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A compact difference scheme for time-fractional Black-Scholes equation with time-dependent parameters under the CEV model: American options

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Abstract The Black-Scholes equation is one of the most important mathematical models in option pricing theory, but this model is far from market realities and cannot show memory effect in the financial market. This paper investigates an American option based on a time-fractional Black-Scholes equation under the constant elasticity of variance (CEV) model, which parameters of interest rate and dividend yield supposed as deterministic functions of time, and the price change of the underlying asset follows a fractal transmission system. This model does not have a closed-form solution; hence, we numerically price the American option by using a compact difference scheme. Also, we compare the time-fractional Black-Scholes equation under the CEV model with its generalized Black-Scholes model as $\alpha = 1$ and $\beta = 0$. Moreover, we demonstrate that the introduced difference scheme is unconditionally stable and convergent using Fourier analysis. The numerical examples illustrate the efficiency and accuracy of the introduced difference scheme.

Keywords. CEV model, Time-dependent parameters, Option pricing, American option, Fractional Black-Scholes equation, Compact difference scheme.

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1. Introduction

According to exercise time, we have two types of options: American and European options. An American option can be exercised at any time before the maturity date, but a European option can only be exercised at the maturity date. The American option must be worth at least as much as its European equivalent and it can never be worth less than its payoff. This option is more valuable than the European option, because investors have the freedom to exercise their option at any time during the contract while holder of European option can only exercise at maturity.

The pricing problem of American options is a very common class of optimal-stopping problems that leads to a free boundary value problem. In 1965, McKean [31] was the first person who worked on the American options and derived a free-boundary problem to determine the price of American options. In 1976, Moerbeke [36] further studied on the properties of the optimal exercise boundary and extended its analysis. Then, numerical methods for solving the free boundary problem are developed by Brennan and Schwartz (in 1977) [4], Schwartz (in 1977) [42] and Courtadon (in 1982) [9].

Analytical solutions of European options can easily be found, while there exists no exact solution for the American options, because its price depends on the history of the underlying asset price and its present value. There are various numerical methods to obtain an optimal solution of the free boundary problem, that one of the most common methods is binomial methods. Also, Muthuraman [38], Chockalingam, and Muthuraman [8] used the moving boundary approach to price American option. Chen et al. [6] introduced a predictor-corrector approach based on the spectral-collocation method to price the American options under the finite moment log-stable model. Moradipour and Yousefi used collocation methods to solve the Black-Scholes equation for American option pricing [37].

The classical Black-Scholes model is under some restrictive assumptions that it makes to weaken. Hence, some developed models are required, such as the fractional Black-Scholes model [17, 19, 32, 51], the Black-Scholes model with the transaction costs [27, 44, 47], the stochastic volatility model [18, 30], and the jump-diffusion model [22, 35]. The classical Black-Scholes models cannot show memory effect in financial systems as well. In the last decade, the fractional Black-Scholes models are used to describe the effect of trend and noise memory in financial pricing. Li et al. [25] priced the European option based on the fractional order stochastic differential equation model and derived the trend memory in stock price process when the Hurst index is between 0.5 and 1. Moreover, they presented a new approach in option pricing that it leads to a better result than the classic model and stochastic model with fractional Brownian motion when the stock prices are simulated by Monte Carlo simulation. Liu and Chang [27] presented a formula for European option pricing with transaction costs based on the fractional Black-Scholes model and showed that the price of the European option decreases with the increase of the Hurst index. Zhang et al. [52] investigated the tempered fractional Black-Scholes equation with the numerical simulation for pricing of a European double barrier option under three models of Finite Moment Log Stable, KoBoL, and CGMY. Mehrdoust et al. [34] considered the mixed fractional Heston model to show long-range dependence and exhibited that the Euler discretization method on this model has strong convergence. Also, they estimated the American put option price under this model. Besides, other classes of fractional Black-Scholes equations are introduced by Jumarie [20] and Farhadi et al. [15].

In this paper, we consider the time-fractional Black-Scholes equation to price the American call options whose parameters: interest rate, dividend yield, and volatility are as functions, while they are considered constant in the classical models, and these assumptions of the problem are closer to the actual model of the market. In many markets, stock price volatility is increasing when the stock price is decreasing. For modeling this phenomenon, we cannot use the classical Black-Scholes model and will have to use the generalized models. Including these models: local volatility models [16] and stochastic volatility models [43]. One of the most famous of these models is the CEV model that was introduced for the first time by Cox [10] in 1975 to record the inverse relationship between stock prices and their volatility. The volatility of this model without introducing any additional random process is a function of the stock price and two parameters of β and δ that are called elasticity of volatility and scale parameter fixing the initial instantaneous volatility, respectively. An important parameter of this model is the elasticity of volatility that controls the relationship between volatility and asset price. This model has been widely used in many areas, including: determining the value of American options [1, 48, 49, 53], Asian options [23, 34], Barrier options [28, 46], and Lookback options [3, 13, 21]. Since the closed-form solution does not exist to price the American option, our main aim of this paper is to obtain an American option price under the CEV model with a compact difference scheme. In the following, we investigate the stability and convergence of the introduced scheme. Staelen and Hendy [14] have already used this scheme to price the double barrier options for time-fractional Black-Scholes model whose parameters are constant.

The main body of this paper is organized as follows: In section 2, we formulate time-fractional Black-Scholes equation under the CEV model for pricing American option. In section 3, we construct the compact difference scheme to price the American option numerically. In section 4, solvability, unconditionally stability and convergence of introduced difference scheme are illustrated using Fourier analysis. In section 5, numerical examples demonstrate the efficiency of compact difference scheme to solve time-fractional Black-Scholes equation with time-dependent parameters. Finally, the paper ends with remarks and conclusions.

2. American option pricing model

In this paper, the underlying asset price is considered under CEV model as follows (see [11])

$$dS_t = (r(t) - D(t))\, S_t dt + \delta S_t^{\beta+1} dW_t, \tag{2.1}$$

where functions $r(t)$ and $D(t)$ are the risk-free interest rate and dividend yield, respectively. W_t is standard Brownian motion, dS_t is the change in the stock price S over the short increment of time dt, also, β and δ ($\delta = \sigma_0 S_0^{-\beta}$) are positive constants.

Noticing that in the case of $\beta > 0$, the local volatility function, $\sigma(S) = \delta S^{\beta}$, does not remain bounded as $S \to +\infty$. Therefore, we consider $\beta < 0$ that $\sigma(S)$ remains bounded and decreases as the asset price increases. In particular, when $\beta = 0$ the volatility $\sigma(S) = \delta S^{\beta}$ is constant and the CEV model (2.1) turns into the lognormal diffusion model which is the generalized Black-Scholes model [2]. Otherwise, the following partial differential equation will be obtained using the Itô Lemma

$$\frac{\partial C(S,t)}{\partial t} + \frac{1}{2}\delta^2 S^{2\beta+2}\frac{\partial^2 C(S,t)}{\partial S^2} + (r(t) - D(t))\, S\frac{\partial C(S,t)}{\partial S} - r(t)C(S,t) = 0,$$

which $C(S,t)$ is the value of the American call option with asset price S_t in the moment t. Under this assumption that the price change of the underlying follows a fractal transmission system, we can replace the time derivative $\frac{\partial C}{\partial t}$ with the fractional derivative $\frac{\partial^{\alpha} C}{\partial t^{\alpha}}$ (see, e.g., [26, 7]) as follows

$$\frac{\partial^{\alpha} C(S,t)}{\partial t^{\alpha}} + \frac{1}{2}\delta^2 S^{2\beta+2}\frac{\partial^2 C(S,t)}{\partial S^2} + (r(t) - D(t))\, S\frac{\partial C(S,t)}{\partial S} - r(t)C(S,t) = 0, \quad (2.2)$$

where $\frac{\partial^{\alpha} C}{\partial t^{\alpha}}$ is the modified right Riemann-Liouville fractional derivative and is defined as

$$\frac{\partial^{\alpha} C(S,t)}{\partial t^{\alpha}} = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_t^T \frac{C(S,\xi)-C(S,T)}{(\xi-t)^{\alpha}}d\xi, & 0<\alpha<1, \\ \frac{\partial C(S,t)}{\partial t}, & \alpha = 1. \end{cases}$$

With assumption, diffusion of the option price depends on the history of the time to maturity. By using change variable $\tau = T - t$ and $V(S,\tau) \equiv C(S, T-\tau) = C(S,t)$, we turn backward problem to forward problem and show the relationship $\frac{\partial^{\alpha} C}{\partial t^{\alpha}}$ with the Caputo fractional derivative. We calculate as follow

$$\begin{aligned} \frac{\partial^{\alpha} C(S,t)}{\partial t^{\alpha}} &= \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_t^T \frac{C(S,\rho)-C(S,T)}{(\rho-t)^{\alpha}}d\rho \\ &\overset{\rho=T-\zeta}{=\!=\!=} \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial \tau}\int_{\tau}^{0} \frac{C(S,T-\zeta)-C(S,T)}{(\tau-\zeta)^{\alpha}}d\zeta \\ &= \frac{-1}{\Gamma(1-\alpha)}\frac{\partial}{\partial \tau}\int_0^{\tau} \frac{V(S,\zeta)-V(S,0)}{(\tau-\zeta)^{\alpha}}d\zeta \end{aligned}$$

$$\begin{aligned} &= \frac{-1}{\Gamma(1-\alpha)}\frac{\partial}{\partial \tau}\int_0^{\tau} [V(S,\zeta)-V(S,0)]\, d\frac{(\tau-\zeta)^{1-\alpha}}{-(1-\alpha)} \\ &= \frac{-1}{\Gamma(1-\alpha)}\frac{\partial}{\partial \tau}\int_0^{\tau} \frac{(\tau-\zeta)^{1-\alpha}}{1-\alpha}\frac{\partial V(S,\zeta)}{\partial \zeta}d\zeta \\ &= \frac{-1}{\Gamma(1-\alpha)}\int_0^{\tau} \frac{1}{(\tau-\zeta)^{\alpha}}\frac{\partial V(S,\zeta)}{\partial \zeta}d\zeta. \end{aligned} \quad (2.3)$$

Furthermore, right side of (2.3) is definition of the left-hand side Caputo fractional as follows [40]

$${}_0^C D_{\tau}^{\alpha} V(S,\tau) := \frac{1}{\Gamma(1-\alpha)}\int_0^{\tau} \frac{1}{(\tau-\zeta)^{\alpha}}\frac{\partial V(S,\zeta)}{\partial \zeta}d\zeta.$$

Thus,

$$\frac{\partial^\alpha C(S,t)}{\partial t^\alpha} = -{}_0^C D_\tau^\alpha V(S,\tau). \tag{2.4}$$

Due to the (2.2), (2.4), $\frac{\partial C}{\partial S} = \frac{\partial V}{\partial S}$ and $\frac{\partial^2 C}{\partial S^2} = \frac{\partial^2 V}{\partial S^2}$ the estimation of American call option can be formulated as a time-fractional free boundary problem:

$$\begin{aligned} {}_0^C D_\tau^\alpha V(S,\tau) = \frac{1}{2}\delta^2 S^{2\beta+2}\frac{\partial^2 V}{\partial S^2} + (r(\tau) - D(\tau))\, S\frac{\partial V}{\partial S} - r(\tau)V, \\ 0 < \tau < T, \quad 0 < S < S_f(\tau), \end{aligned} \tag{2.5}$$

$$V(S,0) = \max(S-K,0), \tag{2.6}$$

$$V(0,\tau) = 0, \tag{2.7}$$

$$V(S_f(\tau),\tau) = S_f(\tau) - K, \tag{2.8}$$

$$\frac{\partial V(S_f(\tau),\tau)}{\partial S} = 1, \tag{2.9}$$

where K is "exercise price" or "strike price" and T is "expiration date" or "maturity date".

3. Compact difference scheme

In this section, we derive a numerical solution for problem (2.5) by using compact difference scheme. At first, we define following uniform time and space mesh for any positive integers M and N ($S_{\max}$ is a sufficiently large number)

$$\Delta\tau = \frac{T}{M}, \quad \tau_n = n\Delta\tau, \quad n = 0,1,\ldots,M,$$

$$\Delta S = \frac{S_{\max}}{N}, \quad S_j = j\Delta S, \quad j = 0,1,\ldots,N.$$

By using above uniform mesh, we can formulate time-fractional derivative ${}_0^C D_\tau^\alpha V(S,\tau)$ of problem (2.5) at point (S_j,τ_n) with notations $V_j^n = V(S_j,\tau_n)$ ($j = 0,1,\ldots,N; n = 0,1\ldots,M$) as

$${}_0^C D_\tau^\alpha V(S_j,\tau_n) = \varphi\sum_{k=1}^{n}\psi_k\left(V_j^{n-k+1} - V_j^{n-k}\right) + O\left(\Delta\tau^{2-\alpha}\right), \tag{3.1}$$

with

$$\varphi = \frac{1}{\Gamma(2-\alpha)}\frac{1}{\Delta\tau^\alpha}, \quad \psi_k = k^{1-\alpha} - (k-1)^{1-\alpha}, \quad k = 1,\ldots,n,$$

and approximate spatial derivatives based on Taylor expansion of $V \in \mathbb{C}^4(0,+\infty)$ as follows

$$\begin{aligned} \frac{\partial V(S_j,\tau_n)}{\partial S} = \underbrace{\frac{V(S_{j+1},\tau_n) - V(S_{j-1},\tau_n)}{2\Delta S}}_{:=\delta_S V(S_j,\tau_n)} - \frac{\Delta S^2}{6}\frac{\partial^3 V(S_j,\tau_n)}{\partial S^3} \\ + O\left(\Delta S^4\right), \end{aligned} \tag{3.2a}$$

$$\begin{aligned}\frac{\partial^2 V(S_j,\tau_n)}{\partial S^2} &= \underbrace{\frac{V(S_{j-1},\tau_n)-2V(S_j,\tau_n)+V(S_{j+1},\tau_n)}{\Delta S^2}}_{:=\delta_S^2 V(S_j,\tau_n)} \\ &\quad -\frac{\Delta S^2}{12}\frac{\partial^4 V(S_j,\tau_n)}{\partial S^4}+O\left(\Delta S^4\right).\end{aligned} \tag{3.2b}$$

By substituting (3.2a) and (3.2b) in (2.5) at grid point (S_j,τ_n), we obtain

$$\begin{aligned}& aS_j^{2\beta+2}\delta_S^2 V(S_j,\tau_n)+b_nS_j\delta_S V(S_j,\tau_n)-P_j^n=g(S_j,\tau_n),\\ & P_j^n=\frac{\Delta S^2}{12}\left(aS_j^{2\beta+2}\frac{\partial^4 V(S_j,\tau_n)}{\partial S^4}+2b_nS_j\frac{\partial^3 V(S_j,\tau_n)}{\partial S^3}\right)+O\left(\Delta S^4\right),\\ & g(S_j,\tau_n)={}_0^C D_\tau^\alpha V(S_j,\tau_n)+c_nV(S_j,\tau_n),\\ & a=\frac{\delta^2}{2},\quad b(\tau)=r(T-\tau)-D(T-\tau),\quad c(\tau)=r(T-\tau).\end{aligned} \tag{3.3}$$

Also, from (2.5), (3.2a) and (3.2b), we have

$$\begin{aligned}\frac{\partial^3 V(S_j,\tau_n)}{\partial S^3} &= -\frac{2\beta+2}{a}S_j^{-2\beta-3}\left[g(S_j,\tau_n)-b_nS_j\delta_S V(S_j,\tau_n)\right]\\ &\quad +\frac{1}{a}S_j^{-2\beta-2}\left[\delta_S g(S_j,\tau_n)-b_n\delta_S V(S_j,\tau_n)-b_nS_j\delta_S^2 V(S_j,\tau_n)\right]\\ &\quad +O\left(\Delta S^2\right),\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\frac{\partial^4 V(S_j,\tau_n)}{\partial S^4} &= \frac{2\beta+2}{a}S_j^{-2\beta-4}\left[2\beta+3+\frac{b_n}{a}S_j^{-2\beta}\right]\\ &\quad \times\left[g(S_j,\tau_n)-b_nS_j\delta_S V(S_j,\tau_n)\right]-\frac{1}{a}S_j^{-2\beta-3}\left[4(\beta+1)+\frac{b_n}{a}S_j^{-2\beta}\right]\\ &\quad \times\left[\delta_S g(S_j,\tau_n)-b_n\delta_S V(S_j,\tau_n)-b_nS_j\delta_S^2 V(S_j,\tau_n)\right]\\ &\quad -\frac{1}{a}S_j^{-2\beta-2}\left[\delta_S^2 g(S_j,\tau_n)-2b_n\delta_S^2 V(S_j,\tau_n)\right]+O\left(\Delta S^2\right).\end{aligned} \tag{3.5}$$

Now, we substitute (3.4) and (3.5) in P_j^n

$$\begin{aligned}P_j^n &= \frac{(\beta+1)}{6S_j^2}\Delta S^2\left[2\beta+3-\frac{b_n}{a}S_j^{-2\beta}\right]g(S_j,\tau_n)\\ &\quad +\frac{\Delta S^2}{12}\left[-\frac{4(\beta+1)}{S_j}+\frac{b_n}{a}S_j^{-2\beta-1}\right]\delta_S g(S_j,\tau_n)+\frac{\Delta S^2}{12}\delta_S^2 g(S_j,\tau_n)\\ &\quad +\frac{b_n(2\beta+1)}{12S_j}\Delta S^2\left[-2\beta-2+\frac{b_n}{a}S_j^{-2\beta}\right]\delta_S V(S_j,\tau_n)\\ &\quad +\frac{b_n}{12}\Delta S^2\left[4\beta+2-\frac{b_n}{a}S_j^{-2\beta}\right]\delta_S^2 V(S_j,\tau_n)+O\left(\Delta S^4\right).\end{aligned}$$

Then, we set P_j^n in Eq. (3.3) and after simplification we have

$$\left[aS_j^{2\beta+2} - \frac{b_n}{12}\Delta S^2\left(4\beta+2-\frac{b_n}{a}S_j^{-2\beta}\right)\right]\delta_S^2 V(S_j,\tau_n) + \left[b_nS_j + \frac{b_n(2\beta+1)}{12S_j}\Delta S^2\left(2\beta+2-\frac{b_n}{a}S_j^{-2\beta}\right)\right]\delta_S V(S_j,\tau_n)$$

$$= \left[1+\frac{(\beta+1)}{6S_j^2}\Delta S^2\left(2\beta+3-\frac{b_n}{a}S_j^{-2\beta}\right)\right]g(S_j,\tau_n) + \frac{\Delta S^2}{12}\left[-\frac{4(\beta+1)}{S_j}+\frac{b_n}{a}S_j^{-2\beta-1}\right]\delta_S g(S_j,\tau_n) + \frac{\Delta S^2}{12}\delta_S^2 g(S_j,\tau_n) + O\left(\Delta S^4\right).$$

We define the coefficients of the above equation as follows

$$\gamma_j^n\delta_S^2 V(S_j,\tau_n) + \xi_j^n\delta_S V(S_j,\tau_n) = \theta_j^n g(S_j,\tau_n) + w_j^n\delta_S g(S_j,\tau_n) + \frac{\Delta S^2}{12}\delta_S^2 g(S_j,\tau_n) + O\left(\Delta S^4\right), \tag{3.6}$$

$$\gamma_j^n := aS_j^{2\beta+2} - \frac{b_n}{12}\Delta S^2\left(4\beta+2-\frac{b_n}{a}S_j^{-2\beta}\right),$$

$$\xi_j^n := b_nS_j + \frac{b_n(2\beta+1)}{12S_j}\Delta S^2\left(2\beta+2-\frac{b_n}{a}S_j^{-2\beta}\right),$$

$$\theta_j^n := 1+\frac{(\beta+1)}{6S_j^2}\Delta S^2\left(2\beta+3-\frac{b_n}{a}S_j^{-2\beta}\right),$$

$$w_j^n := \frac{\Delta S^2}{12}\left(-\frac{4(\beta+1)}{S_j}+\frac{b_n}{a}S_j^{-2\beta-1}\right),$$

then, insert $g(S_j,\tau_n)$ in it

$$\gamma_j^n\delta_S^2 V(S_j,\tau_n) + \xi_j^n\delta_S V(S_j,\tau_n) = \theta_j^n\left[{}_0^C D_\tau^\alpha V(S_j,\tau_n) + c_nV(S_j,\tau_n)\right] + \left[w_j^n\delta_S + \frac{\Delta S^2}{12}\delta_S^2\right]\left[{}_0^C D_\tau^\alpha V(S_j,\tau_n) + c_nV(S_j,\tau_n)\right] + O\left(\Delta S^4\right).$$

Now, substitute $\delta_S^2 V(S_j,\tau_n)$, $\delta_S V(S_j,\tau_n)$ and (3.1) in (3.6)

$$\gamma_j^n\frac{V_{j-1}^n - 2V_j^n + V_{j+1}^n}{\Delta S^2} + \xi_j^n\frac{V_{j+1}^n - V_{j-1}^n}{2\Delta S} = \theta_j^n\left[\varphi\sum_{k=1}^n\psi_k\left(V_j^{n-k+1}-V_j^{n-k}\right) + c_nV(S_j,\tau_n)\right] + \left[w_j^n\delta_S + \frac{\Delta S^2}{12}\delta_S^2\right]\left[\varphi\sum_{k=1}^n\psi_k\left(V_j^{n-k+1}-V_j^{n-k}\right) + c_nV_j^n\right] + R_j^n,$$

where the evaluation R_j^n holds in $|R_j^n| \le C\left(\Delta\tau^{2-\alpha} + \Delta S^4\right)$. Again, by substituting $\delta_S^2 V(S_j,\tau_n)$ and $\delta_S V(S_j,\tau_n)$ on the right hand side above formula and rearranging, we obtain

$$
\begin{aligned}
&\left[\frac{\gamma_j^n}{\Delta S^2} - \frac{\xi_j^n}{2\Delta S} + \frac{w_j^n}{2\Delta S}(\varphi + c_n) - \frac{1}{12}(\varphi + c_n)\right] V_{j-1}^n \\
&\quad - \left[\frac{2\gamma_j^n}{\Delta S^2} + c_n\theta_j^n + \varphi\theta_j^n - \frac{1}{6}(\varphi + c_n)\right] V_j^n \\
&\quad + \left[\frac{\gamma_j^n}{\Delta S^2} + \frac{\xi_j^n}{2\Delta S} - \frac{w_j^n}{2\Delta S}(\varphi + c_n) - \frac{1}{12}(\varphi + c_n)\right] V_{j+1}^n
\end{aligned}
$$

$$
\begin{aligned}
&= -\theta_j^n\varphi V_j^{n-1} + \theta_j^n\varphi\sum_{k=2}^{n}\psi_k\left(V_j^{n-k+1} - V_j^{n-k}\right) \\
&\quad - \frac{w_j^n}{2\Delta S}\varphi\left(V_{j+1}^{n-1} - V_{j-1}^{n-1}\right) \\
&\quad + \frac{w_j^n}{2\Delta S}\varphi\sum_{k=2}^{n}\psi_k\left(V_{j+1}^{n-k+1} - V_{j-1}^{n-k+1} - V_{j+1}^{n-k} + V_{j-1}^{n-k}\right) \\
&\quad - \frac{1}{12}\varphi\left(V_{j-1}^{n-1} - 2V_j^{n-1} + V_{j+1}^{n-1}\right) \\
&\quad + \frac{1}{12}\varphi\sum_{k=2}^{n}\psi_k\Big(V_{j-1}^{n-k+1} - 2V_j^{n-k+1} + V_{j+1}^{n-k+1} - V_{j-1}^{n-k} \\
&\quad + 2V_j^{n-k} - V_{j+1}^{n-k}\Big) + R_j^n.
\end{aligned}
\tag{3.7}
$$

Finally, with eliminating R_j^n and rearranging, we get the following compact difference scheme

$$
\begin{aligned}
&\left[\frac{\gamma_j^n}{\Delta S^2} - \frac{\xi_j^n}{2\Delta S} + (\varphi + c_n)(\frac{w_j^n}{2\Delta S} - \frac{1}{12})\right] \widetilde{V}_{j-1}^n \\
&\quad - \left[\frac{2\gamma_j^n}{\Delta S^2} + c_n\theta_j^n + (\varphi + c_n)(\theta_j^n - \frac{1}{6})\right] \widetilde{V}_j^n \\
&\quad + \left[\frac{\gamma_j^n}{\Delta S^2} + \frac{\xi_j^n}{2\Delta S} - (\varphi + c_n)(\frac{w_j^n}{2\Delta S} + \frac{1}{12})\right] \widetilde{V}_{j+1}^n
\end{aligned}
$$

$$
\begin{aligned}
&= \theta_j^n \varphi \left[\sum_{k=1}^{n-1} (\psi_{k+1} - \psi_k) \widetilde{V}_j^{n-k} - \psi_n \widetilde{V}_j^0 \right] \\
&\quad + \frac{w_j^n}{2\Delta S} \varphi \left[\sum_{k=1}^{n-1} (\psi_{k+1} - \psi_k) \left(\widetilde{V}_{j+1}^{n-k} - \widetilde{V}_{j-1}^{n-k} \right) - \psi_n \left(\widetilde{V}_{j+1}^0 - \widetilde{V}_{j-1}^0 \right) \right] \\
&\quad + \frac{\varphi}{12} \left[\sum_{k=1}^{n-1} (\psi_{k+1} - \psi_k) \left(\widetilde{V}_{j-1}^{n-k} - 2\widetilde{V}_j^{n-k} + \widetilde{V}_{j+1}^{n-k} \right) \right. \\
&\quad \left. - \psi_n \left(\widetilde{V}_{j-1}^0 - 2\widetilde{V}_j^0 + \widetilde{V}_{j+1}^0 \right) \right],
\end{aligned} \tag{3.8}
$$

where $\widetilde{V}$ is the exact solution of the compact difference scheme and V is the exact solution of the differential equation.

An index $J(\tau_n)$ (for $n = 0, 1, \ldots, M$) is found such that (similar to [53])

$$
\begin{aligned}
&\frac{1}{2}\delta^2 S_j^{2\beta+2} \frac{\partial^2 V(S_j, \tau_n)}{\partial S^2} + (r(\tau_n) - D(\tau_n)) S_j \frac{\partial V(S_j, \tau_n)}{\partial S} \\
&\quad - r(\tau_n) V(S_j, \tau_n) - {}_0^C D_\tau^\alpha V(S_j, \tau_n) = 0,
\end{aligned}
$$

where $V(S_j, \tau_n) > \max(S_j - K, 0)$ for $j = 0, 1, \ldots, J(\tau_n)$, and

$$
\begin{aligned}
&\frac{1}{2}\delta^2 S_j^{2\beta+2} \frac{\partial^2 V(S_j, \tau_n)}{\partial S^2} + (r(\tau_n) - D(\tau_n)) S_j \frac{\partial V(S_j, \tau_n)}{\partial S} \\
&\quad - r(\tau_n) V(S_j, \tau_n) - {}_0^C D_\tau^\alpha V(S_j, \tau_n) < 0,
\end{aligned}
$$

where $V(S_j, \tau_n) = \max(S_j - K, 0)$ for $j = J(\tau_n) + 1, J(\tau_n) + 2, \ldots, N$. Therefore, American call option prices are obtained from (3.8) for $j = 0, 1, \ldots, J(\tau_n)$ and are equal $S_j - K$ for $j = J(\tau_n) + 1, \ldots, N$.

4. Stability and convergence analysis

In this section, we investigate that the compact difference scheme is solvable, unconditionally stable and convergent.

4.1. **Solvability.**

Theorem 4.1. *The compact difference scheme* (3.8) *has a unique solution.*

Proof. Matrix form of compact difference scheme (3.8) can be briefly written as $A^n \widetilde{V}^n = d_{n-1}$, where d_{n-1} depends only $\widetilde{V}^{n-1}$, $\widetilde{V}^{n-2}$, $\ldots$ $\widetilde{V}^0$. The tridiagonal coefficient matrix $A^n = (a_{ij}^n)$ from compact difference scheme (3.8) is strictly diagonally dominant since $|a_{ii}^n| > \sum_{j \neq i} \left|a_{ij}^n\right|$, where

$$
\begin{aligned}
&|a_{ii}^n| = \frac{2\gamma_i^n}{\Delta S^2} + c_n \theta_i^n + (\varphi + c_n)(\theta_i^n - \frac{1}{6}), \\
&\sum_{j \neq i} \left|a_{ij}^n\right| = \frac{2\gamma_j^n}{\Delta S^2} - \frac{1}{6}(\varphi + c_n), \quad n = 1, 2, \ldots, M.
\end{aligned}
$$

For each n, the coefficient matrix A^n is nonsingular, so the compact difference scheme (3.8) is uniquely solvable. $\square$

4.2. **Stability.** In this part, we prove that the compact difference scheme (3.8) is unconditionally stable using the Fourier analysis [45, 12, 24]. Suppose $\widehat{V}_j^n$ be a numerical solution of compact difference scheme (3.8). Let

$$\varepsilon_j^n = \widetilde{V}_j^n - \widehat{V}_j^n, \quad j = 0, 1, \dots, N; \quad n = 0, 1, \dots, M,$$

then, ε_j^n satisfies in the following equations

$$\begin{aligned}
&\left[\frac{\gamma_j^n}{\Delta S^2} - \frac{\xi_j^n}{2\Delta S} + (\varphi + c_n)(\frac{w_j^n}{2\Delta S} - \frac{1}{12})\right]\varepsilon_{j-1}^n \\
&\quad - \left[\frac{2\gamma_j^n}{\Delta S^2} + c_n\theta_j^n + (\varphi + c_n)(\theta_j^n - \frac{1}{6})\right]\varepsilon_j^n \\
&\quad + \left[\frac{\gamma_j^n}{\Delta S^2} + \frac{\xi_j^n}{2\Delta S} - (\varphi + c_n)(\frac{w_j^n}{2\Delta S} + \frac{1}{12})\right]\varepsilon_{j+1}^n \\
&= \theta_j^n\varphi\left[\sum_{k=1}^{n-1}(\psi_{k+1} - \psi_k)\,\varepsilon_j^{n-k} - \psi_n\varepsilon_j^0\right] \\
&\quad + \frac{w_j^n}{2\Delta S}\varphi\left[\sum_{k=1}^{n-1}(\psi_{k+1} - \psi_k)\left(\varepsilon_{j+1}^{n-k} - \varepsilon_{j-1}^{n-k}\right) - \psi_n\left(\varepsilon_{j+1}^0 - \varepsilon_{j-1}^0\right)\right] \\
&\quad + \frac{\varphi}{12}\left[\sum_{k=1}^{n-1}(\psi_{k+1} - \psi_k)\left(\varepsilon_{j-1}^{n-k} - 2\varepsilon_j^{n-k} + \varepsilon_{j+1}^{n-k}\right)\right. \\
&\quad \left. - \psi_n\left(\varepsilon_{j-1}^0 - 2\varepsilon_j^0 + \varepsilon_{j+1}^0\right)\right],
\end{aligned} \tag{4.1}$$

and $\varepsilon_0^n = \varepsilon_{J(\tau_n)+1}^n = \cdots = \varepsilon_N^n = 0$.

Now, we define the following grid function

$$\varepsilon^n(S) = \begin{cases} \varepsilon_j^n, & S \in \left(S_j - \frac{\Delta S}{2}, S_j + \frac{\Delta S}{2}\right], \\ 0, & S \in \left[0, \frac{\Delta S}{2}\right] \bigcup \left(S_{J(\tau_n)+1} - \frac{\Delta S}{2}, S_{J(\tau_n)+1} + \frac{\Delta S}{2}\right] \\ & \quad \bigcup \cdots \bigcup \left(S_{\max} - \frac{\Delta S}{2}, S_{\max}\right], \end{cases}$$

and make a Fourier series extension for it with the period $L = S_{\max}$ as follows

$$\varepsilon^n(S) = \sum_{j=-\infty}^{+\infty} \varsigma_j^n e^{i\frac{2\pi jS}{L}} \quad (i^2 = -1), \quad n = 0, 1, \dots, M,$$

$$\varsigma_j^n = \frac{1}{L}\int_0^L \varepsilon^n(S) e^{i\frac{2\pi jS}{L}}\, dS, \quad j \in \mathbb{Z}.$$

We let $\varepsilon^n = \left(\varepsilon_1^n, \varepsilon_2^n, \dots, \varepsilon_{J(\tau_n)}^n\right)^t$, and define the following norm

$$\|\varepsilon^n\|_2^2 = \sum_{j=1}^{J(\tau_n)} \Delta S \left|\varepsilon_j^n\right|^2 = \int_0^L \left|\varepsilon^n(S)\right|^2 dS = \|\varepsilon^n(S)\|_{L^2}^2, \quad n = 0, 1, \dots, M.$$

In addition, by using the Parseval equality

$$\|\varepsilon^n(S)\|_{L^2}^2 = L \sum_{j=-\infty}^{+\infty} \left|\varsigma_j^n\right|^2, \quad n = 0, 1, \dots, M,$$

we obtain

$$\|\varepsilon^n\|_2^2 = \sum_{j=1}^{J(\tau_n)} \Delta S \left|\varepsilon_j^n\right|^2 = L \sum_{j=-\infty}^{+\infty} \left|\varsigma_j^n\right|^2, \quad n = 0, 1, \dots, M. \tag{4.2}$$

According to the above analysis and $S_j = j\Delta S$, we assume that the solution of (3.8) has the form as follows

$$\varepsilon_j^n = \varsigma^n e^{iqj\Delta S}, \quad q = \frac{2\pi l}{L}, \quad l \in \mathbb{Z}.$$

With substituting the above formula into (4.1), we obtain

$$\begin{aligned}
&\left[\frac{\gamma_j^n}{\Delta S^2} - \frac{\xi_j^n}{2\Delta S} + (\varphi + c_n)(\frac{w_j^n}{2\Delta S} - \frac{1}{12})\right] \varsigma^n e^{-iq\Delta S} \\
&\quad - \left[\frac{2\gamma_j^n}{\Delta S^2} + c_n\theta_j^n + (\varphi + c_n)(\theta_j^n - \frac{1}{6})\right] \varsigma^n \\
&\quad + \left[\frac{\gamma_j^n}{\Delta S^2} + \frac{\xi_j^n}{2\Delta S} - (\varphi + c_n)(\frac{w_j^n}{2\Delta S} + \frac{1}{12})\right] \varsigma^n e^{iq\Delta S} \\
&= \theta_j^n \varphi \left[\sum_{k=1}^{n-1} (\psi_{k+1} - \psi_k) \varsigma^{n-k} - \psi_n \varsigma^0\right] \\
&\quad + \frac{w_j^n}{2\Delta S} \varphi \Bigg[\sum_{k=1}^{n-1} (\psi_{k+1} - \psi_k) \varsigma^{n-k} \left(e^{iq\Delta S} - e^{-iq\Delta S}\right) \\
&\quad - \psi_n \varsigma^0 \left(e^{iq\Delta S} - e^{-iq\Delta S}\right)\Bigg] \\
&\quad + \frac{\varphi}{12} \Bigg[\sum_{k=1}^{n-1} (\psi_{k+1} - \psi_k) \varsigma^{n-k} \left(e^{-iq\Delta S} - 2 + e^{iq\Delta S}\right) \\
&\quad - \psi_n \varsigma^0 \left(e^{-iq\Delta S} - 2 + e^{iq\Delta S}\right)\Bigg].
\end{aligned}$$

By using $\sin^2(\frac{q\Delta S}{2}) = -\frac{1}{4}\left(e^{iq\Delta S} - 2 + e^{-iq\Delta S}\right)$, the above relation reduces to

$$\begin{aligned}
&\left[\left(-\frac{4\gamma_j^n}{\Delta S^2} + \frac{1}{3}(\varphi + c_n)\right)\sin^2(\frac{q\Delta S}{2}) - (\varphi + 2c_n)\theta_j^n \right. \\
&\quad \left. + i\left(\frac{\xi_j^n}{\Delta S} - (\varphi + c_n)\frac{w_j^n}{\Delta S}\right)\sin(q\Delta S)\right]\varsigma^n \\
&= \left[\theta_j^n\varphi + i\frac{w_j^n}{\Delta S}\varphi\sin(q\Delta S) - \frac{\varphi}{3}\sin^2(\frac{q\Delta S}{2})\right] \\
&\quad \times \left[\sum_{k=1}^{n-1}(\psi_{k+1} - \psi_k)\varsigma^{n-k} - \psi_n\varsigma^0\right].
\end{aligned}$$

This yields

$$\begin{aligned}
\varsigma^n = &\left[\theta_j^n\varphi + i\frac{w_j^n}{\Delta S}\varphi\sin(q\Delta S) - \frac{\varphi}{3}\sin^2(\frac{q\Delta S}{2})\right] / \\
&\left[\left(-\frac{4\gamma_j^n}{\Delta S^2} + \frac{\varphi + c_n}{3}\right)\sin^2(\frac{q\Delta S}{2}) - (\varphi + 2c_n)\theta_j^n \right. \\
&\left. + i\left(\frac{\xi_j^n}{\Delta S} - (\varphi + c_n)\frac{w_j^n}{\Delta S}\right)\sin(q\Delta S)\right] \\
&\times \left[\sum_{k=1}^{n-1}(\psi_{k+1} - \psi_k)\varsigma^{n-k} - \psi_n\varsigma^0\right].
\end{aligned} \tag{4.3}$$

Lemma 4.2. *The coefficients ψ_n satisfy*

(I) $\psi_n > 0, \quad n = 1, 2, \ldots.$
(II) $1 = \psi_1 > \psi_2 > \ldots > \psi_{n+1}, \quad \psi_{n+1} \to 0, \quad$ *as* $n \to +\infty$.
(III) $\sum_{k=1}^{n}(\psi_k - \psi_{k+1}) + \psi_{n+1} = 1$.

Proof. (I)–(III) are clearly established. □

Lemma 4.3. *The following inequality is established*

$$\begin{aligned}
&\left|\left[\theta_j^n\varphi + i\frac{w_j^n}{\Delta S}\varphi\sin(q\Delta S) - \frac{\varphi}{3}\sin^2(\frac{q\Delta S}{2})\right] / \right. \\
&\quad \left[\left(-\frac{4\gamma_j^n}{\Delta S^2} + \frac{\varphi + c_n}{3}\right)\sin^2(\frac{q\Delta S}{2}) - (\varphi + 2c_n)\theta_j^n \right. \\
&\quad \left.\left. + i\left(\frac{\xi_j^n}{\Delta S} - (\varphi + c_n)\frac{w_j^n}{\Delta S}\right)\sin(q\Delta S)\right]\right| \leq 1.
\end{aligned}$$

Proof. See [14]. □

Lemma 4.4. *Suppose that ς^n $(n = 1, 2, \cdots, N)$ is the solutions of* (4.3)*, we have* $|\varsigma^n| \leq |\varsigma^0|$.

Proof. We prove it by using the mathematical induction. For $n = 1$, (4.3) becomes

$$\begin{aligned}\varsigma^1 =& \left[\left(\theta_j^n \varphi + i \frac{w_j^n}{\Delta S} \varphi \sin(q\Delta S) - \frac{\varphi}{3} \sin^2(\frac{q\Delta S}{2}) \right) \varsigma^0 \right] / \\ & \left[\left(-\frac{4\gamma_j^n}{\Delta S^2} + \frac{\varphi + c_n}{3} \right) \sin^2(\frac{q\Delta S}{2}) - (\varphi + 2c_n)\theta_j^n \right. \\ & \left. + i \left(\frac{\xi_j^n}{\Delta S} - (\varphi + c_n) \frac{w_j^n}{\Delta S} \right) \sin(q\Delta S) \right].\end{aligned}$$

Noticing that $\psi_1 = 1$, according to Lemma 4.3, $\left|\varsigma^1\right| \le \left|\varsigma^0\right|$. Suppose that $\left|\varsigma^k\right| \le \left|\varsigma^0\right|$, $k = 2, 3, \ldots, n-1$, and prove $|\varsigma^n| \le \left|\varsigma^0\right|$. By applying Lemma 4.2, Lemma 4.3 and the relation (4.3) for $n \ge 2$, we get

$$\begin{aligned}|\varsigma^n| =& \left| \left[\theta_j^n \varphi + i \frac{w_j^n}{\Delta S} \varphi \sin(q\Delta S) - \frac{\varphi}{3} \sin^2(\frac{q\Delta S}{2}) \right] / \right. \\ & \left[\left(-\frac{4\gamma_j^n}{\Delta S^2} + \frac{\varphi + c_n}{3} \right) \sin^2(\frac{q\Delta S}{2}) - (\varphi + 2c_n)\theta_j^n \right. \\ & \left. \left. + i \left(\frac{\xi_j^n}{\Delta S} - (\varphi + c_n) \frac{w_j^n}{\Delta S} \right) \sin(q\Delta S) \right] \right| \\ & \times \left| \sum_{k=1}^{n-1} (\psi_{k+1} - \psi_k)\, \varsigma^{n-k} - \psi_n \varsigma^0 \right| \\ \le& \left| \sum_{k=1}^{n-1} (\psi_{k+1} - \psi_k)\, \varsigma^{n-k} - \psi_n \varsigma^0 \right| \\ \le& \left| \sum_{k=1}^{n-1} (\psi_{k+1} - \psi_k)\, \varsigma^{n-k} \right| + \psi_n \left|\varsigma^0\right| \\ \le& \sum_{k=1}^{n-1} (\psi_k - \psi_{k+1}) \left|\varsigma^{n-k}\right| + \psi_n \left|\varsigma^0\right| \\ \le& \left(\sum_{k=1}^{n-1} (\psi_k - \psi_{k+1}) + \psi_n \right) \left|\varsigma^0\right| = \left|\varsigma^0\right|,\end{aligned}$$

thus $|\varsigma^n| \le \left|\varsigma^0\right|$. The proof is finished. □

Theorem 4.5. *The compact difference scheme (3.8) is unconditionally stable.*

Proof. According to Lemma 4.4 and formula (4.2), we derive

$$\|\varepsilon^n\|_2^2 = L \sum_{j=-\infty}^{+\infty} \left|\varsigma_j^n\right|^2 \le L \sum_{j=-\infty}^{+\infty} \left|\varsigma_j^0\right|^2 = \left\|\varepsilon^0\right\|_2^2.$$

This yields $\|\varepsilon^n\|_2 \le \left\|\varepsilon^0\right\|_2$ for $n = 1, 2, \ldots, M$. Thus, the compact difference scheme (3.8) is unconditionally stable. □

4.3. **Convergence.** Now, we investigate the convergence of the compact difference scheme (3.8). We suppose that V_j^n is the exact solution of (2.5) and $\widetilde{V}_j^n$ is the exact solution of (3.8). Let

$$\begin{cases} E_j^n = V_j^n - \widetilde{V}_j^n, \quad j = 0, 1, \ldots, N; \quad n = 0, 1, \ldots, M, \\ R_j^n = O\left(\Delta\tau^{2-\alpha} + \Delta S^4\right). \end{cases} \tag{4.4}$$

Subtracting (3.7) from (3.8) achieve

$$\begin{aligned}
&\left[\frac{\gamma_j^n}{\Delta S^2} - \frac{\xi_j^n}{2\Delta S} + (\varphi + c_n)(\frac{w_j^n}{2\Delta S} - \frac{1}{12})\right] E_{j-1}^n \\
&\quad - \left[\frac{2\gamma_j^n}{\Delta S^2} + c_n\theta_j^n + (\varphi + c_n)(\theta_j^n - \frac{1}{6})\right] E_j^n \\
&\quad + \left[\frac{\gamma_j^n}{\Delta S^2} + \frac{\xi_j^n}{2\Delta S} - (\varphi + c_n)(\frac{w_j^n}{2\Delta S} + \frac{1}{12})\right] E_{j+1}^n \\
&= \theta_j^n \varphi \left[\sum_{k=1}^{n-1} (\psi_{k+1} - \psi_k)\, E_j^{n-k} - \psi_n E_j^0\right] \\
&\quad + \frac{w_j^n}{2\Delta S}\varphi \left[\sum_{k=1}^{n-1} (\psi_{k+1} - \psi_k) \left(E_{j+1}^{n-k} - E_{j-1}^{n-k}\right) - \psi_n \left(E_{j+1}^0 - E_{j-1}^0\right)\right] \\
&\quad + \frac{\varphi}{12}\left[\sum_{k=1}^{n-1} (\psi_{k+1} - \psi_k) \left(E_{j-1}^{n-k} - 2E_j^{n-k} + E_{j+1}^{n-k}\right)\right. \\
&\quad \left. - \psi_n \left(E_{j-1}^0 - 2E_j^0 + E_{j+1}^0\right)\right] + R_j^n,
\end{aligned} \tag{4.5}$$

with the initial-boundary conditions

$$\begin{cases} E_j^0 = 0, \quad j = 0, 1, \ldots, N, \\ E_0^n = E_{J(\tau_n)+1}^n = \cdots = E_N^n = 0, \quad n = 0, 1, \ldots, M. \end{cases}$$

Similar to the stability proof as above, we define the following grid functions

$$E^n(S) = \begin{cases} E_j^n, \quad S \in \left(S_j - \frac{\Delta S}{2}, S_j + \frac{\Delta S}{2}\right], \\ 0, \quad S \in \left[0, \frac{\Delta S}{2}\right] \bigcup \left(S_{J(\tau_n)+1} - \frac{\Delta S}{2}, S_{J(\tau_n)+1} + \frac{\Delta S}{2}\right] \\ \qquad \bigcup \cdots \bigcup \left(S_{\max} - \frac{\Delta S}{2}, S_{\max}\right], \end{cases}$$

and

$$R^n(S) = \begin{cases} R_j^n, \quad S \in \left(S_j - \frac{\Delta S}{2}, S_j + \frac{\Delta S}{2}\right], \\ 0, \quad S \in \left[0, \frac{\Delta S}{2}\right] \bigcup \left(S_{J(\tau_n)+1} - \frac{\Delta S}{2}, S_{J(\tau_n)+1} + \frac{\Delta S}{2}\right] \\ \qquad \bigcup \cdots \bigcup \left(S_{\max} - \frac{\Delta S}{2}, S_{\max}\right]. \end{cases}$$

Therefore, $E^n(S)$ and $R^n(S)$ have the Fourier series expansions as follow, respectively,

$$
\begin{aligned}
E^n(S) &= \sum_{j=-\infty}^{+\infty} \vartheta_j^n e^{i\frac{2\pi j S}{L}}, \\
R^n(S) &= \sum_{j=-\infty}^{+\infty} \nu_j^n e^{i\frac{2\pi j S}{L}}, \quad (i^2=-1), \quad n=0,1,\dots,M,
\end{aligned}
$$

where $L = S_{\max}$ and

$$
\vartheta_j^n = \frac{1}{L}\int_0^L E^n(S) e^{i\frac{2\pi j S}{L}} dS, \quad \nu_j^n = \frac{1}{L}\int_0^L R^n(S) e^{i\frac{2\pi j S}{L}} dS, \quad j \in \mathbb{Z}.
$$

Let $E^n = \left(E_1^n, E_2^n, \dots, E_{J(\tau_n)}^n\right)^t$, $R^n = \left(R_1^n, R_2^n, \dots, R_{J(\tau_n)}^n\right)^t$ and define their corresponding norms as follows, respectively

$$
\|E^n\|_2^2 = \sum_{j=1}^{J(\tau_n)} \Delta S \left|E_j^n\right|^2 = \int_0^L |E^n(S)|^2 \, dS = \|E^n(S)\|_{L^2}^2, \quad n=0,1,\dots,M,
$$

and

$$
\|R^n\|_2^2 = \sum_{j=1}^{J(\tau_n)} \Delta S \left|R_j^n\right|^2 = \int_0^L |R^n(S)|^2 \, dS = \|R^n(S)\|_{L^2}^2, \quad n=0,1,\dots,M. \tag{4.6}
$$

Applying Parseval equality leads to

$$
\|E^n\|_2^2 = \sum_{j=1}^{J(\tau_n)} \Delta S \left|E_j^n\right|^2 = L \sum_{j=-\infty}^{+\infty} \left|\vartheta_j^n\right|^2, \quad n=0,1,\dots,M, \tag{4.7a}
$$

$$
\|R^n\|_2^2 = \sum_{j=1}^{J(\tau_n)} \Delta S \left|R_j^n\right|^2 = L \sum_{j=-\infty}^{+\infty} \left|\nu_j^n\right|^2, \quad n=0,1,\dots,M. \tag{4.7b}
$$

According to the stability analysis and $S_j = j\Delta S$, we suppose that the solution of (4.5) has the form as follows

$$
E_j^n = \vartheta^n e^{iqj\Delta S}, \quad R_j^n = \nu^n e^{iqj\Delta S}, \quad q = \frac{2\pi l}{L}, \quad l \in \mathbb{Z}.
$$

Replacing the above relations into (4.5), we have

$$
\begin{aligned}
&\left[\frac{\gamma_j^n}{\Delta S^2} - \frac{\xi_j^n}{2\Delta S} + (\varphi + c_n)(\frac{w_j^n}{2\Delta S} - \frac{1}{12})\right] \vartheta^n e^{-iq\Delta S} \\
&\quad - \left[\frac{2\gamma_j^n}{\Delta S^2} + c_n\theta_j^n + (\varphi + c_n)(\theta_j^n - \frac{1}{6})\right] \vartheta^n \\
&\quad + \left[\frac{\gamma_j^n}{\Delta S^2} + \frac{\xi_j^n}{2\Delta S} - (\varphi + c_n)(\frac{w_j^n}{2\Delta S} + \frac{1}{12})\right] \vartheta^n e^{iq\Delta S}
\end{aligned}
$$

$$
\begin{aligned}
&= \theta_j^n \varphi \left[\sum_{k=1}^{n-1} \left(\psi_{k+1} - \psi_k\right) \vartheta^{n-k} - \psi_n \vartheta^0 \right] \\
&\quad + \frac{w_j^n}{2\Delta S} \varphi \left[\sum_{k=1}^{n-1} \left(\psi_{k+1} - \psi_k\right) \left(e^{iq\Delta S} - e^{-iq\Delta S}\right) \vartheta^{n-k} \right. \\
&\quad \left. - \psi_n \left(e^{iq\Delta S} - e^{-iq\Delta S}\right) \vartheta^0 \right] \\
&\quad + \frac{\varphi}{12} \left[\sum_{k=1}^{n-1} \left(\psi_{k+1} - \psi_k\right) \left(e^{-iq\Delta S} - 2 + e^{iq\Delta S}\right) \vartheta^{n-k} \right. \\
&\quad \left. - \psi_n \left(e^{-iq\Delta S} - 2 + e^{iq\Delta S}\right) \vartheta^0 \right] + \nu^n.
\end{aligned}
$$

Applying $\sin^2(\frac{q\Delta S}{2}) = -\frac{1}{4}\left(e^{iq\Delta S} - 2 + e^{-iq\Delta S}\right)$, the above relation turns to

$$
\begin{aligned}
&\left[\left(-\frac{4\gamma_j^n}{\Delta S^2} + \frac{1}{3}(\varphi + c_n) \right) \sin^2(\frac{q\Delta S}{2}) - (\varphi + 2c_n)\theta_j^n \right. \\
&\quad \left. + i \left(\frac{\xi_j^n}{\Delta S} - (\varphi + c_n)\frac{w_j^n}{\Delta S} \right) \sin(q\Delta S) \right] \vartheta^n \\
&= \left[\theta_j^n \varphi + i\frac{w_j^n}{\Delta S}\varphi \sin(q\Delta S) - \frac{\varphi}{3}\sin^2(\frac{q\Delta S}{2}) \right] \\
&\quad \times \left[\sum_{k=1}^{n-1} \left(\psi_{k+1} - \psi_k\right) \vartheta^{n-k} - \psi_n \vartheta^0 \right] + \nu^n,
\end{aligned}
$$

in result

$$
\begin{aligned}
\vartheta^n = &\left[\theta_j^n \varphi + i\frac{w_j^n}{\Delta S}\varphi \sin(q\Delta S) - \frac{\varphi}{3}\sin^2(\frac{q\Delta S}{2}) \right] / \\
&\left[\left(-\frac{4\gamma_j^n}{\Delta S^2} + \frac{\varphi + c_n}{3} \right) \sin^2(\frac{q\Delta S}{2}) - (\varphi + 2c_n)\theta_j^n \right. \\
&\left. + i \left(\frac{\xi_j^n}{\Delta S} - (\varphi + c_n)\frac{w_j^n}{\Delta S} \right) \sin(q\Delta S) \right] \times \sum_{k=1}^{n-1} \left(\psi_{k+1} - \psi_k\right) \vartheta^{n-k} \\
&+ \nu^n / \left[\left(-\frac{4\gamma_j^n}{\Delta S^2} + \frac{\varphi + c_n}{3} \right) \sin^2(\frac{q\Delta S}{2}) - (\varphi + 2c_n)\theta_j^n \right. \\
&\left. + i \left(\frac{\xi_j^n}{\Delta S} - (\varphi + c_n)\frac{w_j^n}{\Delta S} \right) \sin(q\Delta S) \right],
\end{aligned}
\tag{4.8}
$$

Noticing that $\vartheta^0 = 0$. From (4.4) and (4.6), we obtain (C_1 is constant)

$$
\begin{aligned}
\|R^n\|_2 \le \sqrt{N\Delta S} C_1 \left(\Delta\tau^{2-\alpha} + \Delta S^4\right) = \sqrt{L} C_1 \left(\Delta\tau^{2-\alpha} + \Delta S^4\right), \\
n = 0, 1, \ldots, M.
\end{aligned}
\tag{4.9}
$$

Because of the convergence of series in the right hand side of (4.7b), there is a positive constant C_2 such that

$$|\nu^n| \equiv \left|\nu_j^n\right| \leq C_2 \Delta\tau \left|\nu_j^1\right| \equiv C_2 \Delta\tau \left|\nu^1\right|, \quad n = 1, 2, \ldots, M. \tag{4.10}$$

Lemma 4.6. *The following relationship is established*

$$\begin{aligned} 1/\Bigg[&\left(\left[-\frac{4\gamma_j^n}{\Delta S^2} + \frac{\varphi + c_n}{3} \right] \sin^2(\frac{q\Delta S}{2}) - (\varphi + 2c_n)\theta_j^n \right)^2 \\ &+ \left(\frac{\xi_j^n}{\Delta S} - (\varphi + c_n)\frac{w_j^n}{\Delta S} \right)^2 \sin^2(q\Delta S) \Bigg] \leq 9. \end{aligned}$$

Lemma 4.7. *Suppose that ϑ^n is a solution of (4.8), then there is a positive constant C_3 such that*

$$|\vartheta^n| \leq C_3 (1 + 3\Delta\tau)^n |\nu^1|, \quad n = 1, 2, \ldots, M.$$

Proof. We show the proof by using the mathematical induction. From (4.8), (4.10) and (4.6), we obtain

$$\begin{aligned} |\vartheta^1|^2 \leq & |\nu^1|^2 / \Bigg[\left(\left[-\frac{4\gamma_j^n}{\Delta S^2} + \frac{\varphi + c_n}{3} \right] \sin^2(\frac{q\Delta S}{2}) - (\varphi + 2c_n)\theta_j^n \right)^2 \\ &+ \left(\frac{\xi_j^n}{\Delta S} - (\varphi + c_n)\frac{w_j^n}{\Delta S} \right)^2 \sin^2(q\Delta S) \Bigg] \\ \leq & 9\Delta\tau^2 C_3^2 |\nu^1|^2, \quad \Rightarrow \quad |\vartheta^1| \leq 3\Delta\tau C_3 |\nu^1| \leq C_3(1 + 3\Delta\tau)|\nu^1|. \end{aligned}$$

Now, we suppose $\left|\vartheta^k\right| \leq C_3(1+3\Delta\tau)^k |\nu^1|$, $k = 2, 3, \ldots, n-1$. Applying (4.10), Lemma 4.3 and (4.6) into (4.8), we prove $|\vartheta^n| \leq C_3(1+3\Delta\tau)^n|\nu^1|$, where $C_3 = \max\{C_2, C_3\}$,

$$\begin{aligned} |\vartheta^n| \leq & \left| \theta_j^n \varphi + i \frac{w_j^n}{\Delta S} \varphi \sin(q\Delta S) - \frac{\varphi}{3} \sin^2(\frac{q\Delta S}{2}) \right| / \\ & \left| \left(-\frac{4\gamma_j^n}{\Delta S^2} + \frac{\varphi + c_n}{3} \right) \sin^2(\frac{q\Delta S}{2}) - (\varphi + 2c_n)\theta_j^n \right. \\ & \left. + i \left(\frac{\xi_j^n}{\Delta S} - (\varphi + c_n)\frac{w_j^n}{\Delta S} \right) \sin(q\Delta S) \right| \times \left| \sum_{k=1}^{n-1} (\psi_{k+1} - \psi_k) \vartheta^{n-k} \right| \\ & + |\nu^n| / \left| \left(-\frac{4\gamma_j^n}{\Delta S^2} + \frac{\varphi + c_n}{3} \right) \sin^2(\frac{q\Delta S}{2}) - (\varphi + 2c_n)\theta_j^n \right. \\ & \left. + i \left(\frac{\xi_j^n}{\Delta S} - (\varphi + c_n)\frac{w_j^n}{\Delta S} \right) \sin(q\Delta S) \right| \end{aligned}$$

$$
\begin{aligned}
&\leq \left|\sum_{k=1}^{n-1} (\psi_{k+1} - \psi_k)\, \vartheta^{n-k}\right| + 3\left|\nu^n\right| \\
&\leq \sum_{k=1}^{n-1} (\psi_k - \psi_{k+1}) \left|\vartheta^{n-k}\right| + 3C_2\Delta\tau \left|\nu^1\right| \\
&\leq \sum_{k=1}^{n-1} (\psi_k - \psi_{k+1})\, C_3(1+3\Delta\tau)^{n-k}|\nu^1| + C_2(1+3\Delta\tau)\left|\nu^1\right| \\
&\leq C_3(1+3\Delta\tau)^{n-1}|\nu^1| \sum_{k=1}^{n-1} (\psi_k - \psi_{k+1}) + C_2(1+3\Delta\tau)\left|\nu^1\right| \\
&\leq C_3(1+3\Delta\tau)^{n-1}|\nu^1|\,(1-\psi_n) + C_2(1+3\Delta\tau)\psi_n\left|\nu^1\right| \\
&\leq C_3(1+3\Delta\tau)^{n}|\nu^1|\,(1-\psi_n+\psi_n) = C_3(1+3\Delta\tau)^{n}|\nu^1|.
\end{aligned}
$$

This ends the proof. □

Theorem 4.8. *Assume that $V(S,\tau)$ is the exact solution of (2.5) and $\widetilde{V}(S,\tau)$ is the exact solution of (3.8), the compact difference scheme (3.8) is convergent, and the convergence order is $O\left(\Delta\tau^{2-\alpha} + \Delta S^4\right)$.*

Proof. Consider Lemma 4.7, combine (4.7a), (4.7b) and (4.9)

$$
\|E^n\|_2 \leq C_3(1+3\Delta\tau)^n \left\|R^1\right\|_2 \leq C_1C_3\sqrt{L}\exp(3n\Delta\tau)\left(\Delta\tau^{2-\alpha} + \Delta S^4\right).
$$

Since $n\Delta\tau \leq T$, we derive $\|E^n\|_2 \leq C\left(\Delta\tau^{2-\alpha} + \Delta S^4\right)$, where

$$
C = C_1C_3\sqrt{L}\exp(3n\Delta\tau),
$$

This finishes the proof. □

5. Numerical examples

Now, we exhibit the accuracy of the introduced scheme with three examples for solving the time-fractional Black-Scholes equation under CEV model in which interest rate and dividend yield are as deterministic time-dependent parameters. We compute the time of running the program by using CPU times of MATLAB R2015a. The CPU times show the low volume of computation and the advantages of the introduced scheme. Furthermore, in all three examples, we will compare the generalized Black-Scholes model with the time-fractional Black-Scholes equation under CEV model when $\alpha = 1$ and $\beta = 0$. We also discuss the Greek letters Δ, Γ and Θ using the figure.

Example 5.1. *Consider the time-fractional Black-Scholes Eq. (2.5) with the parameters: $r(t) = 0.1 + 0.05e^{-t}$, $D(t) = 0.03 + 0.001e^{0.01t}$ [50], $K = 50$, $\sigma_0 = 0.4$, $S_0 = 50$, $T = 3$, $\beta = -0.5$, $\alpha = 0.8$, $N = M = 100$ and $S_{\max} = 3K$.*

Example 5.2. *Price the American call option model (2.5) with the parameters: $r(t) = 0.075 + 0.05t$ [29], $D(t) = 0.05$, $K = 50$, $\sigma_0 = 0.4$, $S_0 = 50$, $T = 3$, $\beta = -0.5$, $\alpha = 0.8$, $N = M = 100$ and $S_{\max} = 3K$.*

Example 5.3. *Obtain the American call option price for model (2.5) with the parameters:* $r(t) = 0.1 + 0.0005t$ [39], $D(t) = 0.02$, $K = 50$, $\sigma_0 = 0.4$, $S_0 = 50$, $T = 3$, $\beta = -0.5$, $\alpha = 0.8$, $N = M = 100$ *and* $S_{\max} = 3K$.

TABLE 1. CPU time to determine option price in expiry date.

N	M	Example 1 CPU time	Example 2 CPU time	Example 3 CPU time
64	64	0.281639 s	0.286932 s	0.292268 s
128	128	1.242377 s	1.221253 s	1.203057 s
256	264	4.881068 s	4.815844 s	4.887891 s
512	512	23.071165 s	22.823704 s	23.102432 s
1024	1024	128.594945 s	130.967890 s	130.980526 s

TABLE 2. Option price for different α in expiry date.

	S	$\alpha = 0.80$	$\alpha = 0.85$	$\alpha = 0.90$	$\alpha = 0.95$
Example 1	30	5.1104	5.3491	5.5929	5.8421
	60	22.4017	22.8049	23.2154	23.6336
	90	49.0168	49.2749	49.5398	49.8124
	120	80.8191	80.9912	81.1681	81.3502
Example 2	30	5.0655	5.3655	5.6727	5.9873
	60	21.5117	22.0903	22.6809	23.2838
	90	46.7652	47.2432	47.7340	48.2382
	120	76.4837	76.8334	77.1894	77.5520
Example 3	30	5.1687	5.4266	5.6891	5.9565
	60	22.4865	22.9615	23.4440	23.9345
	90	48.8599	49.2118	49.5711	49.9387
	120	79.2868	79.5290	79.7757	80.0272

TABLE 3. Option price for different β in expiry date.

	S	$\beta = -0.4$	$\beta = -0.3$	$\beta = -0.2$	$\beta = -0.1$
Example 1	30	4.9234	4.7464	4.5783	4.4180
	60	22.5520	22.7105	22.8762	23.0474
	90	49.3856	49.7679	50.1598	50.5572
	120	81.2265	81.6356	82.0426	82.4436
Example 2	30	4.8568	4.6594	4.4723	4.2943
	60	21.6257	21.7473	21.8753	22.0081
	90	47.0949	47.4342	47.7789	48.1251
	120	76.8457	77.2025	77.5505	77.8865
Example 3	30	4.9648	4.7716	4.5878	4.4126
	60	22.6000	22.7194	22.8439	22.9724
	90	49.1333	49.4172	49.7085	50.0039
	120	79.5577	79.8313	80.1045	80.3747

We compute CPU time of Examples 5.1-5.3 for different N and M in Table 1. This table shows that CPU time is almost 2 min 10 s for $N = M = 1024$. We investigate the effect of each parameter time-fractional derivative order (α), elasticity factor (β), and initial instantaneous volatility (σ_0) on the long memory in Tables 2-4. Table 2 displays that option price is increasing for $\alpha = \{0.80, 0.85, 0.90, 0.95\}$. Table 3 shows that option price is both decreasing and increasing for $\beta = \{-0.4, -0.3, -0.2, -0.1\}$. Table 4 is increasing for $\sigma_0 = \{0.2, 0.3, 0.4, 0.5\}$.

Table 4. Option price for different σ_0 in expiry date.

	S	$\sigma_0 = 0.2$	$\sigma_0 = 0.3$	$\sigma_0 = 0.4$	$\sigma_0 = 0.5$
Example 1	30	1.6013	3.2722	5.1104	7.0233
	60	18.7947	20.2931	22.4017	24.8057
	90	46.5690	47.5341	49.0168	50.8966
	120	78.1671	79.5494	80.8191	82.1093
Example 2	30	1.7016	3.3045	5.0655	6.9024
	60	17.9890	19.4581	21.5117	23.8501
	90	44.4315	45.3374	46.7652	48.5862
	120	73.9721	75.2834	76.4837	77.6991
Example 3	30	1.5689	3.2812	5.1687	7.1194
	60	18.7584	20.3488	22.4865	24.8718
	90	46.9493	47.6299	48.8599	50.5255
	120	77.5483	78.4229	79.2868	80.2690

Table 5. Convergence rate for different N when $K = 40$, $\sigma_0 = 0.4$, $S_0 = 40$, $T = 3$, $\beta = -1$, $\alpha = 0.7$ and $S_{\max} = 3K$.

$r(t) = 0.1 + 0.05e^{-t},\ D(t) = 0.03 + 0.001e^{0.01t}$				
M	N	Error	Rate	CPU time
1024	64	1.5287e-05	–	–
	128	9.9149e-07	3.9466	551.254211 s
	256	6.4936e-08	3.9325	561.611072 s
	512	4.0624e-09	3.9986	597.201253 s
	1024	2.4475e-10	4.0529	660.541270 s
$r(t) = 0.075 + 0.05t,\ D(t) = 0.05$				
M	N	Error	Rate	CPU time
1024	64	1.1963e-05	–	–
	128	7.7641e-07	3.9456	551.784606 s
	256	5.0150e-08	3.9525	569.122115 s
	512	3.2328e-09	3.9554	590.740887 s
	1024	1.8889e-10	4.0972	656.945209 s
$r(t) = 0.1 + 0.0005t,\ D(t) = 0.02$				
M	N	Error	Rate	CPU time
1024	64	6.9990e-06	–	–
	128	2.8956e-07	4.5952	540.760544 s
	256	2.0692e-08	3.8067	550.976153 s
	512	1.3045e-09	3.9875	578.041359 s
	1024	7.5417e-11	4.1124	664.398942 s

The American options have no closed-form solution. Therefore, to illustrate the fourth-order convergence rate in space numerically, we compare our solution with the approximated solution that N and M are large enough (see [5]). We define the discrete Maximum-norm error as

$$e^{N,M} = \max_{\substack{0\le j\le N\\0\le n\le M}} \left\| \bar{V}_j^n - \widehat{V}_j^n \right\|_\infty = \max_{0\le n\le M}\left(\max_{0\le j\le N} \left| \bar{V}_j^n - \widehat{V}_j^n \right| \right),$$

where $\bar{V}_j^n$ is the approximated solution for $N = 2048$ and $M = 1024$ and $\widehat{V}_j^n$ is the our solution. The convergence rate in space is obtained from following relation

$$\text{Rate} = \log_2\left(\frac{e^{N,M}}{e^{2N,M}}\right).$$

Table 6. Comparison of the American put option price by compact difference scheme with Zhou's result at $r = 0.05$, $D = 0$, $K = 40$, $S_0 = 40$, $T = 3$, $S_{\max} = 200$, $N = 800$ and $M = 200$ in expiry date.

$\beta = 0$						
	$\sigma_0 = 0.1$			$\sigma_0 = 0.2$		
α	LTM	FDM	Compact	LTM	FDM	Compact
1.0	1.2189	1.2362	1.2308	3.4116	3.4792	3.4741
0.9	1.1771	1.1912	1.1830	3.2651	3.3157	3.3080
0.7	1.0959	1.1028	1.0879	2.9817	3.0071	2.9927
0.4	0.9778	0.9793	0.9347	2.5770	2.5829	2.5418
0.2	0.9005	0.9002	0.8094	2.3184	2.3191	2.2341
$\beta = -1$						
	$\sigma_0 = 0.1$			$\sigma_0 = 0.2$		
α	LTM	FDM	Compact	LTM	FDM	Compact
1.0	1.1877	1.2020	1.1977	3.3325	3.3834	3.3880
0.9	1.1485	1.1604	1.1528	3.1918	3.2297	3.2300
0.7	1.0722	1.0802	1.0641	2.9208	2.9400	2.9309
0.4	0.9609	0.9657	0.9202	2.5347	2.5397	2.5048
0.2	0.8877	0.8922	0.8017	2.2876	2.2898	2.2158

Table 7. Comparison of the American put option price by compact difference scheme with Pun's result at $\alpha = 1$, $r = 0.05$, $D = 0$, $\sigma_0 = 0.4$, $\beta = -0.1$, $K = 40$, $S_0 = 40$, $T = 1$ and $S_{\max} = 3K$ in expiry date.

		Pun's result			$\alpha = 1$, N=M=240
S	BS sol.	1st-order	2nd-order	RE	Compact
20	19.7979	20	20	20	20
30	12.6119	11.1815	10.7256	10.7299	10.9832
40	7.98526	5.77404	5.36116	5.33151	5.4557
50	5.20209	2.84158	2.58372	2.54957	2.5448
60	3.38609	1.23602	1.18842	1.13693	1.1450

In Table 5, we list the error estimates and convergence rates of the introduced scheme for three $r(t)$ and $D(t)$. This table shows that the obtained convergence rates support Theorem 4.8. The CPU times in this table represent the sum of the run times associated with computing $e^{N,M}$, $e^{2N,M}$ and convergence rate per run.

We can also use the introduced difference scheme to determine the American put option price. Hence, we present two comparisons of the introduced scheme with the results of [53] and [41] in Tables 6 and 7, respectively.

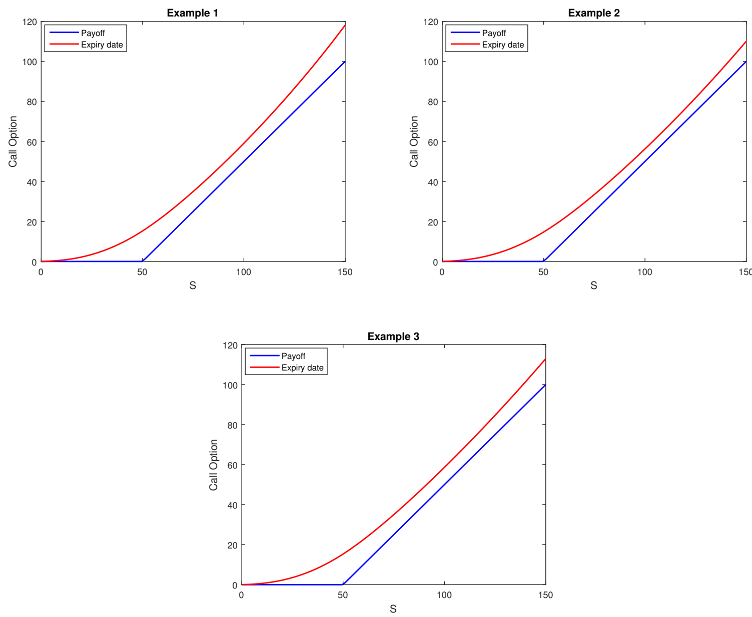

Figure 1. Option price of three examples in expiry date with their payoff.

Figure 1 shows the option price of three examples in the expiry date with their payoff. Figure 2 displays that time-fractional Black-Scholes equations under the CEV model equal to generalized Black-Scholes model in expiry date when $\alpha = 1$ and $\beta = 0$ for all examples. The generalized Black-Scholes model of this figure with an implicit difference scheme is described in Appendix A. Figures 3-5 illustrate the effect of parameters α, β, and σ_0 on option price, respectively. Figures 6-8 represent option price sensitivities relative to the parameters.

6. Conclusion

Due to the limitations of the Black-Scholes model, we need a model that is closer to market realities and show memory effect in financial pricing. In this work, we investigated American call option pricing based on the time-fractional Black-Scholes equation under the CEV model with time-dependent parameters of risk-free interest rate and dividend yield. We presented a compact difference scheme to price the American call option as numerically. We analyzed stability and convergence of the introduced difference scheme using Fourier analysis and showed that the introduced scheme has the fourth-order convergence rate in space. Numerical examples express that the time-fractional Black-Scholes equation under the CEV model coincides with its generalized Black-Scholes equation as $\alpha = 1$ and $\beta = 0$. Also, we observed which American option price is increasing with respect to the time-fractional order derivative (α) and initial instantaneous volatility (σ_0), and the American option price is both

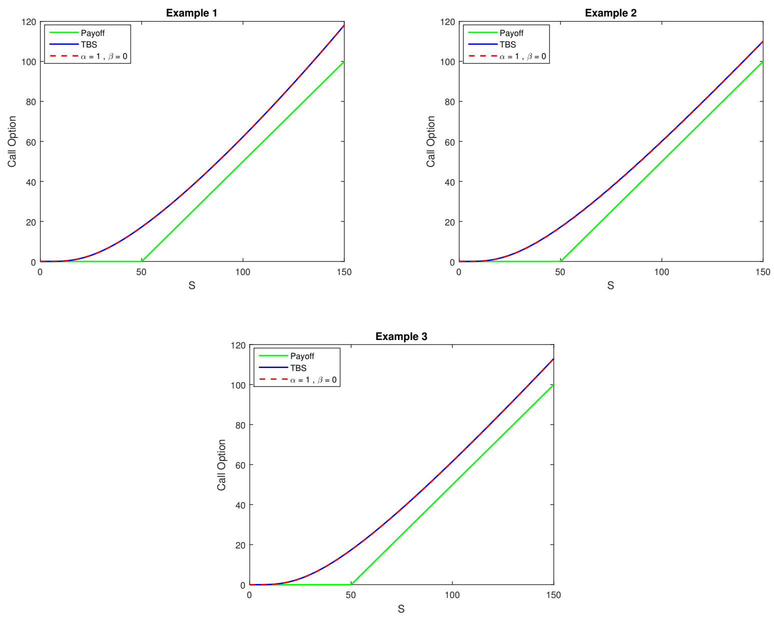

FIGURE 2. Comparison of generalized Black-Scholes model and time-fractional Black-Scholes equation under CEV model in expiry date when $\alpha = 1$ and $\beta = 0$.

decreasing and increasing as elasticity factor (β) is increasing. Moreover, we discussed the American option price sensitivities relative to the underlying asset and time to expire. As a suggestion, we can use the introduced difference scheme of this paper to price other options.

APPENDIX A. GENERALIZED BLACK-SCHOLES MODEL ITS DIFFERENCE SCHEME

The generalized Black-Scholes model for pricing American call option is following form

$$\begin{aligned}
&\frac{\partial C(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S,t)}{\partial S^2} + (r(t) - D(t))\, S \frac{\partial C(S,t)}{\partial S} \\
&\quad - r(t) C(S,t) = 0, \qquad \text{(A.1)} \\
&C(S,T) = \max(S - K, 0), \\
&C(0,t) = 0, \\
&C(S_f(t), t) = S_f(t) - K, \\
&\frac{\partial C(S_f(t), t)}{\partial S} = 1,
\end{aligned}$$

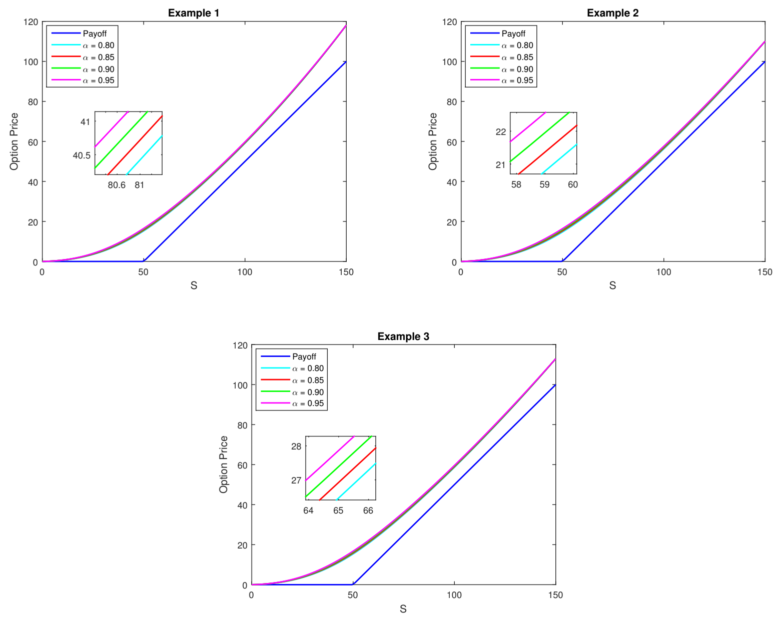

FIGURE 3. Option price based on different α in expiry date.

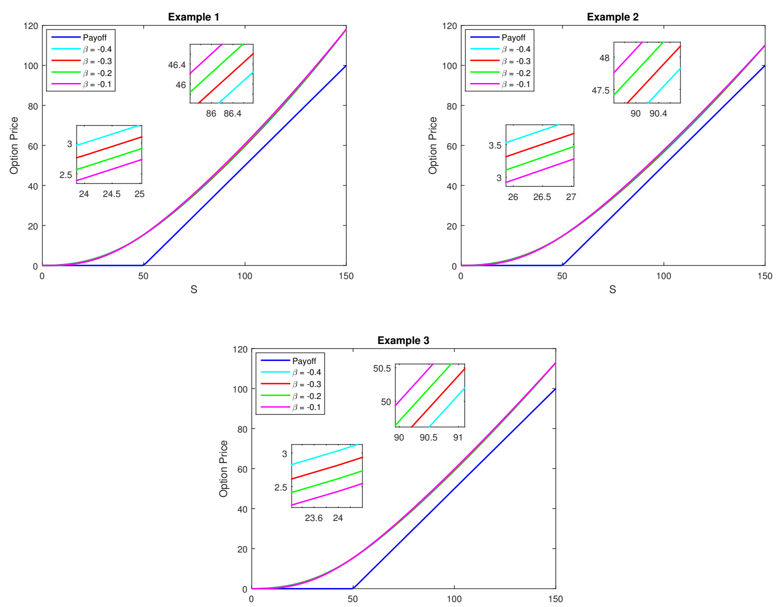

FIGURE 4. Option price based on different β in expiry date.

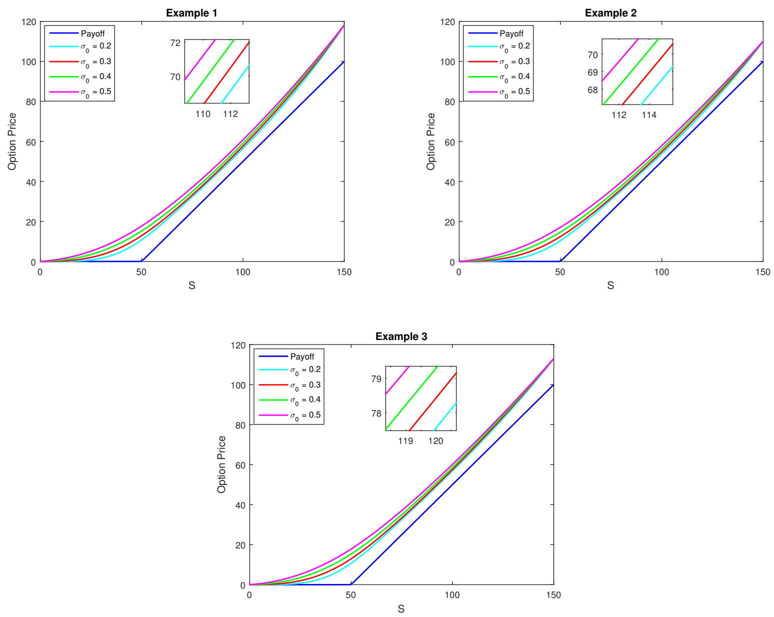

FIGURE 5. Option price based on different σ_0 in expiry date.

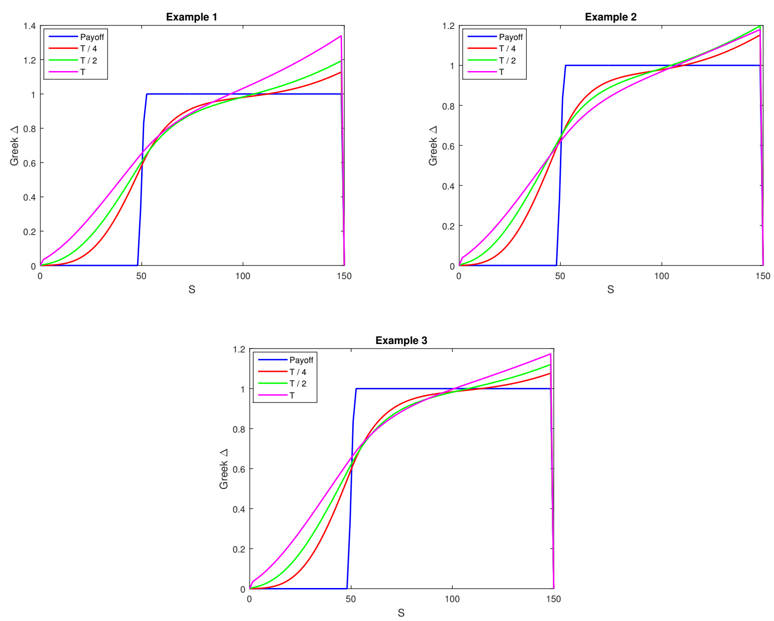

FIGURE 6. American options in the Greeks Δ.

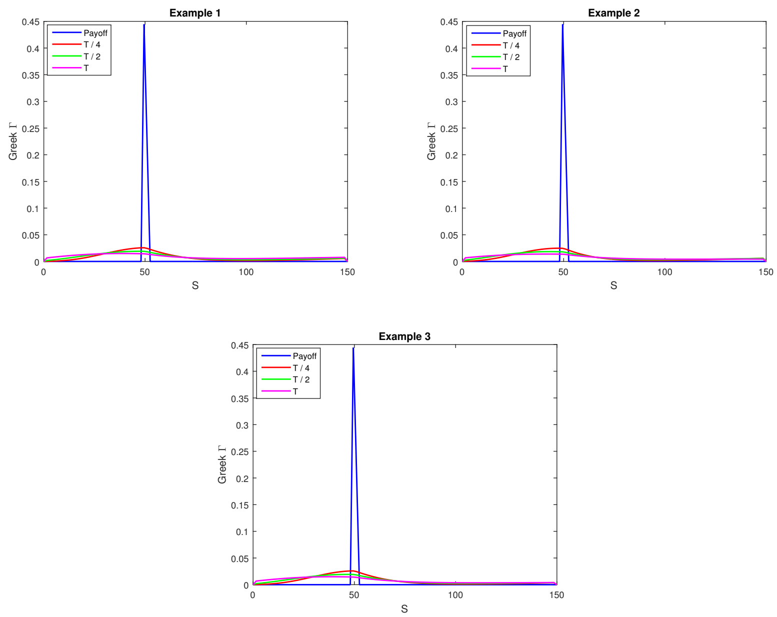

FIGURE 7. American options in the Greeks Γ.

where σ is volatility of underlying asset. Using approximation derivatives,

$$\frac{\partial C(S_j,t_n)}{\partial t} = \frac{C(S_j,t_{n+1}) - C(S_j,t_n)}{\Delta t} + O\left(\Delta t\right),$$

$$\frac{\partial^2 C(S_j,t_n)}{\partial S^2} = \frac{C(S_{j-1},t_n) - 2C(S_j,t_n) + C(S_{j+1},t_n)}{\Delta S^2} + O\left(\Delta S^2\right),$$

$$\frac{\partial C(S_j,t_n)}{\partial S} = \frac{C(S_{j+1},t_n) - C(S_{j-1},t_n)}{2\Delta S} + O\left(\Delta S^2\right),$$

we apply following implicit difference scheme on (A.1) as

$$\begin{aligned}
&\frac{C_j^{n+1} - C_j^n}{\Delta t} + \frac{1}{2}\sigma^2 S_j^2 \left[\frac{C_{j-1}^n - 2C_j^n + C_{j+1}^n}{2\Delta S^2} + \frac{C_{j-1}^{n+1} - 2C_j^{n+1} + C_{j+1}^{n+1}}{2\Delta S^2}\right] \\
&\quad + \left[r\left(t_n\right) - D\left(t_n\right)\right] S_j \left[\frac{C_{j+1}^n - C_{j-1}^n}{4\Delta S} + \frac{C_{j+1}^{n+1} - C_{j-1}^{n+1}}{4\Delta S}\right] \\
&\quad - r\left(t_n\right)\left[\frac{C_j^n + C_j^{n+1}}{2}\right] = 0.
\end{aligned}$$

The matrix form above scheme can be written as

$$A^n C^{n+1} = B^n C^n - F^{n+1}, \quad n \geq 0,$$

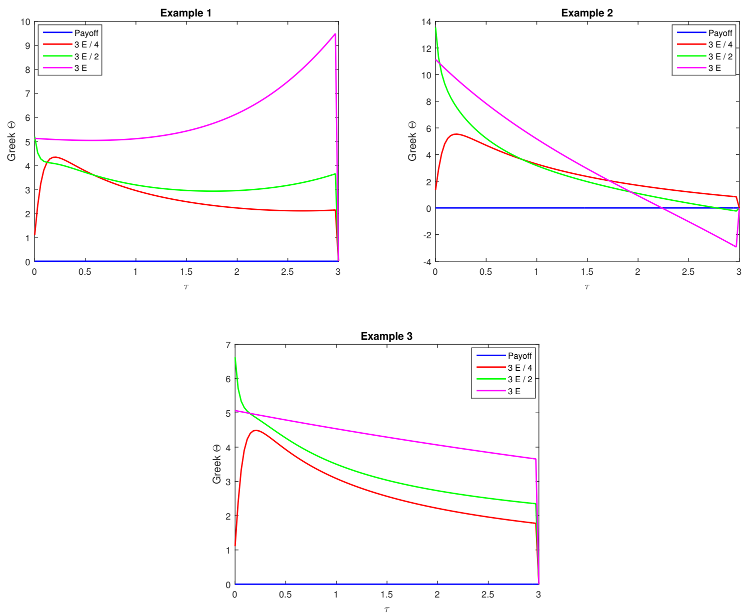

Figure 8. American options in the Greeks Θ.

where

$$A^n = \begin{bmatrix} b_1^n & e_2^n & & & 0 \\ a_1^n & b_2^n & e_3^n & & \\ & \ddots & \ddots & \ddots & \\ & & & & e_{N-1}^n \\ 0 & & & a_{N-2}^n & b_{N-1}^n \end{bmatrix},$$

$$B^n = \begin{bmatrix} d_1^n & -e_2^n & & & 0 \\ -a_1^n & d_2^n & -e_3^n & & \\ & \ddots & \ddots & \ddots & \\ & & & & -e_{N-1}^n \\ 0 & & & -a_{N-2}^n & d_{N-1}^n \end{bmatrix},$$

$$C^n = \left[C_1^n, C_2^n, \ldots, C_{N-1}^n\right]^t,$$
$$F^{n+1} = \left[a_0^n\left(C_0^n + C_0^{n+1}\right), 0, \ldots, 0, e_N^n\left(C_N^n + C_N^{n+1}\right)\right]^t.$$

and

$$\begin{cases} a_{j-1}^{n} = \dfrac{\sigma^2}{4}\dfrac{\Delta t}{\Delta S^2}S_j^2 - \dfrac{\Delta t}{4\Delta S}S_j\left[r\left(t_n\right) - D\left(t_n\right)\right], \\ b_j^n = -\dfrac{\sigma^2}{2}\dfrac{\Delta t}{\Delta S^2}S_j^2 - \dfrac{\Delta t}{2}r\left(t_n\right) + 1, \\ e_{j+1}^{n} = \dfrac{\sigma^2}{4}\dfrac{\Delta t}{\Delta S^2}S_j^2 + \dfrac{\Delta t}{4\Delta S}S_j\left[r\left(t_n\right) - D\left(t_n\right)\right], \\ d_j^n = \dfrac{\sigma^2}{2}\dfrac{\Delta t}{\Delta S^2}S_j^2 + \dfrac{\Delta t}{2}r\left(t_n\right) + 1, \end{cases}, \quad j = 1, \dots, N-1.$$

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