Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 9, No. 2, 2021, pp. 347-357
DOI:10.22034/cmde.2020.30733.1459

Necessary and sufficient conditions for M-stationarity of nonsmooth optimization problems with vanishing constraints

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Abstract We consider a nonsmooth optimization problem with a feasible set defined by vanishing constraints. First, we introduce a constraint qualification for the problem, named NNAMCQ. Then, NNAMCQ is applied to obtain a necessary M-stationary condition. Finally, we present a sufficient condition for M-stationarity, under generalized convexity assumption. Our results are formulated in terms of Mordukhovich subdifferential.

Keywords. Stationary conditions, Vanishing constraints, Nonsmooth optimization, Constraint qualification.

2010 Mathematics Subject Classification. 65L05, 34K06, 34K28.

1. Introduction

Kanzow and his coauthors in 2007 [1, 8] have been introduced a very complicated group of optimization problems, which was named "Mathematical programming with vanishing constraints" (MPVC in brief). They formulized the problem as

$$\begin{array}{lll} \min & u(x) & \\ s.t. & v_\alpha(x) \le 0 & \alpha \in \Delta, \\ & w_\beta(x) = 0 & \beta \in \Omega, \\ & p_\gamma(x) \ge 0, & \gamma \in \Gamma, \\ & p_\gamma(x) q_\gamma(x) \le 0, & \gamma \in \Gamma, \end{array}$$

in which the functions $u,\ v_\alpha,\ w_\beta,\ p_\gamma,\ q_\gamma$ are continuously differentiable from $\mathbb{R}^n$ to $\mathbb{R}$, and index sets $\Delta,\ \Omega,\ \Gamma$ are finite.

Received: 05 December 2018 ; Accepted: 10 October 2019.
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As we know, the product function $p_\gamma q_\gamma$ is not usually convex, even when p_γ and q_γ are convex. Consequently, the feasible set of MPVC is not convex and unlike the convex sets, has several normal cones; for instance, Frechet, Clark, and Proximal normal cones (see, e.g. [13]). These normal cones are applied for presenting several stationary conditions. One of the most important normal cones for a nonconvex set is its Mordukhovich normal cone. M-stationary condition is a stationary condition that is based on Mordukhovich normal cone. This kind of stationary condition for MPVCs is studied in [6, 7]. Our first aim is to consider the MPVCs with nonsmooth functions, and to provide M-stationary condition for its optimal solution.To the best of our knowledge, there is no reference that studies nonsmooth MPVCs. Since the inequality constraints $v_\alpha(x) \leq 0$ and equality constraints $w_\beta(x) = 0$ do not add to the complexity of the problem and just prolong the formulas, we ignore them and only deal with the following problem:

$$
\begin{array}{llll}
(P) & \min & f(x) & \\
 & s.t. & H_i(x) \geq 0, & i \in I := \{1, ..., m\}, \\
 & & G_i(x)H_i(x) \leq 0, & i \in I,
\end{array}
$$

where $f, H_i, G_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ (for $i \in I$) are locally Lipschitz functions. Another problem which is similar to (P) is the following mathematical problem with equilibrium constraints (briefly, MPEC):

$$
\begin{array}{llll}
(P^*) & \min & f(x) & \\
 & s.t. & H_i(x) \geq 0,\ G_i(x) \geq 0, & i \in I, \\
 & & G_i(x)H_i(x) = 0, & i \in I.
\end{array}
$$

Movahedian and Nobakhtian [11] presented a M-stationary condition for optimal solution of nonsmooth MPEC. For extension of their result to (P), we need to introduce a suitable constraint qualification (CQ in short). We will discuss this in section 3. Our next aim is to show that the necessary M-stationarity is also sufficient under certain convexity assumptions.

The structure of subsequent sections of this paper is as follows: in section 2, we present the definitions, theorems and relations of non-smooth analysis. In section 3, we introduce a CQ for (P), named NNAMCQ. Also, M-stationary necessary condition is provided in section 3. Section 4 contains an important sufficient M-stationary condition for optimal solution.

2. Notations and preliminaries

In this section we present some preliminary results on nonsmooth analysis from [13]. We assume $\varphi : \mathbb{R}^p \to \mathbb{R}$ is a locally Lipschitz function, and $x_0 \in \mathbb{R}^p$.

The set

$$
\partial_* \varphi(x_0) := \Big\{ \xi \in \mathbb{R}^n \mid \liminf_{x \to x_0} \frac{\varphi(x) - \varphi(x_0) - \langle \xi, x - x_0 \rangle}{\|x - x_0\|} \geq 0 \Big\},
$$

is called the Fréchet subdifferential of φ at x_0. The Mordukhovich (or limiting, or basic) subdifferential of φ at x_0 is defined as

$$\partial\varphi(x_0) := \limsup_{x\to x_0} \partial_*\varphi(x).$$

We observe that for two locally Lipschitz functions φ_1 and φ_2 from $\mathbb{R}^p$ to $\mathbb{R}$, and for two arbitrary real numbers α and β, the following subadditive formula holds:

$$\partial\big(\alpha\varphi_1 + \beta\varphi_2\big)(x_0) \subseteq \alpha\partial\varphi_1(x_0) + \beta\partial\varphi_2(x_0). \tag{2.1}$$

Notice that the subdifferential $\partial\varphi(x_0)$ is always a compact (not necessarily convex) subset of $\mathbb{R}^p$. Also, always one has

$$\partial\big(-\varphi\big)(x_0) \subseteq -\partial\varphi(x_0). \tag{2.2}$$

The following theorem is useful in what follows.

Theorem 1. [13] *If x_0 is a local minimizer of φ on $\mathbb{R}^p$, then one has $0 \in \partial\varphi(x_0)$.*

Recall also that the normal cone of a closed subset $\mathcal{A} \subseteq \mathbb{R}^p$ at $x_0 \in \mathcal{A}$ is defined by $N(\mathcal{A}, x_0) := \partial\Theta_{\mathcal{A}}(x_0)$, where $\Theta_{\mathcal{A}}(.)$ denotes the indicator function of $\mathcal{A}$, i.e., $\Theta_{\mathcal{A}}(x) := 0$ for $x \in \mathcal{A}$, and $\Theta_{\mathcal{A}}(x) := +\infty$ otherwise.

Let $M : \mathbb{R}^r \rightrightarrows \mathbb{R}^s$ be a set-valued function, and

$$(\overline{y}, \overline{x}) \in \mathrm{Gph}M := \{(y, x) \in \mathbb{R}^r \times \mathbb{R}^s \mid x \in M(y)\}.$$

We say that M is calm at $(\overline{y}, \overline{x})$ if there exist some $L > 0$ and neighborhoods $\mathcal{X}$ and $\mathcal{Y}$ around $\overline{x}$ and $\overline{y}$, respectively, such that

$$d_{M(\overline{y})}(x) \leq L\|y - \overline{y}\|, \qquad \forall y \in \mathcal{Y},\ \forall x \in \mathcal{X} \cap M(y),$$

where, $d_B(a) := \inf_{b\in B} \|a - b\|$ denotes the point-to-set distance between $a \in \mathbb{R}^s$ to $B \subseteq \mathbb{R}^s$ induced by the standard norm $\|.\|$ on $\mathbb{R}^s$.

Also, we associate Mordukhovich coderivative to M as $D^*M(\overline{y}, \overline{x}) : \mathbb{R}^s \rightrightarrows \mathbb{R}^r$ defined by

$$D^*M(\overline{y}, \overline{x})(x^*) := \{y^* \in \mathbb{R}^r \mid (y^*, -x^*) \in N(\mathrm{Gph}M, (\overline{y}, \overline{x}))\}.$$

If M is single-valued, we simply write $D^*M(\overline{y})$ instead of $D^*M(\overline{y}; M\overline{x})$. For single-valued locally Lipschitz function h, it holds as

$$D^*h(\overline{y})(x^*) = \partial_M\langle x^*, h\rangle(\overline{y}), \tag{2.3}$$

where
$\langle x^*, h\rangle(y) := \sum_{k=1}^{s} x_k^* h_k(y)$ for $x^* = (x_1^*, \ldots, x_s^*)$ and $h(y) = (h_1(y), \ldots, h_s(y))$.

Suppose that the set-valued mapping $\widetilde{M} : \mathbb{R}^l \rightrightarrows \mathbb{R}^k$ is defined as

$$\widetilde{M}(y) := \{x \in \widetilde{C} \mid \widetilde{g}(x) + y \in \widetilde{E}\}, \tag{2.4}$$

where the function $\widetilde{g} : \mathbb{R}^k \to \mathbb{R}^l$ is locally Lipschitz and $(\widetilde{C}, \widetilde{E}) \subseteq \mathbb{R}^k \times \mathbb{R}^l$ is closed. The following important theorem will be used in sequel.

Theorem 2. [5, Theorem 4.1] *Consider the multifunction $\widetilde{M}$ given by (2.4) and a pair $(0,\overline{x}) \in Gph\widetilde{M}$. If $\widetilde{M}$ is calm at $(0,\overline{x})$, then*

$$N\big(\widetilde{M}(0),\overline{x}\big) \subseteq \bigcup_{y^* \in N(\widetilde{E},\widetilde{g}(\overline{x}))} D^*\widetilde{g}(\overline{x})(y^*) + N(\widetilde{C},\overline{x}).$$

3. Necessary Condition

At the beginning of this section, we denote the feasible set of (P) by S, i.e.,

$$S := \{x \in \mathbb{R}^n \mid H_i(x) \geq 0, \quad G_i(x)H_i(x) \leq 0, \quad i \in I\}.$$

Throughout this paper, we fix a feasible set $\hat{x} \in S$. Also, we consider following index sets, depending to $\hat{x}$:

$$\begin{aligned}
I_{+0} &:= \{i \in I \mid H_i(\hat{x}) > 0,\ G_i(\hat{x}) = 0\},\\
I_{+-} &:= \{i \in I \mid H_i(\hat{x}) > 0,\ G_i(\hat{x}) < 0\},\\
I_{0+} &:= \{i \in I \mid H_i(\hat{x}) = 0,\ G_i(\hat{x}) > 0\},\\
I_{00} &:= \{i \in I \mid H_i(\hat{x}) = 0,\ G_i(\hat{x}) = 0\},\\
I_{0-} &:= \{i \in I \mid H_i(\hat{x}) = 0,\ G_i(\hat{x}) < 0\}.
\end{aligned}$$

Obviously, we can write I as $I = I_0 \cup I_+$ in which $I_+ := I_{+0} \cup I_{+-}$ and $I_0 := I_{0+} \cup I_{00} \cup I_{0-}$.

Now, we introduce a mention that plays a key role in this paper. The mention of M-stationary point for smooth MPVC is introduced in [6, 7]. Here, we introduce a nonsmooth version of M-stationary point.

Definition 1. *The feasible point $\hat{x}$ is said to be M-stationary point for (P) when there exists a vector $\mu := (\mu_1^G, \mu_1^H, \ldots, \mu_m^G, \mu_m^H) \in \mathbb{R}^{2m}$ such that*

$$0 \in \partial f(\hat{x}) + \sum_{i=1}^{m} \left[\mu_i^G \partial G_i(\hat{x}) - \mu_i^H \partial H_i(\hat{x})\right], \tag{3.1}$$

$$\mu_i^G \geq 0,\ i \in I_{00} \cup I_{+0}; \qquad \mu_i^G = 0,\ i \in I_{0+} \cup I_{0-} \cup I_{+0}, \tag{3.2}$$

$$\mu_i^H \ \textit{free},\ i \in I_{00} \cup I_{0+}; \qquad \mu_i^H \geq 0,\ i \in I_{0-}; \qquad \mu_i^H = 0,\ i \in I_+, \tag{3.3}$$

$$\mu_i^H \mu_i^G = 0, \quad i \in I_{00}. \tag{3.4}$$

It is worth mentioning that the above definition extends the nonsmooth M-stationarity which is presented in [11] for nonsmooth MPECs.

For simplicity in writing, we rearrange the constraints of problem (P) such that the constraints with index $i \in I_{00}$ are first written, then the constraints with index $i \in I_{0+}$, then $i \in I_{0-}$, then $i \in I_{+0}$, and finally $i \in I_{+-}$. We keep this order throughout this paper. Also, we assume that

$$\Psi(x) := \big(G_1(x), H_1(x), \ldots, G_m(x), H_m(x)\big).$$

Now, we consider the following parameterized problem which is parameterized with respect to $y \in \mathbb{R}^{2m}$:

$$\begin{aligned} \widehat{P}(y): \qquad & \min f(x) \\ & s.t.\ \Psi(x) + y \in \mathfrak{D} \\ & x \in \mathbb{R}^n, \end{aligned}$$

in which $\mathfrak{D} := \{(p, q) \in \mathbb{R}^m \times \mathbb{R}^m \mid q_i \geq 0 \text{ and } p_i q_i \leq 0, \quad \forall i \in I\}$.
Obviously, $\widehat{P}(0)$ coincides to problem (P). We denote the feasible set of $\widehat{P}(y)$ by $\widehat{S}(y)$, i.e.,

$$\widehat{S}(y) := \{x \in \mathbb{R}^n \mid \Psi(x) + y \in \mathfrak{D}\}.$$

Therefore, we can consider $\widehat{S}(.)$ as a set-valued mapping from $\mathbb{R}^{2m}$ to $\mathbb{R}^n$.

Theorem 3. *Suppose that $\hat{x}$ is an optimal solution of problem (P) which the mapping $\widehat{S}$ is calm at $(0, \hat{x})$, then*

$$N(S, \hat{x}) \subseteq \bigcup_{\lambda \in N(\mathfrak{D}, \Psi(\hat{x}))} \left[\sum_{i=1}^{m} \left(\lambda_i^H \partial H_i(\hat{x}) + \lambda_i^G \partial G_i(\hat{x}) \right) \right],$$

where $\lambda = \left(\lambda_1^H, \lambda_1^G, \ldots, \lambda_m^H, \lambda_m^G \right)$.

Proof. Taking $\widetilde{g}(x) = \Psi(x)$, $\widetilde{M} = \widehat{S}$, $\widetilde{E} = \mathfrak{D}$, $\widetilde{C} = \mathbb{R}^n$, and $\overline{x} = \hat{x} \in S = \widehat{S}(0)$ in Theorem 2, we deduce that

$$N(S, \hat{x}) \subseteq \bigcup_{\lambda \in N(\mathfrak{D}, \Psi(\hat{x}))} D^* \Psi(\hat{x})(\lambda) + N(\mathbb{R}^n, \hat{x}). \tag{3.5}$$

On the other hand, according to (2.3), for each $\lambda := \left(\lambda_1^H, \lambda_1^G, \ldots, \lambda_m^H, \lambda_m^G \right) \in \mathbb{R}^{2m}$ we have

$$\begin{aligned} D^* \Psi(\hat{x})(\lambda) = \partial \langle \lambda, \Psi(.) \rangle (\hat{x}) &= \partial \left[\sum_{i=1}^{m} \left(\lambda_i^H H_i + \lambda_i^G G_i \right) \right] (\hat{x}) \\ &\subseteq \sum_{i=1}^{m} \left[\lambda_i^H \partial H_i(\hat{x}) + \lambda_i^G \partial G_i(\hat{x}) \right], \end{aligned}$$

in which the last inclusion is written regarding to (2.1).
From the above relation and (3.5) and the fact that $N(\mathbb{R}^n, \hat{x}) = \{0\}$, we conclude that

$$N(S, \hat{x}) \subseteq \bigcup_{\lambda \in N(\mathfrak{D}, \Psi(\hat{x}))} \left[\sum_{i=1}^{m} \left(\lambda_i^H \partial H_i(\hat{x}) + \lambda_i^G \partial G_i(\hat{x}) \right) \right].$$

□

The following theorem states a necessary condition for M-stationarity of (P).

Theorem 4. *Suppose that $\hat{x}$ is an optimal solution of (P). If the set-valued mapping $\widehat{S}$ is calm at $(0, \hat{x})$, then $\hat{x}$ is a M-stationary point for (P).*

Proof. First, we would mention that the minimality of f on S at $\hat{x}$ concludes that $\hat{x}$ is a minimizer of $f + \Theta_S$ on $\mathbb{R}^n$. Applying Theorem 1, virtue of (2.1) and definition of normal cone, we deduce that

$$0 \in \partial(f + \Theta_S)(\hat{x}) \subseteq \partial f(\hat{x}) + \partial\Theta_S(\hat{x}) = \partial f(\hat{x}) + N(S, \hat{x}).$$

From this and Theorem 3 we can find a $\lambda := (\lambda_1^H, \lambda_1^G, \ldots, \lambda_m^H, \lambda_m^G) \in N(\mathfrak{D}, \Psi(\hat{x}))$ such that

$$0 \in \partial f(\hat{x}) + \sum_{i=1}^{m} \left[\lambda_i^H \partial H_i(\hat{x}) + \lambda_i^G \partial G_i(\hat{x})\right].$$

On the other hand, by [6, Lemma 3.2] we have

$$\begin{aligned} N(\mathfrak{D}, \Psi(\hat{x})) = &\prod_{i\in I_{00}} \mathfrak{B} \times \prod_{i\in I_{0+}} (\{0\}\times\mathbb{R}) \times \prod_{i\in I_{0-}} (\{0\}\times\mathbb{R}_-) \\ &\times \prod_{i\in I_{+0}} (\mathbb{R}_+\times\{0\}) \times \prod_{i\in I_{+-}} (\{0\}\times\{0\}), \end{aligned} \tag{3.6}$$

where $\mathfrak{B} := \{(r, s) \in \mathbb{R}^2 \mid r \geq 0,\ rs = 0\}$. As a result

$$\begin{gathered} \lambda_i^G \geq 0,\ i \in I_{00} \cup I_{+0}; \qquad \lambda_i^G = 0,\ i \in I_{0+} \cup I_{0-} \cup I_{+0}, \\ \lambda_i^H \text{ free},\ i \in I_{00} \cup I_{0+}; \qquad \lambda_i^H \leq 0,\ i \in I_{0-}; \qquad \lambda_i^H = 0, i \in I_{+0} \cup I_{+-}, \\ \lambda_i^H \lambda_i^G = 0, \quad i \in I_{00}. \end{gathered}$$

Taking $\mu_i^G := \lambda_i^G$, for each $i \in I$, and

$$\mu_i^H := \begin{cases} -\lambda_i^H, & i \in I_{00}, \\ \lambda_i^H, & i \in I \setminus I_{00}, \end{cases}$$

the result is justified. □

According to Theorems 3 and 4, finding conditions that lead to calmness of mapping $\widetilde{S}(.)$ has great importance. Considering another important applications of calmness, many authors have placed the subject of their research on sufficient conditions for calmness; see e.g., [4, 5, 10, 13, 15]. The following theorem will be useful in this sequel.

Theorem 5. [5, Corollary 3.4] *Consider the set-valued function* $\widehat{M} : \mathbb{R}^p \rightrightarrows \mathbb{R}^k$,

$$\widehat{M}(y) := \{x \in \widehat{C} \mid \widehat{g}(x, y) \in \widehat{E}\},$$

where $\widehat{g} : \mathbb{R}^k \times \mathbb{R}^p \to \mathbb{R}^q$ *is locally Lipschitz and* $\widehat{E} \subseteq \mathbb{R}^q$, $\widehat{C} \subseteq \mathbb{R}^k$ *are closed. Let* $(\bar{y}, \bar{x}) \in Gph\widehat{M}$ *and* $\widehat{C}$ *be regular and semismooth at* $\bar{x}$ *(in the sense of* [5, Definition 2.2]*). Further, assume the following qualification condition holds,*

$$\bigcup_{z^* \in N(\widehat{E}, \widehat{g}(\bar{x}, \bar{y}))\setminus\{0\}} [\partial\langle z^*, \widehat{g}\rangle(\overline{x}, \overline{y})]_x \cap -bd\ N(\widehat{C}, \bar{x}) = \emptyset, \tag{3.7}$$

where $[\ ]_x$ *denotes projection onto the x-component. Then* $\widehat{M}$ *is calm at* $(\bar{y}, \bar{x})$*. It is noteworthy that bd A denotes the topological bound of A.*

Now, we introduce a new CQ for problem (P).

Definition 2. *We say that the "No Nonzero Abnormal Multiplier Constraint Qualification" (NNAMCQ in brief) is satisfied at $\hat{x}$ if the following implication holds:*

$$\left.\begin{array}{ll} 0 \in \sum\limits_{i\in I_{00}\cup I_{+0}} \alpha_i \partial G_i(\hat{x}) - \sum\limits_{i\in I_0} \beta_i \partial H_i(\hat{x}), & \\ \alpha_i \geq 0, & \forall i \in I_{00}\cup I_{+0}, \\ \beta_i \geq 0, & \forall i \in I_{0-}, \\ \alpha_i\beta_i = 0, & \forall i \in I_{00}, \end{array}\right\} \Rightarrow \left\{ \begin{array}{l} \alpha_i = 0,\ \forall i \in I_{00}\cup I_{+0}, \\ \beta_i = 0,\ \forall i \in I_0. \end{array}\right.$$

The NNAMCQ has been introduced by Ye [16] for smooth MPECs (i.e., MPECs with continuously differentiable data). Movahedian and Nobakhtian [11] extended this concept to nonsmooth MPECs. To the best of our knowledge, there is no work available dealing with NNAMCQ for MPVCs, even in smooth case.

Theorem 6. *Suppose that $\hat{x}$ is an optimal solution of (P). If NNAMCQ is satisfied at $\hat{x}$, then $\hat{x}$ is a M-stationary point for (P).*

Proof. According to (3.6), the assumption of NNAMCQ at $\hat{x}$ can be rewritten as

$$\left.\begin{array}{l} 0 \in \sum\limits_{i\in I} \left[\lambda_i^G \partial G_i(\hat{x}) + \lambda_i^H \partial H_i(\hat{x})\right], \\ \lambda := \left(\lambda_1^G, \lambda_1^H, \ldots, \lambda_m^G, \lambda_m^H\right) \in N\left(\mathfrak{D}, \Psi(\hat{x})\right), \end{array}\right\} \Longrightarrow \lambda = 0,$$

which, owing to (2.1), implies that

$$\left.\begin{array}{l} 0 \in \partial\Big[\sum\limits_{i\in I} \left(\lambda_i^G G_i + \lambda_i^H H_i\right)\Big](\hat{x}), \\ \lambda \in N\left(\mathfrak{D}, \Psi(\hat{x})\right), \end{array}\right\} \Longrightarrow \lambda = 0.$$

The above implication yields

$$0 \notin \bigcup_{0\neq\lambda\in N(\mathfrak{D},\Psi(\hat{x}))} \left[\partial\left(\langle \lambda, \Psi(x) + y\rangle\right)(\hat{x}, 0)\right]_x, \tag{3.8}$$

where y is a variable in $\mathbb{R}^{2m}$. Obviously, (3.8) is equivalent to (3.7) by taking $p = q := 2m,\ k := n,\ \widehat{C} := \mathbb{R}^n,\ \widehat{E} := \mathfrak{D},\ \widehat{g}(x, y) := \Psi(x) + y,\ \bar{x} := \hat{x}$ and $\bar{y} := 0$. Therefore, Theorem 5 implies that $\widehat{M}(= \widehat{S})$ is calm at $(0, \hat{x})$, and Theorem 4 completes the proof. □

4. Sufficient Condition

We know from classic nonlinear optimization that necessary optimality conditions are also to be sufficient under certain convexity assumptions. These results can not be applied for (P) since the product function H_iG_i does not satisfy any convexity assumptions, even when H_i and G_i are convex. In this section, we show that M-stationarity is also sufficient optimality condition for nonsmooth MPVC (P), provided that some convexity assumptions hold for data functions $f,\ G_i,\ H_i\ (i \in I)$.

In the remainder of this paper, we shall need the following definition from [14].

Definition 3. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function.*

(i): *φ is said to be ∂-quasiconvex at $\hat{x}$ if for all $x \in \mathbb{R}^n$,*

$$\varphi(x) \leq \varphi(\hat{x}) \Longrightarrow \langle \xi, x - \hat{x} \rangle \leq 0, \qquad \forall \xi \in \partial\varphi(\hat{x}).$$

(ii): *φ is said to be ∂-pseudoconvex at $\hat{x}$ if for all $x \in \mathbb{R}^n$,*

$$\varphi(x) < \varphi(\hat{x}) \Longrightarrow \langle \xi, x - \hat{x} \rangle < 0, \qquad \forall \xi \in \partial\varphi(\hat{x}).$$

It is worth mentioning that some important properties of ∂-quasiconvex and\or ∂-pseudoconvex functions are presented in [14]. Also, they provided some comparisons of these mentions with classic quasiconvex and\or pseudoconvex functions in convex analysis.

The following sufficient condition for M-stationarity is the nonsmooth version of [8, Theorem 3.3].

Theorem 7. *Let $\hat{x}$ be a M-stationary point for (P) with corresponding multipliers $\mu \in \mathbb{R}^{2m}$ satisfying (3.1)-(3.4). We divide the constraint index sets as follows:*

$$I_{00}^{+} := \{i \in I_{00} \mid \mu_i^H > 0\}, \qquad I_{00}^{-} := \{i \in I_{00} \mid \mu_i^H < 0\},$$
$$I_{0-}^{+} := \{i \in I_{0-} \mid \mu_i^H > 0\},$$
$$I_{0+}^{+} := \{i \in I_{0+} \mid \mu_i^H > 0\}, \qquad I_{0+}^{-} := \{i \in I_{0+} \mid \mu_i^H < 0\},$$
$$I_{+0}^{0+} := \{i \in I_{+0} \mid \mu_i^H = 0,\ \mu_i^G > 0\},$$
$$I_{00}^{0+} := \{i \in I_{00} \mid \mu_i^H = 0,\ \mu_i^G > 0\}.$$

Suppose that f is ∂-pseudoconvex at $\hat{x}$. Furthermore, assume that G_i $(i \in I_{+0}^{0+})$, H_i $(i \in I_{0+}^{-})$, $-H_i$ $(i \in I_{0+}^{+} \cup I_{00}^{+} \cup I_{0-}^{+})$ are ∂-quasiconvex. Then, in the case when $I_{0+}^{-} \cup I_{00}^{-} \cup I_{+0}^{0+} \cup I_{00}^{0+} = \emptyset$, $\hat{x}$ is a global minimizer of (P); in the case when $I_{00}^{-} \cup I_{00}^{0+} = \emptyset$, $\hat{x}$ is a local minimizer of (P).

Proof. Owing to the virtue (3.1), we can find some vectors $\xi^f \in \partial f(\hat{x})$, $\xi_i^H \in \partial H_i(\hat{x})$ and $\xi_i^G \in \partial G_i(\hat{x})$, for $i \in I$, such that

$$\xi^f + \sum_{i=1}^{m}(-\lambda_i^H \xi_i^H + \lambda_i^G \xi_i^G) = 0. \tag{4.1}$$

Let $x^\star$ be any feasible point for (P). Since $-H_i(x^\star) \leq 0 = -H_i(\hat{x})$ for $i \in I_{0+}^{+} \cup I_{00}^{+} \cup I_{0-}^{+}$, the ∂-quasiconvexity of $-H_i$ implies that

$$\langle \zeta_i, x^\star - \hat{x} \rangle \leq 0, \qquad \forall \zeta_i \in \partial(-H_i)(\hat{x}),\ i \in I_{0+}^{+} \cup I_{00}^{+} \cup I_{0-}^{+}. \tag{4.2}$$

Taking $\varphi = -H_i$ in inclusion (2.2), we obtain that $\partial H_i(\hat{x}) \subseteq -\partial(-H_i)(\hat{x})$, and hence $\xi_i^H = -\zeta_i$ for some $\zeta_i \in \partial(-H_i)(\hat{x})$. From this, inequality (4.2), and positivity of μ_i^H for $i \in I_{0+}^{+} \cup I_{00}^{+} \cup I_{0-}^{+}$, we get

$$\begin{aligned} &\langle -\xi_i^H, x^\star - \hat{x} \rangle \leq 0, \quad \forall i \in I_{0+}^{+} \cup I_{00}^{+} \cup I_{0-}^{+} \\ &\Rightarrow \sum_{i \in I_{0+}^{+} \cup I_{00}^{+} \cup I_{0-}^{+}} -\mu_i^H \langle \xi_i^H, x^\star - \hat{x} \rangle \leq 0. \end{aligned} \tag{4.3}$$

In the case when $I_{0+}^{-} \cup I_{00}^{-} \cup I_{+0}^{0+} \cup I_{00}^{0+} = \emptyset$, we obviously have

$$I = I_{00}^{+} \cup I_{00}^{-} \cup I_{0+}^{+} \cup I_{0+}^{-} \cup I_{0-}^{+} \cup \left(I_{0-} \setminus I_{0-}^{+}\right) \cup I_{+} = I_{00}^{+} \cup I_{0+}^{+} \cup I_{0-}^{+} \cup \left(I_{0-} \setminus I_{0-}^{+}\right) \cup I_{+}.$$

From this, virtue of (3.3), the fact that $\mu_i^H = 0$ for $i \in \ I_{0-} \setminus I_{0-}^{+}$, and (4.3) we deduce that

$$\sum_{i\in I} -\mu_i \langle \xi_i^H, x^\star - \hat{x}\rangle = \sum_{i\in I_{0+}^{+}\cup I_{00}^{+}\cup I_{0-}^{+}} -\mu_i \langle \xi_i^H, x^\star - \hat{x}\rangle \leq 0. \tag{4.4}$$

On the other hand, we have

$$\begin{aligned} I &= I_{+-} \cup I_{0-} \cup I_{0+} \cup I_{00}^{0+} \cup (I_{00} \setminus I_{00}^{0+}) \cup I_{+0}^{0+} \cup (I_{+0} \setminus I_{+0}^{0+}) \\ &= I_{+-} \cup I_{0-} \cup I_{0+} \cup (I_{00} \setminus I_{00}^{0+}) \cup (I_{+0} \setminus I_{+0}^{0+}). \end{aligned} \tag{4.5}$$

The above equality, (3.2), and the fact that $\mu_i^G = 0$ for all $i \in (I_{00} \setminus I_{00}^{0+}) \cup (I_{+0} \setminus I_{+0}^{0+})$, imply that

$$\sum_{i\in I} -\mu_i^G \langle \xi_i^G, x^\star - \hat{x}\rangle = 0.$$

The last equality and (4.4) yield

$$\langle \sum_{i=1}^{n} \left(-\mu_i^H \xi_i^H + \mu_i^G \xi_i^G\right), x^\star - \hat{x}\rangle \leq 0, \tag{4.6}$$

and thus, in view of the (4.1), we have $\langle \xi^f, x^\star - \hat{x}\rangle \geq 0$, which implies $f(x^\star) \geq f(\hat{x})$ as f is ∂-pseudoconvex at $\hat{x}$. Since $x^\star$ is an arbitrary point in S, then $\hat{x}$ is a global minimizer of (P).

Now, we consider the case that $I_{00}^{-} \cup I_{00}^{0+} = \emptyset$.
Let $i \in I_{0+}$. Since G_i is continuous, there exists a neighborhood U_i for $\hat{x}$ such that $G_i(x) > 0$ for all $x \in U_i$. If $x \in U_i \cap S$, then $G_i(x) > 0,\ H_i(x) \geq 0$ and $G_i(x)H_i(x) \leq 0$, which imply $H_i(x) = 0$. Taking $V_1 = \bigcap_{i\in I_{0+}} U_i$, we deduce that

$$H_i(x) = 0,\ G_i(x) > 0, \quad \forall i \in I_{0+},\ \forall x \in V_1 \cap S.$$

By similar argument, we also obtain that for a neighborhood V_2 of $\hat{x}$ one has

$$H_i(x) > 0,\ G_i(x) \leq 0, \quad \forall i \in I_{+0},\ \forall x \in V_2 \cap S.$$

Taking into account the ∂-quasiconvexity of G_i (for $i \in I_{+0}^{0+}$) and H_i (for $i \in I_{0+}^{-}$), for all $x \in S \cap V_1 \cap V_2$ we have

$$\begin{aligned} &G_i(x) \leq 0 = G_i(\hat{x}) \Rightarrow \langle \xi_i^G, x - \hat{x}\rangle \leq 0, \quad \forall i \in I_{+0}^{0+}, \\ &H_i(x) = 0 \leq H_i(\hat{x}) \Rightarrow \langle \xi_i^H, x - \hat{x}\rangle \leq 0, \quad \forall i \in I_{0+}^{-}. \end{aligned}$$

Thus, we can again show that the inequality (4.6) holds for $x \in S \cap V_1 \cap V_2$, hence $\langle \xi^f, x - \hat{x}\rangle \geq 0$. The ∂-pseudoconvexity of f at $\hat{x}$ justifies the result. □

Remark 1. *We can generalize the results of Theorem (7) by replacing pseudoconvexity (resp. quasiconvexity) with pseudoinvexity (resp. quasiinvexity), which are introduced in* [2, 3] *and extended in* [12]*. Since the proof of this extension is the same as Theorem (7), we omit it.*

Finally, we will explain our result by the following example.

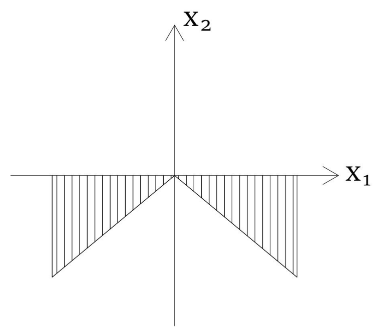

Figure 1.

Example 1. *Consider the following problem,*

$$\begin{aligned} \min \quad & f(x_1,x_2) = x_1^2 + |x_2|, \\ s.t. \quad & x_2 \leq 0, \\ & x_1(|x_2| - |x_1|) \geq 0, \\ & |x_1| - |x_2| \geq 0. \end{aligned}$$

One can consider the above problem as MPVC such that,

$$\begin{aligned} &H_1(x_1,x_2) = -x_2, \qquad H_2(x_1,x_2) = |x_1| - |x_2|, \\ &G_1(x_1,x_2) = -1, \qquad G_2(x_1,x_2) = -x_1. \end{aligned}$$

We have shown the feasible set S in Fig 1. Obviously, S is not convex. We consider $\hat{x} = (0,0)$. Clearly, $I_{0-} = \{1\}$, and $I_{00} = \{2\}$. Now, by choosing $\mu_1^H = \mu_2^H = \frac{1}{2}$ and $\mu_1^G = \mu_2^G = 0$, the conditions (3.2)-(3.4) and the following inclusion are followed

$$\begin{aligned} &(0,0) \in (\{0\} \times [-1,1]) - \lambda_1^H \{(0,-1)\} - \lambda_2^H ([-1,1] \times [-1,1]) \\ &+ \lambda_1^G \{(0,0)\} + \lambda_2^G \{(-1,0)\}. \end{aligned}$$

Thus, $\hat{x}$ is a M-stationary point for problem. Since the convexity assumptions of Theorem 7 hold and $I_{0+}^- \cup I_{00}^- \cup I_{+0}^{0+} \cup I_{00}^{0+} = \emptyset$, thus $\hat{x}$ is a global optimal solution of problem.

5. Conclusion

Motivated by [9], we considered the nonsmooth mathematical programing with vanishing constraints, MPVC for short. We introduced a new constraint qualification for MPVC. This constraint qualification guarantees an optimality condition to hold at a local minimum, named M-stationary condition. Finally, we showed that M-stationary condition (which is weaker than statandard KKT condition) is sufficient optimality condition for an interesting class of MPVCs.

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