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Group-invariant solutions for time-fractional Fornberg-Whitham equation by Lie symmetry analysis

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Abstract This paper is concerned with the time-fractional Fornberg-Whitham equation using Lie symmetry analysis. This equation is used to describe the physical processes of models possessing memory. By employing the classical and nonclassical Lie symmetry analysis, the invariance properties of this equation are investigated. The similarity reductions and new exact solutions are obtained.

Keywords. Lie symmetry analysis, Time-fractional Fornberg-Whitham equation, Group-invariant solution.

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1. Introduction

Over the past few years, significant attention has been devoted to the study of the ordinary and partial differential equations with fractional order and their numerous applications in various fields of science [4, 12, 13]. The fractional derivatives are nonlocal operators arising in mathematical modeling of many systems in physics, chemistry and biochemistry, control, medicine, finance, and other sciences. Fractional differential equations provide an excellent instrument for the description of memory and hereditary properties of a variety of materials and physical processes. This is the main advantage of models involving fractional operators in comparison to integer order models. The Fornberg-Whitham equation was originally introduced by Whitham [20] to study the qualitative behavior of wave breaking. The time-fractional Fornberg-Whitham equation have been widely studied by means of numerical methods [7, 17, 18].

Lie symmetry analysis can be efficiently used to obtain the analytical and exact solutions of the differential equations [2, 6, 8, 11, 15]. One of the possible extensions of Lie symmetry method is nonclassical method that is more general than classical ones. Recently, analytical solutions for Fornberg-Whitham equation with integer order have been obtained by the classical Lie symmetry analysis [3, 10]. Because of the nonlocal nature of fractional differential operators, the extension of existing Lie symmetry analysis method for differential equations with integer order to fractional differential equations is not a straightforward task. Analytical and exact solutions

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for differential equations with fractional order have been obtained by the classical Lie symmetry method [5, 9, 16, 19]. The authors in [1, 14] developed the nonclassical Lie symmetry analysis for differential equations involving fractional derivatives and this method applied to the time-fractional diffusion, Burger's, Airy's, KdV, gas dynamic and Fisher's equations.

In this study, we first intend to employ the classical and nonclassical Lie symmetry analysis to the time-fractional Fornberg-Whitham equation. This equation can be written as

$$D_t^{\alpha} u - u_{xxt} + u_x = u u_{xxx} - u u_x + 3 u_x u_{xx}, \qquad 0 < \alpha \leq 1, \tag{1.1}$$

where $u(x,t)$ is the fluid velocity. Our results provide new exact solutions to this equation. Utilizing nonclassical method, we achieve infinitesimal generators which are new in comparison with obtained symmetries by classical method. The rest of this paper is organized as follows: We recall some preliminaries on invariance analysis for fractional differential equations in Sect.2. In Sect.3, we obtain the new vector fields by classical and nonclassical Lie symmetry method, reduction of order and exact solutions for time-fractional Fornberg-Whitham equation.

2. Invariance analysis for fractional differential equations

In this section we briefly recall the classical and nonclassical Lie symmetry method for fractional differential equations. We first collect a number of definitions and propositions which are essential in our discussion.

Definition 2.1. Let $\alpha > 0$ and $m - 1 < \alpha \leq m$. The operator D_t^{α}, defined by

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-s)^{m-\alpha-1} f(s) ds,$$

is called the Riemann-Liouville fractional differential operator of order α, provided the right-hand side is defined for $t \in (0, T)$ and $\Gamma(\cdot)$ denotes the Gamma function [4].

Proposition 2.2. *(Leibniz's formula). Let $\alpha > 0$ and assume that f and g are analytic on $(-h, h)$ with some $h > 0$. Then,*

$$D_t^{\alpha}[fg](t) = \sum_{n=0}^{\infty} \binom{\alpha}{n} f^{(n)}(t) D_t^{\alpha-n} g(t),$$

for $0 < t < h/2$ [4].

Let us consider the general case of the time-fractional differential equation

$$F(x, t, u, u_x, u_{xx}, \ldots, D_t^{\alpha} u) = 0, \tag{2.1}$$

defined over $M \subset \mathbb{R}^2 \times \mathbb{R}$ and $\alpha > 0$. We consider a one-parameter Lie group of transformations G

$$\begin{aligned} \bar{x} &= x + \varepsilon \xi(x,t,u) + O(\varepsilon^2), \\ \bar{t} &= t + \varepsilon \tau(x,t,u) + O(\varepsilon^2), \\ \bar{u} &= u + \varepsilon \varphi(x,t,u) + O(\varepsilon^2), \end{aligned} \tag{2.2}$$

where ε is the group parameter. This symmetry group will be generated by infinitesimal generator of the form

$$V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \varphi(x,t,u)\frac{\partial}{\partial u}. \tag{2.3}$$

To obtain invariance of the fractional differential equation (2.1) in the classical Lie symmetry method, for every infinitesimal generator V of G we have

$$Pr^{(\alpha,t)}V(F)\big|_{F=0} = 0, \tag{2.4}$$

where

$$Pr^{(\alpha,t)}V = V + \varphi^{x}\frac{\partial}{\partial u_x} + \varphi^{xx}\frac{\partial}{\partial u_{xx}} + \cdots + \varphi^{(\alpha,t)}\frac{\partial}{\partial D_t^{\alpha}u},$$

with

$$\begin{aligned} \varphi^{x} &= \mathcal{D}_x(\varphi) - \mathcal{D}_x(\xi)u_x - \mathcal{D}_x(\tau)u_t,\\ \varphi^{xx} &= \mathcal{D}_x(\varphi^{x}) - \mathcal{D}_x(\xi)u_{xx} - \mathcal{D}_x(\tau)u_{xt},\\ &\cdots\\ \varphi^{(\alpha,t)} &= \mathcal{D}_t^{\alpha}\varphi + \xi D_t^{\alpha}u_x - \mathcal{D}_t^{\alpha}(\xi u_x) - \mathcal{D}_t^{\alpha+1}(\tau u) + \mathcal{D}_t^{\alpha}(\mathcal{D}_t(\tau)u) + \tau D_t^{\alpha+1}u. \end{aligned} \tag{2.5}$$

Here $\mathcal{D}_t^{\alpha}f$ and $\mathcal{D}_t f$ denote the total fractional derivative and the total derivative of f , w.r.t. t, respectively.

Remark 2.3. Because of the lower limit of integral in Definition 2.1 be invariant under the group of transformations (2.2), we have to assume that

$$\tau(x,t,u(x,t))|_{t=0} = 0.$$

Remark 2.4. Using Leibniz's formula for Riemann-Liouville fractional operator, we can write

$$\varphi^{(\alpha,t)} = \mathcal{D}_t^{\alpha}\varphi - \sum_{n=1}^{\infty}\binom{\alpha}{n}\mathcal{D}_t^{n}\xi D_t^{\alpha-n}u_x - \sum_{n=1}^{\infty}\binom{\alpha}{n}\mathcal{D}_t^{n}\tau D_t^{\alpha+1-n}u,$$

whose for $\alpha > 0$, $\mathcal{D}_t^{\alpha}\varphi$ can be computed using the chain rule,

$$\mathcal{D}_t^{\alpha}\varphi(x,t,u(x,t)) = D_t^{\alpha}\varphi + \varphi_u D_t^{\alpha}u - uD_t^{\alpha}\varphi_u + \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^{n}\varphi_u D_t^{\alpha-n}u + \mu,$$

with

$$\mu = \sum_{n=2}^{\infty}\binom{\alpha}{n}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{n}{m}\binom{k}{r}\frac{1}{k!}(-u)^{r}D_t^{m}u^{k-r}D_t^{n-m}(D_u^{k}\varphi),$$

where $\varphi_u = \frac{\partial\varphi}{\partial u}$ and $D_t^{m} = \frac{\partial^{m}}{\partial t^{m}}$.

According to the group of transformations (2.2), the idea of the nonclassical method is that fractional differential equation (2.1) is augmented with the invariant surface condition

$$\Lambda : \xi(x,t,u)u_x + \tau(x,t,u)u_t - \varphi(x,t,u) = 0,$$

and both these equations must be invariant under the group of transformations (2.2). Because for every ξ, τ and φ the invariant surface condition is always invariant under the group of transformations (2.2), then we have

$$Pr^{(\alpha)}V(F)|_{F=0,\Lambda=0} = 0, \tag{2.6}$$

for every infinitesimal generator V of G.

3. Symmetry group of the time-fractional Fornberg-Whitham equation

In this section, let us consider the time-fractional Fornberg-Whitham equation

$$\Delta : D_t^{\alpha}u - u_{xxt} + u_x - uu_{xxx} + uu_x - 3u_xu_{xx} = 0, \qquad 0 < \alpha \leq 1. \tag{3.1}$$

We now determine all classical infinitesimal generators (2.3) so that corresponding group of transformations (2.2) is a symmetry group of Eq. (3.1). According to (2.4) and applying $Pr^{(\alpha)}V$ to (3.1), we find that ξ, τ and φ must satisfy the symmetry conditions

$$\varphi\left(-u_{xxx}+u_x\right)+\varphi^x\left(1+u-3u_{xx}\right)-3\varphi^{xx}u_x-\varphi^{xxx}u-\varphi^{xxt}+\varphi^{(\alpha,t)}\big|_{\Delta=0} = 0. \tag{3.2}$$

Here φ^x, φ^{xx} and $\varphi^{(\alpha,t)}$ are defined by (2.5) and $\varphi^{xxx}, \varphi^{xxt}$ are

$$\begin{aligned}\varphi^{xxx} &= \mathcal{D}_x(\varphi^{xx}) - \mathcal{D}_x(\xi)u_{xxx} - \mathcal{D}_x(\tau)u_{xxt},\\ \varphi^{xxt} &= \mathcal{D}_t(\varphi^{xx}) - \mathcal{D}_t(\xi)u_{xxx} - \mathcal{D}_t(\tau)u_{xxt}.\end{aligned}$$

Substituting these expressions into (3.2) and replacing $D_t^{\alpha}u$ by $u_{xxt} - u_x + uu_{xxx} - uu_x + 3u_xu_{xx}$ wherever it occurs, we obtain the classical determining equations for the symmetry group. From the coefficients of $1, u_t, u_x, u_x^2, u_xu_t, \ldots$ we find that

$$\begin{aligned}
&1 : \varphi_x + \varphi_x u - \varphi_{xxx}u - \varphi_{xxt} = 0, \quad u_t : \tau_x + \tau_x u - \tau_{xxx}u + \varphi_{xxu} - \tau_{xxt} = 0,\\
&u_x : \varphi - \xi_x - \xi_x u - 3\varphi_{xx} - 3\varphi_{xxu}u + \xi_{xxx}u - 2\varphi_{xtu} + \xi_{xxt} + \alpha\tau_t u + \alpha\tau_t = 0,\\
&u_x^2 : \xi_u + \xi_u u + 6\varphi_{xu} - 3\xi_{xx} + 3\varphi_{xuu}u - 3\xi_{xxu}u + \varphi_{tuu} - 2\xi_{xtu} = 0,\\
&u_xu_t : -\tau_u - \tau_u u + 3\tau_{xx} + 3\tau_{xxu}u - 2\varphi_{xuu} + \xi_{xxu} + 2\tau_{xtu} + \alpha\tau_u u + \alpha\tau_u = 0,\\
&u_{xx} : 3\varphi_x + 3\varphi_{xu}u - 3\xi_{xx}u + \varphi_{tu} - 2\xi_{xt} = 0,\\
&u_tu_{xx} : 3\tau_x + 3\tau_{xu} - \varphi_{uu} + 2\xi_{xu} + \tau_{tu} = 0,
\end{aligned}$$

$$\begin{aligned}
&u_xu_{xx} : 3\xi_x - \varphi_u - \varphi_{uu}u + 3\xi_{xu}u + \xi_{tu} - \alpha\tau_t = 0,\\
&u_xu_tu_{xx} : (2-\alpha)\tau_u + \tau_{uu}u + \xi_{uu} = 0,\\
&u_x^2u_{xx} : 2\xi_u + \xi_{uu}u = 0, \qquad u_x^3 : 3\varphi_{uu} - 6\xi_{xu} + \varphi_{uuu}u - 3\xi_{xuu}u - \xi_{tuu} = 0,\\
&u_x^2u_t : 6\tau_{xu} + 3\tau_{xuu}u - \varphi_{uuu} + 2\xi_{xuu} + \tau_{tuu} = 0, \qquad u_x^4 : 3\xi_{uu} + \xi_{uuu}u = 0,\\
&u_x^3u_t : 3\tau_{uu} + \tau_{uuu}u + \xi_{uuu} = 0, \quad u_xu_{xt} : 3\tau_x + 3\tau_{xu}u - \varphi_{uu} + 2\xi_{xu} + \tau_{tu} = 0,\\
&u_x^2u_{xt} : 2\tau_u + \tau_{uu}u + \xi_{uu} = 0, \qquad u_{xt} : 3\tau_{xx}u - 2\varphi_{xu} + \xi_{xx} + 2\tau_{xt} = 0,
\end{aligned}$$

$$
\begin{aligned}
&u_{xxx}: 3\xi_x u + \xi_t - \alpha\tau_t u - \varphi = 0, \qquad u_{xxt}: 3\tau_x u + 2\xi_x + (1-\alpha)\tau_t = 0,\\
&u_{xx}^2: \xi_u = 0, \qquad u_x u_{xxx}: \xi_u u = 0, \qquad u_t u_{xxx}: (1-\alpha)\tau_u u + \xi_u = 0,\\
&u_{xx}u_{xt}: \tau_u u + \xi_u = 0, \quad u_x u_t^2: \tau_{xuu} = 0, \quad u_t^2: \tau_{xxu} = 0, \quad u_{tt}: \tau_{xx} = 0,\\
&u_x u_{tt}: \tau_{xu} = 0, \quad u_x^2 u_t^2: \tau_{uuu} = 0, \quad u_x^2 u_{tt}: \tau_{uu} = 0, \quad u_{xtt}: \tau_x = 0,\\
&u_{xt}^2: \tau_u = 0, \qquad D_t^\alpha \varphi - u D_t^\alpha \varphi_u = 0, \qquad \mathcal{D}_t^n \xi = 0, \quad n \in \mathbb{N},\\
&\binom{\alpha}{n+1}\mathcal{D}_t^{n+1}\tau - \binom{\alpha}{n} D_t^n \varphi_u = 0, \quad n \in \mathbb{N}.
\end{aligned}
$$

After solving the system of this determining equations and in view of Remark 2.3, we arrive at the following solution

$$
\xi = c_1, \qquad \tau = 0, \qquad \varphi = 0,
$$

where c_1 is arbitrary constant, i.e. the classical infinitesimal generator of Eq. (3.1) is $\frac{\partial}{\partial x}$. Using this infinitesimal generator with invariant solution $u(x,t) = f(t)$, Eq. (3.1) is reduced to the fractional ordinary differential equation $D_t^\alpha f(t) = 0$. The solution of this equation has the form $f(t) = ct^{\alpha-1}$ where c is arbitrary constant. To get more infinitesimal generators and, consequently, the new exact solutions of Eq. (3.1), we apply the nonclassical method. To obtain the determining equations, we reconsider invariant surface condition $\Lambda: \xi u_x + \tau u_t - \varphi = 0$. Applying $Pr^{(\alpha,t)}V$ to Eq.(3.1) and according to (2.6), the nonclassical invariance is given by

$$
\varphi\left(-u_{xxx}+u_x\right)+\varphi^x\left(1+u-3u_{xx}\right)-3\varphi^{xx}u_x-\varphi^{xxx}u-\varphi^{xxt}+\varphi^{(\alpha,t)}\big|_{\Delta=0,\Lambda=0}=0. \quad (3.3)
$$

We can distinguish two different cases: $\xi \neq 0$ and $\xi = 0$. In the case $\xi \neq 0$, without loss of generality, we may assume that $\xi = 1$ and hence we have

$$
\begin{aligned}
u_x &= \varphi - \tau u_t, \qquad u_{xt} = \varphi_t + (\varphi_u - \tau_t)u_t - \tau_u u_t^2 - \tau u_{tt},\\
u_{xx} &= (\varphi_x + \varphi\varphi_u - \tau\varphi_t) + (-2\tau\varphi_u - \tau_x - \tau_u\varphi + \tau\tau_t)u_t + 2\tau\tau_u u_t^2 + \tau^2 u_{tt},\\
u_{xtt} &= \varphi_{tt} + (2\varphi_{tu} - \tau_{tt})u_t + (\varphi_{uu} - 2\tau_{tu})u_t^2 - \tau_{uu}u_t^3 + (\varphi_u - 2\tau_t)u_{tt}\\
&\quad -3\tau_u u_t u_{tt} - \tau u_{ttt},\\
u_{xxt} &= \varphi_{xt} + \varphi\varphi_{tu} + \varphi_t\varphi_u - \tau_t\varphi_t - \tau\varphi_{tt} + [\varphi_{xu} + \varphi_u^2 + \varphi\varphi_{uu} - 2\tau_u\varphi_t - 3\tau\varphi_{tu}\\
&\quad -2\tau_t\varphi_u - \tau_{xt} - \tau_{tu}\varphi + \tau_t^2 + \tau\tau_{tt}]u_t - [3\tau_u\varphi_u + 2\tau\varphi_{uu} + \tau_{xu} + \tau_{uu}\varphi - 3\tau_t\tau_u\\
&\quad -3\tau\tau_{tu}]u_t^2 + [2\tau_u^2 + 2\tau\tau_{uu}]u_t^3 - [2\tau\varphi_u + \tau_x + \tau_u\varphi - 3\tau\tau_t]u_{tt} + \tau^2 u_{ttt},
\end{aligned}
$$

$$
\begin{aligned}
u_{xxx} &= \varphi_{xx} + 2\varphi\varphi_{xu} + \varphi_x\varphi_u + \varphi\varphi_u^2 + \varphi^2\varphi_{uu} - 2\tau\varphi_t\varphi_u - 2\tau_x\varphi_t - 2\tau_u\varphi\varphi_t\\
&\quad -\tau\varphi_{xt} - \tau\varphi\varphi_{tu} + \tau\tau_t\varphi_t + \tau^2\varphi_{tt} + [-3\tau\varphi_{xu} - 3\tau\varphi\varphi_{uu} - 3\tau\varphi_u^2 - 3\tau_x\varphi_u\\
&\quad -4\tau_u\varphi\varphi_u + 3\tau\tau_t\varphi_u - \tau_{xx} - 2\tau_{xu}\varphi + 2\tau_x\tau_t - \tau_{uu}\varphi^2 - \tau_u\varphi_x + 5\tau\tau_u\varphi_t\\
&\quad +2\tau_t\tau_u\varphi + 3\tau^2\varphi_{tu} + \tau\tau_{xt} + \tau\tau_{tu}\varphi - \tau\tau_t^2 - \tau^2\tau_{tt}]u_t + [3\tau^2\varphi_{uu} + 9\tau\tau_u\varphi_u\\
&\quad +3\tau\tau_{xu} + 3\tau_x\tau_u + 3\tau\tau_{uu}\varphi + 3\tau_u^2\varphi - 6\tau\tau_t\tau_u - 3\tau^2\tau_{tu}]u_t^2 - [3\tau^2\tau_{uu}\\
&\quad +6\tau\tau_u^2]u_t^3 + [3\tau^2\varphi_u + 3\tau\tau_x + 3\tau\tau_u\varphi - 3\tau^2\tau_t]u_{tt} - 9\tau^2\tau_u u_t u_{tt} - \tau^3 u_{ttt}.
\end{aligned}
$$

After substituting $\varphi^x, \varphi^{xx}, \varphi^{xxx}, \varphi^{xxt}$ into (3.3) with $\xi = 1$, then replacing $D_t^\alpha u$ by $u_{xxt} - u_x + uu_{xxx} - uu_x + 3u_x u_{xx}$ similar to the classical case and also re-substituting the expressions $u_x, u_{xt}, u_{xx}, u_{xtt}, u_{xxt}, u_{xxx}$ wherever it occurs, we obtain the nonclassical determining equations for the symmetry group. To analyze these, we work the order of the derivatives which appear. The coefficient of u_{tt}^2 is

$$\tau^2\tau_u(1-\tau u)=0.$$

But in view of Remark 2.3, $1-\tau u=0$ is impossible, therefore we conclude $\tau=0$ or $\tau_u=0$. If $\tau=0$, we equate to zero the coefficients of the remaining partial derivatives of u and obtain the system of determining equations:

$$\begin{aligned}
1:\ &\varphi_x+\varphi_x u-\varphi_{xxx}u-\varphi_{xxt}+\varphi^2-4\varphi\varphi_{xx}-3\varphi\varphi_{xxu}u-2\varphi\varphi_{xtu}-8\varphi^2\varphi_{xu}\\
&-3\varphi^2\varphi_{xuu}u-\varphi^2\varphi_{tuu}-3\varphi_x^2-7\varphi\varphi_x\varphi_u-3\varphi_x\varphi_{xu}u-3\varphi\varphi_u\varphi_{xu}u\\
&-\varphi_x\varphi_{tu}-\varphi\varphi_u\varphi_{tu}-4\varphi^2\varphi_u^2-3\varphi\varphi_x\varphi_{uu}u-3\varphi^2\varphi_u\varphi_{uu}u-3\varphi^3\varphi_{uu}\\
&-\varphi^3\varphi_{uuu}u-2\varphi\varphi_t\varphi_{uu}-2\varphi_t\varphi_{xu}-\varphi^3\varphi_{uu}=0,\\
u_t:\ &\varphi_{xxu}+2\varphi\varphi_{xuu}+\varphi_x\varphi_{uu}+3\varphi\varphi_u\varphi_{uu}+\varphi^2\varphi_{uuu}+2\varphi_u\varphi_{xu}=0,\\
&D_t^\alpha\varphi-uD_t^\alpha\varphi_u=0,\qquad D_t^n\varphi_u=0,\quad n\in\mathbb{N}.
\end{aligned}$$

After solving the resulting system, we conclude that $\varphi=\pm\frac{1}{2}u$. To illustrate the invariant solution, Assuming $\varphi=\frac{1}{2}u$ and in view of the invariant surface condition, we have $u_x=\frac{1}{2}u$. Differentiating this equation with respect to x leads to the result

$$u_{xx}=\frac{1}{4}u,$$

the general solution of this equation is

$$u(x,t)=e^{-\frac{1}{2}x}f(t)+e^{\frac{1}{2}x}g(t),$$

where f,g are arbitrary functions. Substituting this solution into the fractional Fornberg-Whitham equation (3.1), the reduced fractional ordinary differential equations are

$$D_t^\alpha f(t)-\tfrac{1}{4}f'(t)-\tfrac{1}{2}f(t)=0, \tag{3.4}$$

$$D_t^\alpha g(t)-\tfrac{1}{4}g'(t)+\tfrac{1}{2}g(t)=0. \tag{3.5}$$

The solutions of these equations can be written as

$$\begin{aligned}
f(t)&=c_2\textstyle\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{2}\right)^{k+n}\frac{t^{(n+1)\alpha-k-1}}{\Gamma[(n+1)\alpha-k]},\\
g(t)&=c_3\textstyle\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\left(-\frac{1}{2}\right)^{k+n}\frac{t^{(n+1)\alpha-k-1}}{\Gamma[(n+1)\alpha-k]},
\end{aligned}$$

where c_2,c_3 are arbitrary constants. Thus

$$u(x,t)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{2}\right)^{k+n}\left[c_2e^{-\frac{1}{2}x}+c_3(-1)^{k+n}e^{\frac{1}{2}x}\right]\frac{t^{(n+1)\alpha-k-1}}{\Gamma[(n+1)\alpha-k]},$$

is the invariant solution of Eq. (3.1).
However, if we let $\alpha=\frac{1}{2}$, the solutions of equations (3.4)-(3.5) can be written in

terms of the error function as

$$f(t)=(2+\sqrt{2})e^{(6+4\sqrt{2})t}\mathrm{erf}\left[-(2+\sqrt{2})t^{\frac{1}{2}}\right]-(2-\sqrt{2})e^{(6-4\sqrt{2})t}\mathrm{erf}\left[-(2-\sqrt{2})t^{\frac{1}{2}}\right],$$

$$g(t)=(2+\sqrt{6})e^{(10+4\sqrt{6})t}\mathrm{erf}\left[-(2+\sqrt{6})t^{\frac{1}{2}}\right]-(2-\sqrt{6})e^{(10-4\sqrt{6})t}\mathrm{erf}\left[-(2-\sqrt{6})t^{\frac{1}{2}}\right],$$

[13] where the error function erf defined by

$$\mathrm{erf}(t)=\frac{2}{\sqrt{\pi}}\int_0^t e^{-s^2}ds.$$

Then the invariant solution of Eq. (3.1) is

$$\begin{aligned}u(x,t)=e^{-\frac{1}{2}x}&\Big\{(2+\sqrt{2})e^{(6+4\sqrt{2})t}\mathrm{erf}\left[-(2+\sqrt{2})t^{\frac{1}{2}}\right]-(2-\sqrt{2})e^{(6-4\sqrt{2})t}\\ &\times\mathrm{erf}\left[-(2-\sqrt{2})t^{\frac{1}{2}}\right]\Big\}+e^{\frac{1}{2}x}\Big\{(2+\sqrt{6})e^{(10+4\sqrt{6})t}\mathrm{erf}\left[-(2+\sqrt{6})t^{\frac{1}{2}}\right]\\ &-e^{\frac{1}{2}x}(2-\sqrt{6})e^{(10-4\sqrt{6})t}\mathrm{erf}\left[-(2-\sqrt{6})t^{\frac{1}{2}}\right]\Big\}.\end{aligned}$$

If also $\tau \neq 0$ and $\tau_u = 0$, unfortunately the set of determining equations for the nonclassical symmetry group of Eq. (3.1) are incompatible.
In the case $\xi = 0$ and $\tau \neq 0$, similar to the previous case, we have $\tau u_t = \varphi$. Differentiating this equation and getting $u_{tt}, u_{xt}, u_{xtt}, u_{xxt}$ and also re-substituting these expressions wherever it occurs, we obtain the nonclassical determining equations which are difficult to solve.

4. Conclusions

The method of classical and nonclassical Lie symmetry analysis are applied to time-fractional Fornberg-Whitham equation. By employing these methods and some technical calculations, new infinitesimal generators are obtained for the equation. We have demonstrated that for time-fractional Fornberg-Whitham equation, the nonclassical method is more general than the classical Lie symmetry method; it means that this method gives new solutions which have not obtained by the classical method.

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