k)})\right\}, \quad k = 0, 1, 2, \ldots,$$

then

$$\frac{f(m(\boldsymbol{x}^{(k)}))\boldsymbol{f}'(\boldsymbol{x}^{(k)})}{\left(\boldsymbol{f}'(\boldsymbol{x}^{(k)})\right)^2 - \frac{1}{2}f(m(\boldsymbol{x}^{(k)}))\boldsymbol{f}''(\boldsymbol{x}^{(k)})},$$

consists entirely of elements of the same sign. Thus, the midpoint of $\boldsymbol{x}^{(k)}$ is not contained in $\boldsymbol{x}^{(k+1)}$ (see Figure 1).

FIGURE 1. *Geometric interpretation of interval Halley method.*



Therefore, $w(\boldsymbol{x}^{(k+1)}) < \frac{1}{2}w(\boldsymbol{x}^{(k)})$ and the convergence is proved. □

Theorem 3.2. *Let $f \in C^2(\boldsymbol{x}^{(0)})$ and suppose that*

$$0 \notin \boldsymbol{f}'(\boldsymbol{x}^{(k)}),\ \left\{\left(\boldsymbol{f}'(\boldsymbol{x}^{(k)})\right)^2 - \frac{1}{2}f(m(\boldsymbol{x}^{(k)}))\boldsymbol{f}''(\boldsymbol{x}^{(k)})\right\}, \quad k = 0, 1, 2, \ldots.$$

(i) *If $\boldsymbol{H}(\mathbf{x}^{(k)}) \cap \boldsymbol{x}^{(k)} = \emptyset$ for $k = 0, 1, 2, \ldots$, then $\boldsymbol{x}^{(k)}$ contains no roots of f.*
(ii) *If $\boldsymbol{H}(\boldsymbol{x}^{(k)}) \subset \boldsymbol{x}^{(k)}$, then $\boldsymbol{x}^{(k)}$ contains exactly one root of f.*
In this case,

$$rad(\boldsymbol{x}^{(k+1)}) = O\left(\left(rad(\boldsymbol{x}^{(k)})\right)^2\right). \tag{3.4}$$

Proof. (i) Suppose that $\boldsymbol{x}^{(0)}$ contains a root x^* of f, then Theorem 3.1 results in $x^* \in \boldsymbol{H}(\boldsymbol{x}^{(k)})$ and $x^* \in \boldsymbol{H}(\boldsymbol{x}^{(k)}) \cap \boldsymbol{x}^{(k)}$. Therefore, if $\boldsymbol{H}(\boldsymbol{x}^{(k)}) \cap \boldsymbol{x}^{(k)} = \emptyset$, it follows that $\boldsymbol{x}^{(0)}$ cannot contain a root of f.
(ii) Since $0 \notin \boldsymbol{f}'(\boldsymbol{x}^{(k)})$, then $f'(x) \neq 0$ for all $x \in \boldsymbol{x}^{(k)}$ and f is monotonic on $\boldsymbol{x}^{(k)}$. Therefore, since f is continuous on $\boldsymbol{x}^{(0)}$, there can be at most one root in $\boldsymbol{x}^{(0)}$. In other words, it has at most one zero in $\boldsymbol{x}^{(k)}$. Hence, it is sufficient to find a zero $x^* \in \boldsymbol{x}^{(k)}$. By using the Theorem 3.1 , it is clear that f has exactly one root in $\boldsymbol{x}^{(k)}$. To prove (3.4), we consider the Mean Value Theorem as follows:

$$f(m(\boldsymbol{x}^{(k)})) = f'(\delta)(m(\boldsymbol{x}^{(k)}) - x^*), \tag{3.5}$$

where δ is between $m(\boldsymbol{x}^{(k)})$ and x^*. Since $\boldsymbol{H}(\boldsymbol{x}^{(k)}) \subset \boldsymbol{x}^{(k)}$, from the formulas (3.3) and (3.5), we get

$$\boldsymbol{x}^{(k+1)} = m(\boldsymbol{x}^{(k)}) - \lambda(m(\boldsymbol{x}^{(k)}) - x^*)f'(\delta),$$

where

$$\lambda = \frac{\boldsymbol{f}'(\boldsymbol{x}^{(k)})}{\left(\boldsymbol{f}'(\boldsymbol{x}^{(k)})\right)^2 - \frac{1}{2}f(m(\boldsymbol{x}^{(k)}))\boldsymbol{f}''(\boldsymbol{x}^{(k)})}.$$

Therefore

$$rad(\boldsymbol{x}^{(k+1)}) = rad(\lambda)\,|m(\boldsymbol{x}^{(k)}) - x^*||f'(\delta)|. \tag{3.6}$$

Since $w(\boldsymbol{x}) = 2\,rad(\boldsymbol{x}), \forall \boldsymbol{x} \in \mathbb{IR}$, using Lemma 1.6, we obtain

$$\begin{aligned} rad(\lambda) = {}& \frac{1}{|\left(\boldsymbol{f}'(\boldsymbol{x}^{(k)})\right)^2 - \frac{1}{2}f(m(\boldsymbol{x}^{(k)}))\boldsymbol{f}''(\boldsymbol{x}^{(k)})|} rad(\boldsymbol{f}'(\boldsymbol{x}^{(k)})) \\ & + |\boldsymbol{f}'(\boldsymbol{x}^{(k)})| rad\left(\frac{1}{\left(\boldsymbol{f}'(\boldsymbol{x}^{(k)})\right)^2 - \frac{1}{2}f(m(\boldsymbol{x}^{(k)}))\boldsymbol{f}''(\boldsymbol{x}^{(k)})}\right). \end{aligned} \tag{3.7}$$

Also, from Lemma 1.3, we see that

$$rad\left(\boldsymbol{f}'(\boldsymbol{x}^{(k)})\right) \leq L_1 rad(\boldsymbol{x}^{(k)}), \tag{3.8}$$

$$rad\left(\boldsymbol{f}''(\boldsymbol{x}^{(k)})\right) \leq L_2 rad(\boldsymbol{x}^{(k)}). \tag{3.9}$$

Let

$$|\left(\boldsymbol{f}'(\boldsymbol{x}^{(k)})\right)^2 - \frac{1}{2}f(m(\boldsymbol{x}^{(k)}))\boldsymbol{f}''(\boldsymbol{x}^{(k)})| \geq K_1, \tag{3.10}$$

$$|\boldsymbol{f}'(\boldsymbol{x}^{(k)})| \leq K_2, \tag{3.11}$$

$$|f'(\delta)| \leq K_3. \tag{3.12}$$

It is clear that

$$|m(\boldsymbol{x}^{(k)}) - x^*| \leq rad(\boldsymbol{x}^{(k)}). \tag{3.13}$$

By using Lemma 1.6 and from (3.5), (3.8)-(3.13), we get

$$
\begin{aligned}
& rad\left(\frac{1}{\left(\boldsymbol{f}'(\boldsymbol{x}^{(k)})\right)^2-\frac{1}{2}f(m(\boldsymbol{x}^{(k)}))\boldsymbol{f}''(\boldsymbol{x}^{(k)})}\right)\\
&\leq \frac{rad\left(\left(\boldsymbol{f}'(\boldsymbol{x}^{(k)})\right)^2-\frac{1}{2}f(m(\boldsymbol{x}^{(k)}))\boldsymbol{f}''(\boldsymbol{x}^{(k)})\right)}{\left|\left(\boldsymbol{f}'(\boldsymbol{x}^{(k)})\right)^2-\frac{1}{2}f(m(\boldsymbol{x}^{(k)}))\boldsymbol{f}''(\boldsymbol{x}^{(k)})\right|^2}\\
&\leq \frac{1}{K_1^2}\left(rad\left(\left(\boldsymbol{f}'(\boldsymbol{x}^{(k)})\right)^2\right)+\frac{1}{2}|f(m(\boldsymbol{x}^{(k)}))|rad\left(\boldsymbol{f}''(\boldsymbol{x}^{(k)})\right)\right)\\
&\leq \frac{1}{K_1^2}\left(2|\boldsymbol{f}'(\boldsymbol{x}^{(k)})|rad\left(\boldsymbol{f}'(\boldsymbol{x}^{(k)})\right)+\frac{1}{2}|f'(\delta)||m(\boldsymbol{x}^{(k)})-x^*|rad\left(\boldsymbol{f}''(\boldsymbol{x}^{(k)})\right)\right)\\
&\leq \frac{1}{K_1^2}\left(2K_2L_1 rad(\boldsymbol{x}^{(k)})+\frac{K_3L_2}{2}\left(rad(\boldsymbol{x}^{(k)})\right)^2\right)\\
&= \frac{1}{K_1^2}\left(2K_2L_1+\frac{K_3L_2}{2}rad(\boldsymbol{x}^{(k)})\right)rad(\boldsymbol{x}^{(k)}).
\end{aligned} \tag{3.14}
$$

Therefore, from (3.7), (3.8), (3.10), (3.11) and (3.14), we clearly have

$$
\begin{aligned}
rad(\lambda) &\leq \frac{L_1}{K_1}rad(\boldsymbol{x}^{(k)})+\frac{K_2}{K_1^2}\left(2K_2L_1+\frac{K_3L_2}{2}rad(\boldsymbol{x}^{(k)})\right)rad(\boldsymbol{x}^{(k)})\\
&= \left(\frac{L_1}{K_1}+\frac{K_2}{K_1^2}\left(2K_2L_1+\frac{K_3L_2}{2}rad(\boldsymbol{x}^{(k)})\right)\right)rad(\boldsymbol{x}^{(k)}).
\end{aligned} \tag{3.15}
$$

Now, applying (3.12), (3.13), and (3.15) in formula (3.6) gives (3.4). □

3.2. **Interval extension of the modified Halley method.** Let $\boldsymbol{y}^{(0)}=[y_1^{(0)},y_2^{(0)}]$ be an interval and $x^*\in\boldsymbol{y}^{(0)}\subseteq\boldsymbol{x}^{(0)}$. From (2.1), we have

$$
\boldsymbol{y}^{(0)}=\left\{m(\boldsymbol{x}^{(0)})-\frac{f(m(\boldsymbol{x}^{(0)}))}{\boldsymbol{f}'(\boldsymbol{x}^{(0)})}\right\}\cap\boldsymbol{x}^{(0)}. \tag{3.16}
$$

By writing the first order Taylor series expansion of $f(y)$ around the point x^*, we have

$$
f(y)=f(x^*)+(y-x^*)f'(\beta),
$$

for any $y\in\boldsymbol{y}^{(0)}$ and for some β between y and x^*. Since $f(x^*)=0$ and $f'(\beta)\neq 0$, we get

$$
x^*=y-\frac{f(y)}{f'(\beta)}. \tag{3.17}
$$

Now by considering the second order Taylor series expansion of $f(x^*)$ around the point y, we have

$$
0=f(x^*)=f(y)+(x^*-y)f'(y)+\frac{(x^*-y)^2}{2}f''(y),
$$

for any $y \in \boldsymbol{y}^{(0)}$. Therefore

$$0 = f(y) + (x^* - y)\left(f'(y) + \frac{(x^* - y)}{2} f''(y)\right). \tag{3.18}$$

By substituting (3.17) into (3.18), we get

$$\begin{aligned} 0 &= f(y) + (x^* - y)\left(f'(y) + \frac{(x^* - y)}{2} f''(y)\right) \\ &= f(y) + (x^* - y)\left(f'(y) - \frac{f(y)}{2f'(\beta)} f''(y)\right) \\ &= f(y) + (x^* - y)\left(\frac{2f'(y)f'(\beta) - f(y)f''(y)}{2f'(\beta)}\right), \end{aligned}$$

thus

$$x^* = y - \frac{2f(y)f'(\beta)}{2f'(y)f'(\beta) - f(y)f''(y)}. \tag{3.19}$$

Now, we write the first order Taylor series expansion of $f'(x)$ around the point y:

$$f'(x) = f'(y) + (x - y)f''(y), \tag{3.20}$$

for any $x \in \boldsymbol{x}^{(0)}$. Since x^* and y are arbitrary, we can assume that the two values are sufficiently close together. Thus, using (3.1) and substituting $x - y \approx x - x^*$ into (3.20) give us

$$f''(y) = \frac{f'(x) - f'(y)}{x - y} \approx \frac{f'(x) - f'(y)}{x - x^*} = \frac{(f'(x) - f'(y))\, f'(\xi)}{f(x)}. \tag{3.21}$$

By substituting (3.21) into (3.19), we obtain

$$x^* \approx y - \frac{2f(x)f(y)f'(\beta)}{2f(x)f'(y)f'(\beta) - f'(x)f'(\xi)f(y) + f'(\xi)f'(y)f(y)}.$$

If $\boldsymbol{f}'(\boldsymbol{x}^{(0)})$ denotes the interval extension of $f'(\xi), f'(x)$ and $\boldsymbol{f}'(\boldsymbol{y}^{(0)})$ denotes the interval extension of $f'(y), f'(\beta)$, then

$$x^* \in y - \frac{2f(x)f(y)\boldsymbol{f}'(\boldsymbol{y}^{(0)})}{2f(x)(\boldsymbol{f}'(\boldsymbol{y}^{(0)}))^2 - (\boldsymbol{f}'(\boldsymbol{x}^{(0)}))^2 f(y) + \boldsymbol{f}'(\boldsymbol{x}^{(0)})\boldsymbol{f}'(\boldsymbol{y}^{(0)})f(y)},$$

for any $x \in \boldsymbol{x}^{(0)}$ and $y \in \boldsymbol{y}^{(0)}$, in particular for $x = m(\boldsymbol{x}^{(0)}) = (x_1^{(0)} + x_2^{(0)})/2$ and $y = m(\boldsymbol{y}^{(0)}) = (y_1^{(0)} + y_2^{(0)})/2$. Therefore

$$\begin{aligned} x^* \in m(\boldsymbol{y}^{(0)}) - 2f(m(\boldsymbol{x}^{(0)}))f(m(\boldsymbol{y}^{(0)}))\boldsymbol{f}'(\boldsymbol{y}^{(0)}) / \Big(& 2f(m(\boldsymbol{x}^{(0)}))(\boldsymbol{f}'(\boldsymbol{y}^{(0)}))^2 \\ & - (\boldsymbol{f}'(\boldsymbol{x}^{(0)}))^2 f(m(\boldsymbol{y}^{(0)})) + \boldsymbol{f}'(\boldsymbol{x}^{(0)})\boldsymbol{f}'(\boldsymbol{y}^{(0)})f(m(\boldsymbol{y}^{(0)}))\Big). \end{aligned}$$

Let

$$\boldsymbol{M}(\boldsymbol{x}^{(0)},\boldsymbol{y}^{(0)}) = m(\boldsymbol{y}^{(0)}) - \Big(2f(m(\boldsymbol{x}^{(0)}))f(m(\boldsymbol{y}^{(0)}))\boldsymbol{f}'(\boldsymbol{y}^{(0)}) \\ / \Big(2f(m(\boldsymbol{x}^{(0)}))(\boldsymbol{f}'(\boldsymbol{y}^{(0)}))^2 - (\boldsymbol{f}'(\boldsymbol{x}^{(0)}))^2 f(m(\boldsymbol{y}^{(0)})) \\ +\boldsymbol{f}'(\boldsymbol{x}^{(0)})\boldsymbol{f}'(\boldsymbol{y}^{(0)})f(m(\boldsymbol{y}^{(0)}))\Big)\Big),$$

therefore, $x^* \in \boldsymbol{M}(\boldsymbol{x}^{(0)},\boldsymbol{y}^{(0)})$. Since $x^* \in \boldsymbol{y}^{(0)}$, it is clear that $x^* \in \boldsymbol{M}(\boldsymbol{x}^{(0)},\boldsymbol{y}^{(0)}) \cap \boldsymbol{y}^{(0)}$. Define

$$\boldsymbol{x}^{(1)} = \boldsymbol{M}(\boldsymbol{x}^{(0)},\boldsymbol{y}^{(0)}) \cap \boldsymbol{y}^{(0)}. \tag{3.22}$$

From (3.16) and (3.22), we obtain

$$\begin{cases} \boldsymbol{y}^{(0)} = \left\{ m(\boldsymbol{x}^{(0)}) - \dfrac{f(m(\boldsymbol{x}^{(0)}))}{\boldsymbol{f}'(\boldsymbol{x}^{(0)})} \right\} \cap \boldsymbol{x}^{(0)}, \\ \boldsymbol{M}(\boldsymbol{x}^{(0)},\boldsymbol{y}^{(0)}) = m(\boldsymbol{y}^{(0)}) - \big(2f(m(\boldsymbol{x}^{(0)}))f(m(\boldsymbol{y}^{(0)}))\boldsymbol{f}'(\boldsymbol{y}^{(0)}) \\ \qquad / \Big(2f(m(\boldsymbol{x}^{(0)}))(\boldsymbol{f}'(\boldsymbol{y}^{(0)}))^2 - (\boldsymbol{f}'(\boldsymbol{x}^{(0)}))^2 f(m(\boldsymbol{y}^{(0)})) \\ \qquad +\boldsymbol{f}'(\boldsymbol{x}^{(0)})\boldsymbol{f}'(\boldsymbol{y}^{(0)})f(m(\boldsymbol{y}^{(0)})))\big)\big), \\ \boldsymbol{x}^{(1)} = \boldsymbol{M}(\boldsymbol{x}^{(0)},\boldsymbol{y}^{(0)}) \cap \boldsymbol{y}^{(0)}. \end{cases}$$

Now by continuing this process, we see that

$$\begin{cases} \boldsymbol{y}^{(k)} = \left\{ m(\boldsymbol{x}^{(k)}) - \dfrac{f(m(\boldsymbol{x}^{(k)}))}{\boldsymbol{f}'(\boldsymbol{x}^{(k)})} \right\} \cap \boldsymbol{x}^{(k)}, \\ \boldsymbol{M}(\boldsymbol{x}^{(k)},\boldsymbol{y}^{(k)}) = m(\boldsymbol{y}^{(k)}) - \big(2f(m(\boldsymbol{x}^{(k)}))f(m(\boldsymbol{y}^{(k)}))\boldsymbol{f}'(\boldsymbol{y}^{(k)}) \\ \qquad / \Big(2f(m(\boldsymbol{x}^{(k)}))(\boldsymbol{f}'(\boldsymbol{y}^{(k)}))^2 - (\boldsymbol{f}'(\boldsymbol{x}^{(k)}))^2 f(m(\boldsymbol{y}^{(k)})) \\ \qquad +\boldsymbol{f}'(\boldsymbol{x}^{(k)})\boldsymbol{f}'(\boldsymbol{y}^{(k)})f(m(\boldsymbol{y}^{(k)})))\big)\big), \\ \boldsymbol{x}^{(k+1)} = \boldsymbol{M}(\boldsymbol{x}^{(k)},\boldsymbol{y}^{(k)}) \cap \boldsymbol{y}^{(k)}. \end{cases} \tag{3.23}$$

Thus, an interval extension of the modified Halley method for computing the enclosure solutions of nonlinear equations is produced.

Theorem 3.3. *Let $f \in C(\boldsymbol{x}^{(0)})$ and suppose*

$$0 \notin \boldsymbol{f}'(\boldsymbol{x}^{(k)}), \Big\{2f(m(\boldsymbol{x}^{(k)}))(\boldsymbol{f}'(\boldsymbol{y}^{(k)}))^2 - (\boldsymbol{f}'(\boldsymbol{x}^{(k)}))^2 f(m(\boldsymbol{y}^{(k)})) \\ +\boldsymbol{f}'(\boldsymbol{x}^{(k)})\boldsymbol{f}'(\boldsymbol{y}^{(k)})f(m(\boldsymbol{y}^{(k)}))\Big\}, \qquad k = 0, 1, 2, \ldots.$$

If an interval $\boldsymbol{x}^{(0)}$ contains a root x^* of f, then so do intervals $\boldsymbol{x}^{(k)}, k = 1, 2, \ldots$. Besides, the nested interval sequence $\{\boldsymbol{x}^{(k)}\}$ of the form (3.23) converges to x^*.

Proof. The proof of this theorem is similar to the proof of Theorem 3.1. Hence, it is omitted. □

Theorem 3.4. *Let $f \in C(\boldsymbol{x}^{(0)})$ and suppose that*

$$0 \notin \boldsymbol{f}'(\boldsymbol{x}^{(k)}), \Big\{2f(m(\boldsymbol{x}^{(k)}))(\boldsymbol{f}'(\boldsymbol{y}^{(k)}))^2 - (\boldsymbol{f}'(\boldsymbol{x}^{(k)}))^2 f(m(\boldsymbol{y}^{(k)})) + \boldsymbol{f}'(\boldsymbol{x}^{(k)})\boldsymbol{f}'(\boldsymbol{y}^{(k)})f(m(\boldsymbol{y}^{(k)}))\Big\}, \qquad k = 0, 1, 2, \ldots.$$

Then the following statements hold:

(i) *If $\boldsymbol{M}(\boldsymbol{x}^{(k)}, \boldsymbol{y}^{(k)}) \cap \boldsymbol{y}^{(k)} = \emptyset$ for $k = 0, 1, 2, \ldots,$ then $\boldsymbol{x}^{(k)}$ contains no roots of f.*

(ii) *If $\boldsymbol{M}(\boldsymbol{x}^{(k)}, \boldsymbol{y}^{(k)}) \subset \boldsymbol{y}^{(k)}$, then $\boldsymbol{x}^{(k)}$ contains exactly one root of f.*
In this case,

$$rad(\boldsymbol{x}^{(k+1)}) = O\left((rad(\boldsymbol{x}^{(k)}))^5\right). \tag{3.24}$$

Proof. The proof of (i) and (ii) for the nested interval sequence of the form (3.23) is entirely analogous to the proof of Theorem 3.2. We show only (3.24). Using the Mean Value Theorem gives

$$f(m(\boldsymbol{y}^{(k)})) = f'(\sigma)(m(\boldsymbol{y}^{(k)}) - x^*), \tag{3.25}$$

where σ is between $m(\boldsymbol{y}^{(k)})$ and x^*. Since $\boldsymbol{M}(\boldsymbol{x}^{(k)}, \boldsymbol{y}^{(k)}) \subset \boldsymbol{x}^{(k)}$, from (3.5), (3.23), and (3.25), we get

$$\boldsymbol{x}^{(k+1)} = m(\boldsymbol{y}^{(k)}) - \gamma(m(\boldsymbol{x}^{(k)}) - x^*)f'(\delta)(m(\boldsymbol{y}^{(k)}) - x^*)f'(\sigma),$$

where

$$\gamma = 2\boldsymbol{f}'(\boldsymbol{y}^{(k)})/\left(2f(m(\boldsymbol{x}^{(k)}))(\boldsymbol{f}'(\boldsymbol{y}^{(k)}))^2 - (\boldsymbol{f}'(\boldsymbol{x}^{(k)}))^2 f(m(\boldsymbol{y}^{(k)})) + \boldsymbol{f}'(\boldsymbol{x}^{(k)})\boldsymbol{f}'(\boldsymbol{y}^{(k)})f(m(\boldsymbol{y}^{(k)}))\right).$$

Therefore

$$rad(\boldsymbol{x}^{(k+1)}) = rad(\gamma)\,|m(\boldsymbol{x}^{(k)}) - x^*||f'(\delta)|\,|m(\boldsymbol{y}^{(k)}) - x^*||f'(\sigma)|. \tag{3.26}$$

Since $w(\boldsymbol{x}) = 2\,rad(\boldsymbol{x}), \forall \boldsymbol{x} \in \mathbb{IR}$, using Lemma 1.6, we obtain

$$\begin{aligned} rad(\gamma) = 2rad\left(\boldsymbol{f}'(\boldsymbol{y}^{(k)})\right) / \Big|&2f(m(\boldsymbol{x}^{(k)}))(\boldsymbol{f}'(\boldsymbol{y}^{(k)}))^2 \\ &-(\boldsymbol{f}'(\boldsymbol{x}^{(k)}))^2 f(m(\boldsymbol{y}^{(k)})) + \boldsymbol{f}'(\boldsymbol{x}^{(k)})\boldsymbol{f}'(\boldsymbol{y}^{(k)})f(m(\boldsymbol{y}^{(k)}))\Big| \\ &+ 2|\boldsymbol{f}'(\boldsymbol{y}^{(k)})|rad\left(1/\left(2f(m(\boldsymbol{x}^{(k)}))(\boldsymbol{f}'(\boldsymbol{y}^{(k)}))^2\right.\right. \\ &\left.\left.-(\boldsymbol{f}'(\boldsymbol{x}^{(k)}))^2 f(m(\boldsymbol{y}^{(k)})) + \boldsymbol{f}'(\boldsymbol{x}^{(k)})\boldsymbol{f}'(\boldsymbol{y}^{(k)})f(m(\boldsymbol{y}^{(k)}))\right)\right). \end{aligned} \tag{3.27}$$

Also, from Lemma 1.3, we have

$$rad\left(\boldsymbol{f}'(\boldsymbol{y}^{(k)})\right) \le L_3 rad(\boldsymbol{y}^{(k)}). \tag{3.28}$$

Let

$$\Big|2f(m(\boldsymbol{x}^{(k)}))(\boldsymbol{f}'(\boldsymbol{y}^{(k)}))^2 - (\boldsymbol{f}'(\boldsymbol{x}^{(k)}))^2 f(m(\boldsymbol{y}^{(k)})) + \boldsymbol{f}'(\boldsymbol{x}^{(k)})\boldsymbol{f}'(\boldsymbol{y}^{(k)})f(m(\boldsymbol{y}^{(k)}))\Big| \ge K_4, \tag{3.29}$$

$$|f'(\sigma)| \leq K_5, \tag{3.30}$$

$$|\boldsymbol{f}'(\boldsymbol{y}^{(k)})| \leq K_6. \tag{3.31}$$

It is clear that

$$|m(\boldsymbol{y}^{(k)}) - x^*| \leq rad(\boldsymbol{y}^{(k)}). \tag{3.32}$$

Using Lemma 1.6 and from (2.2), (3.5), (3.8), (3.11)-(3.13), (3.25), (3.28)-(3.32), we get

$$\begin{aligned}
& rad\,(1\,/\left(2f(m(\boldsymbol{x}^{(k)}))(\boldsymbol{f}'(\boldsymbol{y}^{(k)}))^2 - (\boldsymbol{f}'(\boldsymbol{x}^{(k)}))^2 f(m(\boldsymbol{y}^{(k)}))\right.\\
&\left.\left. +\boldsymbol{f}'(\boldsymbol{x}^{(k)})\boldsymbol{f}'(\boldsymbol{y}^{(k)})f(m(\boldsymbol{y}^{(k)}))\right)\right)\\
&\leq 1/\left|2f(m(\boldsymbol{x}^{(k)}))(\boldsymbol{f}'(\boldsymbol{y}^{(k)}))^2 - (\boldsymbol{f}'(\boldsymbol{x}^{(k)}))^2 f(m(\boldsymbol{y}^{(k)}))\right.\\
&\left. +\boldsymbol{f}'(\boldsymbol{x}^{(k)})\boldsymbol{f}'(\boldsymbol{y}^{(k)})f(m(\boldsymbol{y}^{(k)}))\right|^2\\
&\times rad\left(2f(m(\boldsymbol{x}^{(k)}))(\boldsymbol{f}'(\boldsymbol{y}^{(k)}))^2 - (\boldsymbol{f}'(\boldsymbol{x}^{(k)}))^2 f(m(\boldsymbol{y}^{(k)}))\right.\\
&\left. +\boldsymbol{f}'(\boldsymbol{x}^{(k)})\boldsymbol{f}'(\boldsymbol{y}^{(k)})f(m(\boldsymbol{y}^{(k)}))\right)\\
&\leq \frac{1}{K_4^2}\left(2|f(m(\boldsymbol{x}^{(k)}))|rad\left((\boldsymbol{f}'(\boldsymbol{y}^{(k)}))^2\right) + |f(m(\boldsymbol{y}^{(k)}))|rad\left((\boldsymbol{f}'(\boldsymbol{x}^{(k)}))^2\right)\right.\\
&\left. +|f(m(\boldsymbol{y}^{(k)}))|rad(\boldsymbol{f}'(\boldsymbol{x}^{(k)})\boldsymbol{f}'(\boldsymbol{y}^{(k)}))\right)\\
&\leq \frac{1}{K_4^2}\left(4|m(\boldsymbol{x}^{(k)}) - x^*||f'(\delta)||\boldsymbol{f}'(\boldsymbol{y}^{(k)})|rad(\boldsymbol{f}'(\boldsymbol{y}^{(k)}))\right.\\
&+ 2|m(\boldsymbol{y}^{(k)}) - x^*||f'(\sigma)||\boldsymbol{f}'(\boldsymbol{x}^{(k)})|rad(\boldsymbol{f}'(\boldsymbol{x}^{(k)}))\\
&+ |m(\boldsymbol{y}^{(k)}) - x^*||f'(\sigma)|\left(|\boldsymbol{f}'(\boldsymbol{x}^{(k)})|rad(\boldsymbol{f}'(\boldsymbol{y}^{(k)}))\right.\\
&\left.\left. +|\boldsymbol{f}'(\boldsymbol{y}^{(k)})|rad(\boldsymbol{f}'(\boldsymbol{x}^{(k)}))\right)\right)\\
&\leq \frac{1}{K_4^2}\left(4rad(\boldsymbol{x}^{(k)})K_3K_6L_3rad(\boldsymbol{y}^{(k)}) + 2rad(\boldsymbol{y}^{(k)})K_5K_2L_1rad(\boldsymbol{x}^{(k)})\right.\\
&\left. +rad(\boldsymbol{y}^{(k)})K_5(K_2L_3rad(\boldsymbol{y}^{(k)}) + K_6L_1rad(\boldsymbol{x}^{(k)}))\right)\\
&\leq \frac{1}{K_4^2}\left(4K_3K_6L_3C\,(rad(\boldsymbol{x}^{(k)}))^3 + 2K_5K_2L_1C\,(rad(\boldsymbol{x}^{(k)}))^3\right.\\
&\left. +C\,(rad(\boldsymbol{x}^{(k)}))^2K_5\left(K_2L_3C(rad(\boldsymbol{x}^{(k)}))^2 + K_6L_1rad(\boldsymbol{x}^{(k)})\right)\right)\\
&= \frac{1}{K_4^2}\,(4K_3K_6L_3C + 2K_5K_2L_1C\\
&+CK_5\left(K_2L_3C\,rad(\boldsymbol{x}^{(k)}) + K_6L_1\right))\,(rad(\boldsymbol{x}^{(k)}))^3.
\end{aligned} \tag{3.33}$$

Therefore, from (2.2), (3.27)-(3.29), (3.31), (3.33), we clearly have

$$\begin{aligned}
rad(\gamma) \leq\ & \frac{2L_3}{K_4} rad(\boldsymbol{y}^{(k)}) + \frac{2K_6}{K_4^2}\,(4K_3K_6L_3C + 2K_5K_2L_1C \\
& + CK_5\left(K_2L_3C rad(\boldsymbol{x}^{(k)}) + K_6L_1\right)\Big)\,(rad(\boldsymbol{x}^{(k)}))^3 \\
\leq\ & \frac{2L_3}{K_4} C(rad(\boldsymbol{x}^{(k)}))^2 + \frac{2K_6}{K_4^2}\,(4K_3K_6L_3C + 2K_5K_2L_1C \\
& + CK_5\left(K_2L_3C rad(\boldsymbol{x}^{(k)}) + K_6L_1\right)\Big)\,(rad(\boldsymbol{x}^{(k)}))^3 \\
=\ & \Bigg(\frac{2L_3}{K_4} C + \frac{2K_6}{K_4^2}\,(4K_3K_6L_3C + 2K_5K_2L_1C \\
& + CK_5\left(K_2L_3C rad(\boldsymbol{x}^{(k)}) + K_6L_1\right)\Big)\, rad(\boldsymbol{x}^{(k)})\Bigg)\,(rad(\boldsymbol{x}^{(k)}))^2. \qquad (3.34)
\end{aligned}$$

Now, applying (2.2), (3.12), (3.13), (3.30), (3.32), and (3.34) in formula (3.26) gives (3.24). □

4. Numerical results

In this section we have compared the interval methods (3.3) and (3.23) together with the interval Newton method (2.1), using the examples listed in Table 1.

Table 1. Tested functions and initial intervals

Example i	Function f_i	Root x^*	Initial interval $\boldsymbol{x}^{(0)}$	Root enclosures
1	$x^2 - e^x - 3x + 2$	0.25753028543986079	$[0, 1]$	$[0.25753028543986, 0.25753028543987]$
2	$x^5 + x^4 + 4x^2 - 15$	1.3474280989683053	$[1.25, 1.5]$	$[1.34742809896830, 1.34742809896831]$
3	$\log(x^2 + x + 2) - x + 1$	4.1525907367571583	$[4, 4.25]$	$[4.15259073675715, 4.15259073675716]$
4	$(x-5)^2 - e^x$	2.1173913386948322	$[2, 2.25]$	$[2.11739133869483, 2.11739133869484]$
5	$\cos x + x - x^2 + x^5$	-0.5333964635678204	$[-0.6, -0.45]$	$[-0.53339646356783, -0.53339646356782]$
6	$e^x - \sin^3 x$	-3.4623979938206757	$[-3.5, -3.25]$	$[-3.46239799382068, -3.46239799382067]$
7	$e^{-x} + \cos x$	1.746139530408012285	$[1.5, 2]$	$[1.74613953040801, 1.74613953040802]$
8	$(x+2)e^x - 1$	-0.44285440100238854	$[-0.5, 0]$	$[-0.44285440100239, -0.44285440100238]$
9	$\cos x - x$	0.73908513321516067	$[0.5, 1]$	$[0.73908513321516, 0.73908513321517]$
10	$x^5 - 10$	1.5848931924611134	$[1, 1.75]$	$[1.58489319246111, 1.58489319246112]$
11	$x^3 + \sin\left(\frac{x}{\sqrt{3}}\right) - \frac{1}{4}$	0.3568342187225045	$[0.3, 0.4]$	$[0.35683421872250, 0.35683421872251]$
12	$(x-1)e^{-2x} + x^3$	0.5391809932576055	$[0.5, 0.6]$	$[0.53918099325760, 0.53918099325761]$
13	$x^2 \sin x + e^{x \cos x \sin x} + 4x^3 - 15$	1.4322415985999165	$[1.4, 1.5]$	$[1.43224159859991, 1.43224159859992]$
14	$xe^{x^2-1} + \cos x + \log(x^2 + x + 2)$	-1.0634448437881119	$[-1.2, -1]$	$[-1.06344484378812, -1.06344484378811]$
15	$\sin^2(x^2+1) - \frac{\sqrt{x+1}}{3}$	1.1684762578039694	$[1, 1.2]$	$[1.16847625780396, 1.16847625780397]$

All examples in this section are tested on an Intel(R) Core(TM) i5-2450M CPU @ 2.50GHz Processor with 4 GB of RAM using Matlab R2015a and version 8 of INTLAB [15] on Windows 7 (64 bit) operating system. The results of comparisons are displayed in Table 2, where "IT" shows the number of iterations.

According to Table 2, the number of iterations reveals that the interval extension of the modified Halley method (3.23) requires a few numbers of iterations to obtain enclosure of roots of nonlinear equations in the contrast to the other methods. In fact, the interval extension of the modified Halley method (3.23) is better than the interval extension of the Halley method (3.3) and the interval Newton method (2.1).

TABLE 2. Comparison of results for the interval methods

	(2.1)	(3.3)	(3.23)
$f_1(x),\ \boldsymbol{x}^{(0)} = [0, 1]$ IT	5	5	3
$f_2(x),\ \boldsymbol{x}^{(0)} = [1.25, 1.5]$ IT	4	5	2
$f_3(x),\ \boldsymbol{x}^{(0)} = [4, 4.25]$ IT	5	4	2
$f_4(x),\ \boldsymbol{x}^{(0)} = [2, 2.25]$ IT	3	4	2
$f_5(x),\ \boldsymbol{x}^{(0)} = [-0.6, -0.45]$ IT	4	4	2
$f_6(x),\ \boldsymbol{x}^{(0)} = [-3.5, -3.25]$ IT	5	6	3
$f_7(x),\ \boldsymbol{x}^{(0)} = [1.5, 2]$ IT	3	3	2
$f_8(x),\ \boldsymbol{x}^{(0)} = [-0.5, 0]$ IT	4	5	3
$f_9(x),\ \boldsymbol{x}^{(0)} = [0.5, 1]$ IT	4	4	2
$f_{10}(x),\ \boldsymbol{x}^{(0)} = [1, 1.75]$ IT	5	6	3
$f_{11}(x),\ \boldsymbol{x}^{(0)} = [0.3, 0.4]$ IT	3	4	2
$f_{12}(x),\ \boldsymbol{x}^{(0)} = [0.5, 0.6]$ IT	4	4	2
$f_{13}(x),\ \boldsymbol{x}^{(0)} = [1.4, 1.5]$ IT	3	4	2
$f_{14}(x),\ \boldsymbol{x}^{(0)} = [-1.2, -1]$ IT	4	5	2
$f_{15}(x),\ \boldsymbol{x}^{(0)} = [1, 1.2]$ IT	5	6	3

5. Conclusion

In this paper, interval extensions of the Halley method and its modified method which calculate enclosures of roots of given nonlinear equations were produced. Also, error bound and convergence rate were studied. These algorithms were tested using some examples via INTLAB. Numerical results show that the interval modified Halley method is better than the interval Halley method and interval Newton method.

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