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Global dynamics and numerical bifurcation of a bioeconomic system

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Abstract A predator-prey model was extended to include nonlinear harvesting of the predator guided by its population, such that harvesting is only implemented if the predator population exceeds an economic threshold. Theoretical results showed that the harvesting system undergoes multiple bifurcations, including fold, supercritical Hopf, Bogdanov-Takens and cusp bifurcations. We determine stability and dynamical behaviors of the equilibrium of this system. Numerical simulation results are given to support our theoretical results.

Keywords. Hopf bifurcation, Bogdanov-Takens bifurcation, Dynamical behavior, Cusp bifurcations.

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1. Introduction

Mathematical models play a vital role in investigating the optimal management and exploitation of renewable resources. The dynamical relationships between species and their complex properties are the heart of many ecological and biological processes. Predator-prey dynamics are well studied in the process of control of some ecosystems. Typically, a predator-prey model focuses on interactions between two species taking into account some aspects that are considered nodal to explain the dynamics. Recently, nonsmooth dynamic systems have been used to study threshold policies in various applied fields [9, 14, 15, 19] such as plant disease control and human infectious disease control. A number of researchers have examined the influence of harvesting strategies on the interaction of different species [3, 8]. For example, predator-prey models with linear and constant harvesting regimes were considered by researchers [7, 9]. Continuous harvesting models have been widely studied but their assumptions are questionable since regardless of the population of the predator and prey, they may

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lead to the extinction of the predator or the prey. Predator-prey models with different functional responses or (and) harvesting are refined so as to better reflect the specific characteristics of the different populations or economical need. The overexploitation of some fish stocks may have consequences for the whole ecosystem which are difficult to be forecasted, and may eventually lead to depletion of some species, and thus decreasing yields, up to the danger of unexpected extinction of resources. For these reasons, central institutions usually enforce forms of regulation either by imposing harvesting restrictions, such as constant efforts, individual fishing quotas, taxations, or by limiting the kinds of fish to be caught or the regions where exploitation is allowed (see e.g. [4, 6]). The objective is to maximize the monetary social benefit as well as conservation of the ecosystem. More and more complex bifurcation phenomena are discovered and studied in predator-prey models, see, for example, homoclinic bifurcation and Hopf bifurcations of codimension 1 in [17], cusp bifurcation of codimension 2 in[10, 18], Bogdanov-Takens bifurcation of codimension 3 (cusp case) in [11, 22], Bogdanov-Takens bifurcation of codimension 3 (focus case) in [18], Bogdanov-Takens bifurcation of codimension 3 (saddle case) in [5], Hopf bifurcation of codimension 2 in [11], etc. In this work we will use the term complexity to describe the ecological complexity found in nature as well as the of dynamical complexity of the models. In recent years, a significant number of the published papers on the mathematical Continuous and discrete time models of biology, physics, power electrical system,... discussed the systems of differential equations and the associated numerical methods. Mathematical models on prey predator systems create a major interest during the last few decades. Study of such system with discrete models and continuous models can be found in [10, 17, 20, 21]. In this paper werely heavily on advanced numerical methods [1], namely numerical continuation to obtain results that cannot be obtained analytically. Numerical bifurcation analysis techniques are very powerful and efficient in physics, biology, engineering, and economics [9, 12, 13, 17]. In this paper, we extend a predator-prey model to include nonlinear harvesting of the predator population and extend the simple ordinary differential equation model into a predator population guided harvesting model in which the harvesting strategy is implemented according to an economic threshold. We investigate the dynamical complexities of a predator-prey model with switching between a traditional model (the free system) and a model with a nonlinear harvesting regime for the predator population (the harvesting system). This system first was studied in [16], in which some local bifurcations were studied analytically. The system is shown as follows:

$$\begin{cases} \dfrac{dx}{dt} = rx(1-\dfrac{x}{K}) - \beta xy, \\ \dfrac{dy}{dt} = k\beta xy - dy - \dfrac{qy}{1+wy}, \end{cases} \tag{1.1}$$

where x and y represent the densities of the prey and the predator, respectively, r, K, β, q, w, k and d are positive constants. The prey grows logistically with carrying capacity K and intrinsic growth rate r in the absence of predation. The term $\frac{qy}{1+wy}$ which is a saturation function, represents the harvesting of the predator. w is a suitable constant and q is the rate of harvesting. The parameter β is the per-capita rate of predation of the predator, k is a constant conversion rate of eaten prey into new predator abundance and d is the death rate of the predator.

This paper is organized as follows: In section 2, we consider the mathematical model and discussed some basic dynamical results like positivity, boundedness of solution and existence of possible equilibrium and stability of equilibrium In section 3, Hopf bifurcation, cusp bifurcation and Bogdanov-Takens bifurcation of the interior equilibrium point of the model system is discussed. Numerical simulation results are included to support our analytical results in section 4. The paper concludes with a brief discussion and we summarize our results in section 5.

2. Equilibria of the bioeconomic model and its stability analysis

The system (1.1) has trivial equilibrium point $E_0 = (0,0)$, axial equilibrium point $E_1 = (\gamma, 0)$ and co-existing equilibrium point $E_* = (x_*, y_*)$ where $x_* = \frac{\delta+(\alpha\delta-\beta)\xi}{\beta(1-c')-\delta}$ and $y_* = (\frac{1+\alpha\xi+x_*}{(1-c')})(1-\frac{x_*}{\gamma})$. The Jacobian matrix of system (1.1) at (x, y) takes the form:

$$J = \begin{pmatrix} 1-\frac{2x}{\gamma}-\frac{(1-c')(1+\alpha\xi)y}{(1+\alpha\xi+x)^2} & -\frac{(1-c')x}{1+\alpha\xi+x} \\ \frac{\beta y[(1-c')(1+\alpha\xi)-\xi]}{(1+\alpha\xi+x)^2} & \frac{\beta[(1-c')x+\xi]}{1+\alpha\xi+x}-\delta \end{pmatrix}.$$

At trivial equilibrium point $(0,0)$, the Jacobian matrix is given by

$$J = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\beta\xi}{1+\alpha\xi}-\delta \end{pmatrix}.$$

It has always one positive eigenvalue 1 and other eigenvalue $\frac{\beta\xi}{1+\alpha\xi}-\delta$.

Lemma 2.1. *The equilibrium $E_0 = (0,0)$ of the system (1.1) is always a saddle point if $\frac{\beta\xi}{1+\alpha\xi} < \delta$, and the equilibrium E_0 is unstable if $\frac{\beta\xi}{1+\alpha\xi} > \delta$.*

At the axial equilibrium point $(\gamma, 0)$, the Jacobian matrix is given by

$$J = \begin{pmatrix} -1 & -\frac{(1-c')\gamma}{1+\alpha\xi+\gamma} \\ 0 & \frac{\beta[(1-c')\gamma+\xi]}{1+\alpha\xi+\gamma}-\delta \end{pmatrix}.$$

One negative eigenvalue is -1 and other eigenvalue is $\frac{\beta[(1-c')\gamma+\xi]}{1+\alpha\xi+\gamma}-\delta$.

Lemma 2.2. *Equilibrium point $E_\gamma = (\gamma, 0)$ is saddle for $\frac{\beta[(1-c')\gamma+\xi]}{1+\alpha\xi+\gamma} > \delta$ and stable for $\frac{\beta[(1-c')\gamma+\xi]}{1+\alpha\xi+\gamma} < \delta$.*

At the co-existing equilibrium point $(\frac{\delta+(\alpha\delta-\beta)\xi}{\beta(1-c')-\delta}, (\frac{1+\alpha\xi+x_*}{(1-c')})(1-\frac{x_*}{\gamma}))$, the Jacobian matrix is given by

$$J = \begin{pmatrix} \frac{x_*}{\gamma}(\frac{\gamma-x_*}{1+\alpha\xi+x_*}-1) & -\frac{(1-c')x_*}{1+\alpha\xi+x_*} \\ \frac{\beta y_*[(1-c')(1+\alpha\xi)-\xi]}{(1+\alpha\xi+x_*)^2} & 0 \end{pmatrix}.$$

Theorem 2.3. *When* $1 > c'$ *and* $\frac{\beta[(1-c')\gamma+\xi]}{1+\alpha\xi+\gamma} = \delta$*, the degenerate equilibrium point* $E_\gamma = (\gamma, 0)$ *of the system (1.1) is a saddle-node.The system (1.1) has trivial equilibrium point* $E_0 = (0, 0)$*, axial equilibrium point* $E_1 = (K, 0)$ *and co-existing equilibrium point* $E_* = (x_*, y_*)$ *where* $x_* = \frac{d}{k\beta}$ *and* $y_* = \frac{r}{\beta}(1 - \frac{d}{k\beta K})$*.*

The Jacobian matrix of system (1.1) at (x, y) takes the form:

$$J = \begin{pmatrix} r - \frac{2rx}{K} - \beta y & -\beta x \\ k\beta y & k\beta x - d - \frac{q}{(1+wy)^2} \end{pmatrix}.$$

2.1. **Trivial equilibrium point.** At trivial equilibrium point $(0, 0)$, the Jacobian matrix is given by

$$J = \begin{pmatrix} r & 0 \\ 0 & -(d+q) \end{pmatrix}.$$

It has always one positive eigenvalue r and other eigenvalue $-(d + q)$.

Lemma 2.4. *The equilibrium* $E_0 = (0, 0)$ *of system (1.1) is always a saddle point.*

2.2. **Axial equilibrium point.** At the axial equilibrium point $(K, 0)$, the Jacobian matrix is given by

$$J = \begin{pmatrix} -r & -\beta K \\ 0 & k\beta K - d - q \end{pmatrix}.$$

One negative eigenvalue is $-r$ and other eigenvalue is $k\beta K - d - q$.

Lemma 2.5. *Equilibrium point* $(K, 0)$ *is saddle for* $k\beta K > d + q$ *and stable for* $k\beta K < d + q$ *.*

2.3. **Co-existing equilibrium point.** At the co-existing equilibrium point $(\frac{d}{k\beta}, \frac{r}{\beta}(1-\frac{d}{k\beta K}))$, the Jacobian matrix is given by

$$J = \begin{pmatrix} \frac{-rd}{k\beta K} & -\frac{d}{k} \\ kr(1-\frac{d}{k\beta K}) & \frac{q}{[1+\frac{wr}{\beta}(1-\frac{d}{k\beta K})]^2} \end{pmatrix}.$$

One negative eigenvalue is $-r$ and other eigenvalue is $k\beta K - d - q$. Now, let $traceJ|_{(x_*,y_*)} = -\frac{rd}{k\beta K} + \frac{q}{[1+\frac{wr}{\beta}(1-\frac{d}{k\beta K})]^2}$, and $detJ|_{(x_*,y_*)} = -\frac{rdq}{k\beta K[1+\frac{wr}{\beta}(1-\frac{d}{k\beta K})]^2} + \frac{dr}{(1-\frac{d}{k\beta K})}$

Lemma 2.6. *Define* $\Theta = traceJ|_{(x_*,y_*)}$, $\Delta = detJ|_{(x_*,y_*)}$ *then the following statements hold:*

(a) *choose* $\Theta > 0$*, and* $\Delta > 0$*. The eigenvalues of* $J|_{(x_*,y_*)}$ *are both either real number with the positive signs or complex conjugate with non-zero positive real part. Therefore the equilibrium point* (x_*, y_*) *in either case unstable node or unstable spiral.*

(b) *choose* $\Theta < 0$*, and* $\Delta > 0$*. The eigenvalues of* $J|_{(x_*,y_*)}$ *are both either real numbers with the negative signs or complex conjugate with non-zero negative real part. Therefore theequilibrium point* (x_*, y_*) *in either case stable node or stable spiral.*

(c) *we choose* $\Theta < 0$*, or* $\Theta < 0$ *and* $\Delta < 0$ *The eigenvalues of* $J|_{(x_*,y_*)}$ *are both real numbers with opposite sign to each other. Therefore the equilibrium point* (x_*, y_*) *is saddle node.*

(d) *we choose* $\Theta = 0$, *and* $\Delta > 0$. *The eigenvalues of* $J|_{(x_*,y_*)}$ *are purely complex conjugate to each other. Therefore the equilibrium point* (x_*, y_*) *is center.*
(e) *we choose* $\Theta = 0$, *and* $\Delta < 0$. *The eigenvalues of* $J|_{(x_*,y_*)}$ *are both real numbers with same magnitude and opposite sign. Therefore the equilibrium point* (x_*, y_*) *is saddle node.*

3. Bifurcations

The critical parameter value at which qualitative change of dynamics occur is called bifurcation point. Qualitatively different dynamical behaviour may appear in the model with the variation of model parameters. To identify the possible qualitatively different dynamical behaviour with the variation of parameters q, β, K, we do bifurcation analysis of the system (1.1) with respect to q, β, K,.

3.1. **The cusp and Bogdanov-Takens bifurcations analysis.** We first prove the unique positive equilibrium of the system (1.1) is a cusp of codimension 2, then discuss the Bogdanov-Takens bifurcation of system (1.1). Taking the transformation $x_{1*} = x - x_1, x_{2*} = y - y_1$ and expand the model (1.1) in a power series around the origin, we obtain

$$\begin{cases} \dfrac{dx}{dt} = a_1 x_{1*} + b_1 x_{2*} + p_{11} x_{1*}^2 + p_{12} x_{1*} x_{2*} + p_{22} x_{2*}^2 + O(\|x\|^3), \\ \dfrac{dy}{dt} = c_1 x_{1*} + d_1 x_{2*} + q_{11} x_{1*}^2 + q_{12} x_{1*} x_{2*} + q_{22} x_{2*}^2 + O(\|x\|^3), \end{cases} \tag{3.1}$$

where

$$a_1 = \frac{\partial f}{\partial x}|_{(x_1,y_1)} = -r,\ b_1 = \frac{\partial f}{\partial y}|_{(x_1,y_1)} = -\beta K,\ c_1 = \frac{\partial g}{\partial x}|_{(x_1,y_1)} = 0,$$
$$d_1 = \frac{\partial g}{\partial y}|_{(x_1,y_1)} = k\beta K - d - q,\ p_{11} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}|_{(x_1,y_1)} = -\frac{r}{K},$$
$$p_{12} = \frac{\partial^2 f}{\partial x \partial y}|_{(x_1,y_1)} = -\beta,\ p_{22} = \frac{1}{2}\frac{\partial^2 f}{\partial y^2}|_{(x_1,y_1)} = 0\ , q_{11} = \frac{1}{2}\frac{\partial^2 g}{\partial x^2}|_{(x_1,y_1)} = 0,$$
$$q_{12} = \frac{\partial^2 g}{\partial x \partial y}|_{(x_1,y_1)} = k\beta,\ q_{22} = \frac{1}{2}\frac{\partial^2 g}{\partial y^2}|_{(x_1,y_1)} = qw.$$

Then we must have

$$a_1 + d_1 = 0,$$
$$a_1 d_1 + b_1 c_1 = 0.$$

We now use the following transformation

$$y_1 = x_{1*},$$
$$y_2 = -r x_{1*} - \beta K x_{2*},$$

Then model (3.1) reduces to

$$\begin{cases} \dfrac{dy_1}{dt} = y_2 + \alpha_{11}y_1^2 + \alpha_{12}y_1y_2 + \alpha_{22}y_2^2 + O(\|y\|^3), \\ \dfrac{dy_2}{dt} = \beta_{11}y_1^2 + \beta_{12}y_1y_2 + \beta_{22}y_2^2 + O(\|y\|^3), \end{cases} \tag{3.2}$$

where

$$\alpha_{11} = -\frac{1+r}{K},\ \alpha_{12} = \frac{1}{K}, \alpha_{22} = 0,\ \beta_{11} = 2\frac{r^2}{K} + \frac{2r^2qw}{\beta K} + rk\beta,$$
$$\beta_{12} = -\frac{r^2\beta - 2r^2qw}{-\beta K} - k\beta,\ \beta_{22} = -\frac{qw}{\beta K}.$$

There exists a C^∞ invertible transformation given by

$$z_1 = y_1 + (-\frac{1}{K} - \frac{qw}{\beta K})y_1^2,$$
$$z_2 = y_2 - \frac{qw}{\beta K}y_1y_2.$$

such that model (3.2) reduces to

$$\begin{cases} \dfrac{dz_1}{dt} = z_2 + O(\|z\|^3)), \\ \dfrac{dz_2}{dt} = \rho_1 z_1^2 + \rho_2 z_1 z_2 + O(\|z\|^3), \end{cases} \tag{3.3}$$

where $\rho_1 = \beta_{11}, \rho_2 = -\frac{a_1}{b_1}(p_{12} + 2q_{22}) + 2p_{11} + q_{12} = k\beta - \frac{r}{K} - \frac{2qwr}{\beta k}$. If$\rho_1\rho_2 \neq 0$ (non-degeneracy condition) hence the equilibrium E_1^* is a cusp of codimension 2.

Theorem 3.1. *System (1.1) has a cusp point of codimension 2 at the equilibrium E_1.*

Theorem 3.2. *If we choose q and β as bifurcation parameters, then the system (1.1) undergoes Bogdanov-Takens bifurcation in a small neighborhood of E_1^*.*

Proof. Consider the following system

$$\begin{cases} \dfrac{dx}{dt} = rx(1 - \dfrac{x}{K}) - (\beta + \lambda_1)xy = f(x,y), \\ \dfrac{dy}{dt} = k(\beta + \lambda_1)xy - dy - \dfrac{(q+\lambda_2)y}{1+wy} = g(x,y), \end{cases} \tag{3.4}$$

where (λ_1, λ_2) is a parameter vector in a small neighborhood of $(0,0)$. In this case, with the help of the transformation $x = x_1 + x_1^*,\ y = x_2 + y_1^*,\ q = q^* + \lambda_1$ and

$\beta = \beta^* + \lambda_2$. System (3.4) can be written as

$$\begin{cases} \dfrac{dx_1}{dt} = p_0(\lambda) + a_2(\lambda)x_1 + b_2(\lambda)x_2 + p'_{11}x_1^2 + p'_{12}(\lambda)x_1x_2 + p'_{22}(\lambda)x_2^2 + O(\|x\|^3), \\ \dfrac{dx_2}{dt} = q'_0(\lambda) + c_2(\lambda)x_1 + d_2(\lambda)x_2 + q'_{11}(\lambda)x_1^2 + q'_{12}(\lambda)x_1x_2 + q'_{22}(\lambda)x_2^2 + O(\|x\|^3), \end{cases} \tag{3.5}$$

where
$a_2(\lambda) = \frac{\partial f}{\partial x}|_{(x_1^*,y_1^*)} = -r, \quad b_2(\lambda) = \frac{\partial f}{\partial y}|_{(x_1^*,y_1^*)} = -(\beta+\lambda_1)K,$
$c_2(\lambda) = \frac{\partial g}{\partial x}|_{(x_1^*,y_1^*)} - 0, \quad d_2(\lambda) = \frac{\partial g}{\partial x}|_{(x_1^*,y_1^*)} = k(\beta+\lambda_1)K - d - (q+\lambda_2),$
$p'_{11}(\lambda) = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}|_{(x_1^*,y_1^*)} = -\frac{r}{K}, \quad p'_{12}(\lambda) = \frac{\partial^2 f}{\partial x \partial y}|_{(x_1^*,y_1^*)} = -\beta + \lambda_1,$
$p'_{22}(\lambda) = \frac{1}{2}\frac{\partial^2 f}{\partial y^2}|_{(x_1^*,y_1^*)} = 0, \quad q'_{11}(\lambda) = \frac{1}{2}\frac{\partial^2 g}{\partial x^2}|_{(x_1^*,y_1^*)} = 0,$
$q'_{12}(\lambda) = \frac{\partial^2 g}{\partial x \partial y}|_{(x_1^*,y_1^*)} = k(\beta+\lambda_1), \quad q'_{22}(\lambda) = \frac{1}{2}\frac{\partial^2 f}{\partial y^2}|_{(x_1^*,y_1^*)} = (q+\lambda_2 w)$ and
$p_0(\lambda) = -\lambda_1 x_1^* y_1^*,\ q_0(\lambda) = -\lambda_2 \frac{q}{1+wy_1^*}$. Making the affine transformation

$$y_1 = x_1,\ y_2 = a_2x_1 + b_2x_2.$$

We have

$$\begin{cases} \dfrac{dy_1}{dt} = p_0(\lambda) + y_2 + \alpha_{11}(\lambda)y_1^2 + \alpha_{12}(\lambda)y_1y_2 + \alpha_{22}(\lambda)y_2^2 + O(\|y\|), \\ \dfrac{dy_2}{dt} = q\prime_0(\lambda) + c_3(\lambda)y_1 + d_3(\lambda)y_2 + \beta_{11}(\lambda)y_1^2 + \beta_{12}y_1y_2 + \beta_{22}y_2^2 + O(\|y\|), \end{cases} \tag{3.6}$$

where

$$q_0\prime(\lambda) = p_0a_2 + b_2q_0,\ c_3 = b_2c_2 - a_2d_2,\ d_3 = a_2 + d_2,$$
$$\alpha_{11} = \frac{p'_{22}a_2^2}{b_2} - \frac{p'_{12}a_2}{b_2} + p'_{11},\ \alpha_{12} = -\frac{2p'_{22}a_2}{b_2^2} + \frac{p'_{12}}{b_2},\ \alpha_{22} = \frac{p'_{22}}{b_2^2},$$
$$\beta_{11} = b_2q'_{11} + a_2(p'_{11} - q'_{12}) - \frac{a_2^2(p'_{12} - q'_{22})}{b_2} + \frac{p'_{22}a_2^3}{b_2^2},$$
$$\beta_{12} = -(2\frac{p'_{22}a_2^2}{b_2^2} - \frac{a_2(p'_{12} - q'_{22})}{b_2} - q'_{12}),\ \beta_{22} = \frac{p'_{22}a_2}{b_2^2} + \frac{q'_{22}}{b_2}.$$

The functions $q_0\prime(\lambda)$, α_{kl}, β_{kl}, are smooth functions of λ. We have $q\prime_0(\lambda^*) = c_3(\lambda^*) = d_3(\lambda^*) = 0$. Now, we consider the following transformation

$$z_1 = y_1,\ z_2 = p_0(\lambda) + y_2 + \alpha_{11}(\lambda)y_1^2 + \alpha_{12}(\lambda)y_1y_2 + \alpha_{22}y_2^2 + O(\|y\|).$$

This transformation brings (3.6) into the following

$$\begin{cases} \dfrac{dz_1}{dt} = z_2, \\ \dfrac{dz_2}{dt} = g_{00}(\lambda) + g_{10}(\lambda)z_1 + g_{01}(\lambda)z_2 + g_{20}(\lambda)z_1^2 + g_{11}(\lambda)z_1z_2 + g_{02}(\lambda)z_2^2 + O(\|z\|^3), \end{cases} \tag{3.7}$$

where $g_{00}(0) = 0$, $g_{10}(0) = 0$, $g_{01}(0) = 0$, and $z = (z_1, z_2)$. Furthermore, we also have

$$\begin{aligned}
g_{00}(\lambda) &= q_0\prime(\lambda) - p_0(\lambda)d_3(\lambda) + ...,\\
g_{10}(\lambda) &= c_3(\lambda) + \alpha_{12}(\lambda)q_0\prime(\lambda) - \beta_{12}(\lambda)p_0(\lambda) + ...,\\
g_{01}(\lambda) &= d_3(\lambda) + 2\alpha_{22}(\lambda)q_0\prime(\lambda) - \alpha_{12}(\lambda)p_0(\lambda) - 2\beta_{22}(\lambda)p_0(\lambda),\\
g_{20}(\lambda) &= \beta_{11}(\lambda) - \alpha_{11}(\lambda)d_3(\lambda) + c_3(\lambda)\alpha_{12}(\lambda) + ...,\\
g_{02}(\lambda) &= \alpha_{12}(\lambda) + \beta_{22}(\lambda) - \alpha_{22}(\lambda)d_2(\lambda) + ...,\\
g_{11}(\lambda) &= \beta_{12}(\lambda) + 2\alpha_{11}(\lambda) + 2\alpha_{22}(\lambda)c_3(\lambda) - \alpha_{12}(\lambda)d_3(\lambda) + ...,
\end{aligned}$$

correspondingly,

$$\begin{aligned}
&g_{00}(\lambda^*) = 0, \quad g_{10}(\lambda^*) = 0, \quad g_{01}(\lambda^*) = 0, \quad g_{20}(\lambda^*) = \beta_{11}(\lambda^*),\\
&g_{02}(\lambda^*) = \alpha_{11}(\lambda^*) + \beta_{22}, \quad g_{11}(\lambda^*) = \beta_{12}(\lambda^*) + 2\alpha_{11}(\lambda^*).
\end{aligned}$$

Again, we can write (3.7) as of the following form:

$$\begin{cases}
\dfrac{dz_1}{dt} = z_2,\\
\dfrac{dz_2}{dt} = (g_{00}(\lambda) + g_{10}(\lambda)z_1 + g_{20}(\lambda)z_1^2 + O(\|z\|^3))\\
+(g_{01}(\lambda)z_2 + (g_{11}(\lambda)z_1 + O(\|z\|^2))z_2 + (g_{02}(\lambda) + O(\|z\|))z_2^2\\
= \mu(z_1, \lambda) + \nu(z_1, \lambda)z_2 + \Phi(z, \lambda)z_2^2,
\end{cases} \tag{3.8}$$

where μ, ν, Φ are smooth functions and satisfy the following

$$\begin{aligned}
&\mu(0, \lambda^*) = g_{00}(\lambda^*) = 0, \quad \frac{\partial\mu}{\partial z_1}|_{(0,\lambda^*)} = g_{10}(\lambda^*) = 0,\\
&\frac{\partial^2\mu}{\partial z_1^2}|_{(0,\lambda^*)} = g_{20}(\lambda^*) = \beta_{11}(\lambda^*) = \rho_1 \neq 0\\
&\nu(0, \lambda^*) = g_{01}(\lambda^*) = 0, \quad \frac{\partial\nu}{\partial z_1}|_{(0,\lambda^*)} = g_{11}(\lambda^*) = \beta_{11}(\lambda^*) + 2\alpha_{11}(\lambda^*) = \rho_2 \neq 0.
\end{aligned}$$

Since $\mu(0, \lambda^*) = 0$, $\frac{\partial\nu}{\partial z_1}|_{(0,\lambda^*)} = \rho_2 \neq 0$ (due to the nondegeneracy assumption), it follows from the Implicit function theorem that there exists a C^∞ function z_1 defined in a small neighbourhood of $\lambda = \lambda^*$ such that $\phi(\lambda^*) = 0$, $\nu(\phi, \lambda) = 0$ for any $\lambda \in N(\lambda^*)$. We now use a parameter-dependent shift of co-ordinates in the z_1-direction to annihilate the z_2 term on the RHS of the second equation of (3.8)

$$z_1 = u_1 + \phi(\lambda), \; z_2 = u_2.$$

The above transformation brings the system (3.8) to the following system

$$\begin{cases} \dfrac{du_1}{dt} = u_2, \\ \dfrac{du_2}{dt} = (h_{00}(\lambda) + h_{10}(\lambda)u_1 + h_{20}(\lambda)u_1^2 + O(\|u_1\|^3)) \\ +(h_{01}(\lambda)u_2 + (h_{11}(\lambda)u_1 + O(\|u\|^2))u_2 + (h_{02}(\lambda) + O(\|u\|))u_2^2 \\ = \overline{\mu}(u_1, \lambda) + \overline{\nu}(u_1, \lambda)u_2 + \overline{\Phi}(u, \lambda)u_2^2, \end{cases} \tag{3.9}$$

where $u = (u_1, u_2)$,

$$h_{00} = g_{00} + g_{10}\phi + ..., \; h_{10} = g_{10} + 2g_{20}\phi + ...,$$
$$h_{20} = g_{20} + ..., \; h_{01} = g_{01} + g_{11}\phi + ...,$$
$$h_{11} = g_{11} + ..., \; h_{02} = g_{02} + ...$$

The coefficient of u_2 term on the RHS of the second equations of (3.9) is given by

$$h_{01} = \overline{\nu}(0, \lambda) = g_{01} + g_{11}\phi + O(\|\phi\|^2) = [d_2 + 2\alpha_{22}q_0\prime - \alpha_{12}p_0 - 2\beta_{22}p_0 + ...]$$
$$+[\beta_{12} + 2\alpha_{11} + \alpha_{22}c_2 - \alpha_{12}d_2 + ...]\phi.$$

Thus we have the following

$$h_{01}(0, \lambda) = g_{01}(\lambda^*) = 0, \; \frac{\partial h_{01}}{\partial \phi}|_{(0,\lambda^*)} = \beta_{12}(\lambda^*) + 2\alpha_{11}(\lambda^*) = \rho_2 \neq 0.$$

Let for $\lambda \in N(\lambda^*)$, $\phi(\lambda) \in M$. Then in the region$M$, $\phi(\lambda)$ can be approximated by

$$\phi(\lambda) \approx -\frac{g_{01}(\lambda)}{\rho_2}.$$

Thus, (3.9) reduces to the following

$$\begin{cases} \dfrac{du_1}{dt} = u_2, \\ \dfrac{du_2}{dt} = h_{00}(\lambda) + h_{10}(\lambda)u_1 + h_{20}(\lambda)u_1^2 + h_{11}(\lambda)u_1u_2 + h_{02}(\lambda)u_2^2 + O(\|u\|)^3. \end{cases} \tag{3.10}$$

We now introduce a new time scale, defined by $dt = (1 + \psi u_1)d\tau$, where $\psi = \psi(\lambda)$ is a smooth function to be defined later. With this transformation, (3.10) reduces to

$$\begin{cases} \dfrac{du_1}{d\tau} = u_2(1 + \psi u_1), \\ \dfrac{du_2}{d\tau} = h_{00} + (h_{10} + h_{00}\psi)u_1 + (h_{20} + h_{10}\psi)u_1^2 + h_{11}u_1u_2 + h_{02}u_2^2 + O(\|u\|)^3, \end{cases} \tag{3.11}$$

assume

$$\nu_1 = u_1, \; \nu_2 = u_2(1 + \psi u_1).$$

then we obtain,

$$\begin{cases} \dfrac{d\nu_1}{d\tau} = \nu_2, \\ \dfrac{d\nu_2}{d\tau} = l_{00}(\lambda) + l_{10}(\lambda)\nu_1 + l_{20}(\lambda)\nu_1^2 + l_{11}(\lambda)\nu_1\nu_2 + l_{02}(\lambda)\nu_2^2 + O(\|\nu\|)^3, \end{cases} \tag{3.12}$$

where

$$\begin{aligned} l_{00}(\lambda) &= h_{00},\ l_{10}(\lambda) = h_{10} + 2h_{00}\psi(\lambda), \\ l_{20} &= h_{20} + 2h_{00}\psi(\lambda) + h_{00}(\lambda)\psi(\lambda)^2, \\ l_{11}(\lambda) &= h_{11}(\lambda),\ l_{02}(\lambda) = h_{02} + \psi(\lambda). \end{aligned}$$

Now, we take $\psi(\lambda) = -h_{02}(\lambda)$ in order to get rid of ν_2^2-term. We then have

$$\begin{cases} \dfrac{d\nu_1}{d\tau} = \nu_2, \\ \dfrac{d\nu_2}{d\tau} = \beta_1(\lambda) + \beta_2(\lambda)\nu_1 + \eta(\lambda)\nu_1^2 + \zeta(\lambda)\nu_1\nu_2 + O(\|\nu\|)^3, \end{cases} \tag{3.13}$$

where $v = (v_1, v_2)$,

$$\begin{aligned} \beta_1(\lambda) = h_{00}(\lambda),\ \beta_2(\lambda) &= h_{10}(\lambda) - 2h_{00}(\lambda)h_{02}(\lambda), \\ \eta(\lambda) = h_{20}(\lambda) - 2h_{10}(\lambda)h_{02}(\lambda) &+ h_{02}^2(\lambda)h_{00}(\lambda) \neq 0, \\ \zeta(\lambda) &= h_{11}(\lambda) \neq 0. \end{aligned}$$

We now introduce a new time scale given by

$$t = |\frac{\eta(\lambda)}{\zeta(\lambda)}|\tau.$$

With the new stable variables $\xi_1 = \frac{\eta(\lambda)}{\zeta^2(\lambda)}\nu_1$ and $\xi_2 = \frac{\eta^2(\lambda)}{\zeta^3(\lambda)}\nu_2$ such that $s = sign\frac{\eta(\lambda)}{\zeta(\lambda)} = sign\frac{\eta(\lambda^*)}{\zeta(\lambda^*)} = \frac{\rho_2}{g_{20}(\lambda^*)} = \pm 1$. This yields (3.13) into the form

$$\begin{cases} \dfrac{d\xi_1}{d\tau} = \xi_2, \\ \dfrac{d\xi_2}{d\tau} = \mu_1 + \mu_2\xi_1 + \xi_1^2 + s\xi_1\xi_2 + O(\|\xi\|)^3, \end{cases} \tag{3.14}$$

where

$$\mu_1(\lambda) = \frac{\eta(\lambda)}{\zeta^2(\lambda)}\beta_1(\lambda),\ \mu_2(\lambda) = \frac{\eta(\lambda)}{\zeta^2(\lambda)}\beta_2(\lambda).$$

The system (3.14) is locally topologically equivalent near the origin for small $\|\mu\|$ to the system

$$\begin{cases} \dfrac{d\xi_1}{d\tau} = \xi_2, \\ \dfrac{d\xi_2}{d\tau} = \mu_1 + \mu_2\xi_1 + \xi_1^2 + s\xi_1\xi_2, \end{cases} \tag{3.15}$$

where $s = \pm 1$. We have obtained the generic normal from of the Bogdanov-Takens bifurcation for the system(3.15)

$$rank(\frac{\partial(\mu_1, \mu_2)}{\partial \lambda})_{\lambda=\lambda^*} = 2,$$

$$J = \begin{vmatrix} \frac{\partial \mu_1}{\partial \lambda_2} & \frac{\partial \mu_1}{\partial \lambda_1} \\ \frac{\partial \mu_2}{\partial \lambda_2} & \frac{\partial \mu_2}{\partial \lambda_1} \end{vmatrix} \neq 0.$$

□

3.2. **Hopf bifurcation.** We study the existence of a Hopf bifurcation and stability in a small neighborhood of E_* when parameters q, β vary. In this section, we consider system (1.1). It is easy to see that the determinant of $J|_{E_*}$ is positive if

$$\frac{1}{1 - \frac{d}{k\beta K}} > -\frac{q}{k\beta K[1 + \frac{wr}{\beta}(1 - \frac{d}{k\beta K})]^2}.$$

Theorem 3.3. *The system* (1.1) *undergoes Hopf bifurcation with respect to the parameter* q *around the equilibrium point* (x_*, y_*)*, if* $q_* = \frac{rd}{k\beta K}[1 + \frac{wr}{\beta}(1 - \frac{d}{k\beta K})]^2$.

Proof. The characteristic equation is given by

$$\lambda^2 - \Theta\lambda + \Delta = 0,$$

then the solutions of the characteristic equation give

$$\lambda_{1,2} = \frac{1}{2}tr(J|_{E_*}) \pm \sqrt{(tr(J|_{E_*}))^2 - 4det(J|_{E_*})}.$$

We know that, if $\Theta = 0$, then both the eigenvalues will be purely imaginary provided $\Delta > 0$. Therefore, from the implicit function theorem a Hopf bifurcation occurs where a periodic orbit is created as the stability of the equilibrium point (x_*, y_*) changes. Using the above two conditions it is found that the critical value of the Hopf bifurcation parameter is $q_* = \frac{rd}{k\beta K}[1 + \frac{wr}{\beta}(1 - \frac{d}{k\beta K})]^2$.
It is clear that the given conditions:
(a) $\Theta = 0$.
(b) $\Delta > 0$.
(c) $\frac{d\Theta}{dq}|_{q=q_*} \neq 0$. Guarantee the existence of Hopf bifurcation of the system (1.1) around (x_*, y_*).
Obviously,

$$\frac{d\Theta}{dq}|_{q=q_*} = \frac{1}{[1 + \frac{wr}{\beta}(1 - \frac{d}{k\beta K})]^2} \neq 0.$$

□

Theorem 3.4. *Define* $\beta_* = \frac{d-r}{kx_* - y_*} + \frac{2rx}{K(kx-y)} + \frac{q}{(1+wy)^2(kx-y)}$ *then a supercritical Hopf bifurcation occurs around the positive equilibrium* E_*.

Proof. The proof is similar to that of the Theorem (3.3) and

$$\frac{d\Theta}{d\beta}|_{\beta=\beta_*} = kx - y \neq 0.$$

□

4. Numerical simulation

In this section we present computer simulation of some solutions of the system (1.1). Beside verification of our analytical findings, these numerical solutions are very important from practical point of view. The phase portraits were calculated with ode45 of MATLAB. This is done by calculating the solutions forward and backward in time for intial values located on a equally spaced grid in the first quadrant. We use the following symbols in the bifurcation diagrams of this paper: LP:limit point, H: Hopf bifurcations of an equilibrium point, Lpc: for the tangent bifurcations of limit cycles, BT: for the Bogdanov-Takens, GH: for the Generalized Hopf, CP:cusp point.

4.1. **Continuation Curve of Equilibrium Point (one-parameter bifurcation diagram).** Analytical studies can never be completed without numerical verification of the derived results. The main aim of this section is to study the pattern of bifurcation that takes place as we vary the parameters β, q. This is actually done by studying the change in the eigenvalue of the Jacobian matrix and also following the continuation algorithm. To start with, we consider a set of fixed point initial solution, $(x, y) = (10, 2)$, corresponding to a parameter set of values, $r = 0.8, k = 0.61, K = 15, \beta = 0.3, d = 0.5, q = 1.1, w = 1$. The characteristics of Hopf point, the limit cycle and the general bifurcation may be explored. To compute curve of equilibrium from the equilibrium point we take parameter $q = 1.1$ as the free parameter with fixed $r = 0.8, k = 0.61, K = 15, \beta = 0.3, d = 0.5, w = 1$. It is evident that the system has two Hopf point in a neighborhood of E_1, E_*, as predicted by the theory, with purely imaginary eigenvalues $\pm i0.115727$, $\pm i0.645976$ and fold point in a small neighborhood of E_1. For this Hopf point the first Lyapunov coefficient is in Table 1 indicating two supercritical Hopf bifurcation. It being negative implies that a stable limit cycle bifurcates from the equilibrium when this loses stability. From Figs. (4.2), (4.2), (6), (7), (8),(4.2), (4.2) it is evident that the system has a Hopf point at:

label = H , x = (5.019706 1.774275 1.161328)
First Lyapunov coefficient = -1.808411e-003
label = H , x = (11.522757 0.618177 2.603103)
First Lyapunov coefficient = -1.558876e-001 .
label = LP, x = (11.678620 0.590468 2.603894)
a=-2.768846e+000
label = BP, x = (15.000000 -0.000000 2.245000)

To compute curve of equilibrium from the equilibrium point we take $\beta = 0.3$ as the free parameter with fixed $r = 0.8, k = 0.61, K = 15, \beta = 0.3, d = 0.5, w = 1$. It is evident that the system has a Hopf point in a small neighborhood of E_* with purely imaginary eigenvalues $\pm i0.661621$ and fold point in a small neighborhood of E_1. For this Hopf point the first Lyapunov coefficient ℓ_1 is in Table 1 indicating a subcritical Hopf bifurcation. From Figs. (4.2), (4.2), (4.2), (4.2), it is evident that the system has a Hopf point

label = H , x = (4.344504 1.741618 0.326302)
First Lyapunov coefficient = -1.725231e-003.
label = BP, x = (15.000000 -0.000000 2.245000)
label = LP, x = (11.881131 1.285986 0.129348)
a=-6.708999e+002
label = BP, x = (8.196721 0.000000 2.245000)

By selecting Hopf point in the one-parameter bifurcation diagram of the equilibrium as initial point, we can plot the Limit cycles and bifurcations of limit cycles starting from the Hopf point . Figs. (5), (6), (7).

When we take $\beta = 0.3$ as the free parameter :
Limit point cycle (period = 9.496654e+000, parameter = 3.263018e-001)
Normal form coefficient = -1.273765e-001
Limit point cycle (period = 9.496654e+000, parameter = 3.263018e-001)
Normal form coefficient = -1.262238e-001
Limit point cycle (period = 9.496654e+000, parameter = 3.263018e-001)
Normal form coefficient = -1.263172e-001

When we take $q = 1.1$ as the free parameter :
Limit point cycle (period = 9.726653e+000, parameter = 1.161328e-000)
Normal form coefficient = -1.546788e-001
Limit point cycle (period = 9.726653e+000, parameter = 1.161328e-000)
Normal form coefficient = -1.576773e-001

4.2. **Two-parameter bifurcation diagram.** By selecting Hopf point in the one-parameter bifurcation diagram of the equilibrium as initial point, and taking K, q as the free parameter. Fig. (4.2), (4.2).
label = BT, x = (11.845560 0.594337 15.242823 2.658935 0.000000)
(a,b)=(4.224022e-002, 1.019134e-001)
label = GH , x = (3.319008 2.413592 34.972675 0.366546 0.434025)
12=-2.618638e-005.

By selecting fold point in the one-parameter bifurcation diagram of the equilibrium as initial point, and taking β, q as the free parameter. Fig. (10).
label = CP , x = (14.997674 0.001908 0.059019 0.040068)
c=-1.787529e+002
label = BT, x = (11.779936 0.536928 0.319851 2.763966)
(a,b)=(4.224022e-002, -1.105923e-001)

label = BT, x = (12.421062 0.326629 0.421100 3.5694391)
(a,b)=(-4.156444e-002, 1.822598e-001).

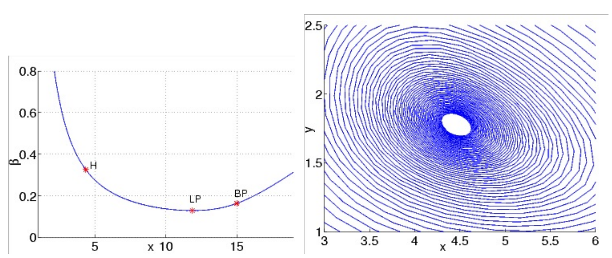
We notice that the numerical bifurcat.

lable	eigenvalues	ℓ_1, a	free parameter
H	$\lambda_{1,2} = 2.00916e - 008 \pm i0.661621$	$\ell_1 = -1.725231e + 003$	β
H	$\lambda_{1,2} = -2.12598e - 006 \pm i0.115727$	$\ell_1 = -1.558876e + 001$	q
H	$\lambda_{1,2} = -4.63381e - 007 \pm i0.645976$	$\ell_1 = -1.808367e - 003$	q
LP	$\lambda_{1,2} = -0.387576, 2.45078e - 006$	$a = -6.708999e - 002$	β
LP	$\lambda_{1,2} = -0.0150474, 1.79065e - 007$	$a = 2.768846e - 000$	q

Table 1. One-parameter bifurcation points and eigenvalues.

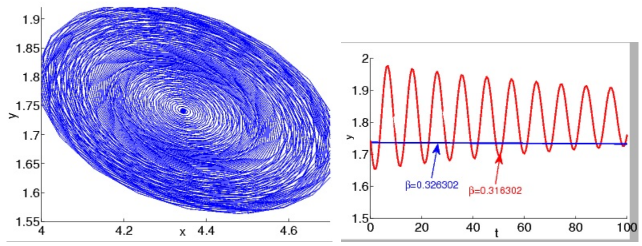
lable	eigenvalues	normal form coefficient	free parameter
BT	$\lambda_{1,2} = -1.017e - 015 \pm i0.00031$	$4.1777e - 002, -9.75644e - 002$	q, K
BT	$\lambda_{1,2} = -2.594e - 006, -1.545e - 010$	$-4.1564e - 0022, 1.8225e - 001$	q, β
BT	$\lambda_{1,2} = 2.749e - 008 \pm i4.668e - 008$	$4.2240e - 002, -1.1059e - 001$	q, β
CP	$\lambda_{1,2} = -0.7997, -5.207e - 005$	$-1.7875e - 002$	q, β
GH	$\lambda_{1,2} = -1.734e - 017 \pm i0.6588$	$-2.6186e - 005$	q, K
LPC	$\mu_{1,2} = 1, 1$	$-1.2631e - 001$	β
LPC	$\mu_{1,2} = 1, 1$	$-1.5467e - 001$	q
LPC	$\mu_{1,2} = 5.139e - 008, 0.9991$	$5.2533e + 000$	q

Table 2. Two-parameter bifurcation points, limit cycles and eigenvalues.



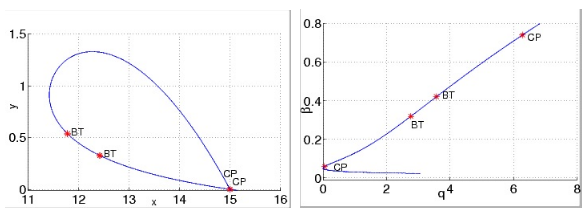
(a) (b)

Figure 1. (a)Bifurcation diagram of the equilibrium in a small neighborhood of E_1, E_* with the parameter β undergoing a supercritical Hopf bifurcation. (b)Trajectories of system (1.1), when $\beta = 0.316302$. E_* is locally asymptotically unstable.



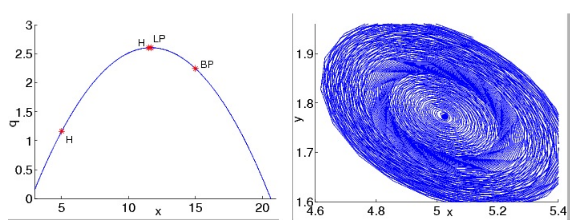
(a) (b)

FIGURE 2. (a)Trajectories of system (1.1), when $\beta = 0.326302$. E_* is locally asymptotically stable. (b) Hopf bifurcation occurs at E_* and bifurcating periodic solution for system (1.1) with $\beta = 0.316302$, $\beta = 0.326302$



(a) (b)

FIGURE 3. (a) Fold curve in model (1.1), two-parameter bifurcation diagram, when we take β, q as the free parameter with fixed d, r, K, w, k in axis x, y. (b) Fold curve in model (1.1), two-parameter bifurcation diagram, when we take β, q as the free parameter with fixed d, r, K, w, k.



(a) (b)

FIGURE 4. (a) Continuation curves of equilibrium with the variation of the parameter q in a small neighborhood of E_1, E_*. (b) Trajectories of system (1.1), when $q = 1.161321$.

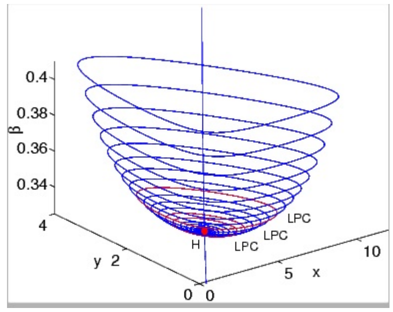


FIGURE 5. The family of limit cycles and bifurcations of limit cycles starting from the Hopf point with the variation of the parameter $\beta = 0.326302$.

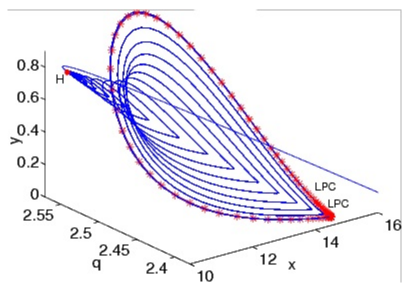


FIGURE 6. The family of limit cycles starting from the Hopf point with the variation of the parameter $q = 2.603103$.

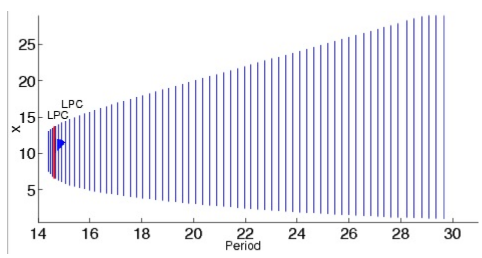


FIGURE 7. Limit cycle bifurcation from supercritical Hopf point $HP = (11.522757, 0.618177)$.

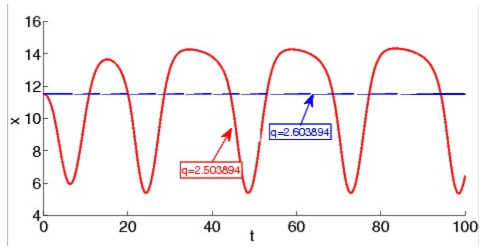


FIGURE 8. periodic solution for system (1.1) with $q = 2.502894$

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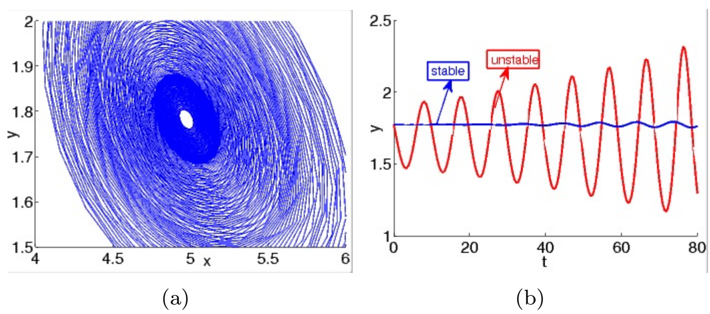

FIGURE 9. (a) Trajectories of system (1.1), when $q = 1.151321$. (b)Periodic solution for system (1.1) with $q = 1.151321$, $q = 1.161321$.

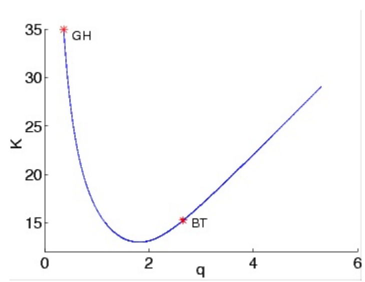

FIGURE 10. Hopf curve in model (1.1), two-parameter bifurcation diagram, when we take K, q as the free parameter with fixed d, r, β, w, k.

Conclusion

In this paper, we have studied the dynamics of a predator-prey system with harvesting of the predator guided by its population. We obtain conditions that aect the persistence of the system. Local asymptotic stability of various equilibrium solutions is explored to understand the dynamics of the model system. Based on the normal form theory, we find that it is a cusp point of co-dimension 2 so that BogdanovTakens bifurcation may occur. When the sum of the death rate and harvesting rate of the predator is above a critical value the prey only equilibrium is stable, otherwise, it is unstable and the positive equilibrium exists. It has at most one interior equilibrium which is a stable node or focus whenever it exists. Therefore, increasing the linear harvesting rate can lead to the predators extinction after coexisting with the prey as an equilibrium. In contrast, by the introduction of the nonlinear harvesting term, our proposed model (1.1), can exhibit much richer behaviors, i.e., numerous bifurcations may occur including the, Hopf,and Bogdanov-Takens bifurcations. In summary, our analysis of the dynamics of the nonsmooth dynamics system can assist decision-making in a variety of field but especially in fishery management to improve implementation of harvesting strategies. For a specific problem, our finding suggests that we can choose a reasonable economic harvest threshold and harvesting rate so that the predator and prey populations can coexist and stabilize at ideal values. This harvest regime can also be extended to other discontinuous harvesting policies and the switching harvest policy can also be used to tackle other problems in ecology which we leave for future work.

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