Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 9, No. 2, 2021, pp. 446-466
DOI:10.22034/cmde.2020.34992.1598

A new numerical Bernoulli polynomial method for solving fractional optimal control problems with vector components

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Abstract In this paper, a numerical method is developed and analyzed for solving a class of fractional optimal control problems (FOCPs) with vector state and control functions using polynomial approximation. The fractional derivative is considered in the Caputo sense. To implement the proposed numerical procedure, the Ritz spectral method with Bernoulli polynomials basis is applied. By applying the Bernoulli polynomials and using the numerical estimation of the unknown functions, the FOCP is reduced to solve a system of algebraic equations. By rigorous proofs, the convergence of the numerical method is derived for the given FOCP. Moreover, a new fractional operational matrix compatible with the proposed spectral method is formed to ease the complexity in the numerical computations. At last, several test problems are provided to show the applicability and effectiveness of the proposed scheme numerically.

Keywords. Fractional derivative, Optimal control problem, Bernoulli operational matrix, Spectral Ritz method, Convergence.

2010 Mathematics Subject Classification. 49J15, 65N35,26A33,11B68.

1. Introduction

Nowadays, the role played by fractional order calculus in modeling signal processing, optimal control, optimization problems and other fields of science is very effective. Because of long-range interaction of fractional and integer calculus, it will bring about more accurate and precise behavioral description of many processes and practical plants in many contexts such as biophysics and biochemistry [11, 12, 15, 25], disease control [18, 44], optimal control problems [1, 10, 13, 30, 31] and so on. In fact, to include memory effects, i.e. the influence of the behavior of the system in the

Received: 6 August 2019 ; Accepted: 5 February 2020.
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past, the fractional derivative is more suitable tool to be considered for system definition [16]. Besides, a vivid application of fractional calculus is in the optimal control problems [4,34,41,43,50]. A fractional optimal control problem consists of finding an optimal control function along with optimal state function minimizing the given performance index as well as satisfying the dynamical system constraint. Obviously, most of the fractional optimal control problems do not have analytical solution. Alternatively, many researchers have focused on numerical solution of the fractional optimal control problems, instead [5,20–22,38,53]. In the literature, several papers address the numerical solution of FOCPs using polynomials and orthogonal functions [23,32,38].

The next issue in FOCPs is to find an optimal solution. In addition, determining an optimal controller in real systems like cryptography and secure data transmission instances is vital because of providing high speed technology [2]. Owing to their high order of accuracy, spectral and pseudospectral methods have been used for solving a wide variety of fractional optimal control problems as well as variational problems [14, 26, 35, 51]. The key point in these methods is to consider the interpolating polynomial basis on the overall interval rather than just on the equally spaced points on subintervals [8, 9].

Since a system description using differential equation having derivative of order greater than one can be stated with a system of differential equations, many researchers have considered the vector functions in their work instead of scalar state functions. Sayevand and Rostami studied the fractional optimal control problems (FOCPs) and obtain the necessary and sufficient optimality conditions for vector function [48]. Pommaret and Quadrat studied controllable linear multidimensional systems by transforming linear quadratic optimal problem into that of a variational problem without constraints [37].

In this work, by considering the spectral Ritz method, an estimated state vector function is constructed. An interesting active research field of fractional calculus is the FOCPs in which the dynamical system as well as cost function involve not only integer order derivatives but also fractional order derivatives or integrals [6, 27]. On the other hand, some dynamical models such as light amplification in Erbium-doped fiber amplifier which is one of the most commonly used type of fiber amplifiers in metro optical networks may have both of the fractional and integer order derivatives [36].

By using the Ritz method, all the given initial-boundary conditions can be easily met. Therefore, using the proposed auxiliary functions, the given initial and boundary conditions are also assumed in the approximate function. By considering the dynamical system constraint, the approximate control function is also identified. In the next part of the paper, to reduce the computational complexity on fractional derivative of the Bernoulli basis, a new fractional operational derivative matrix of Caputo type is also constructed. It is known that the Bernoulli polynomials are complete [29].

There are special attentions for fractional operational matrix of Bernstein polynomials [33, 47, 52], Legendre polynomials [32, 39, 46], Laguerre polynomials [28], Bernoulli polynomials and Bernoulli wavelets [17, 40, 42, 49]. By applying the new formed operational derivative matrix, the arisen computational complexity is simplified. Substituting the approximate state function and control input into the cost function and estimating the integral yields an optimization problem without any constraint. By utilizing the given tools and applying the necessary conditions of optimality, the problem is converted into a system of algebraic equations. To find the optimal solution, the numerical methods like Newton's iterative method can be applied.

The outline of this paper is as follows: In section 2, some important and basic definitions are described. In section 3, the Bernoulli fractional operational matrix of Caputo derivative is constructed. In section 4, we define the main problem of fractional optimal control. Section 5 is devoted to the convergence analysis of the proposed scheme. The numerical findings and illustrative test problems are provided in section 6 to show the accuracy of the numerical method. In section 7, the main results are summarized.

2. Preliminaries

2.1. **Basic definitions.** In this part, we recall some basic definitions needed in sequel. Let $x : [a, b] \to \mathbb{R}$ and $\alpha > 0$ be the order of fractional derivative or integral. First the Riemann-Liouville fractional integral is defined. Next, the fractional derivatives are also defined. For $t \in [a, b]$ we have the following definitions:

Definition 2.1. The left and right Riemann-Liouville fractional integral of order $\alpha > 0$ of the function $x(t)$ on $t \in [a, b]$ are defined as follows, respectively

$$
\begin{aligned}
{}_aI_t^\alpha x(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} x(\tau)\mathrm{d}\tau,\\
{}_tI_b^\alpha x(t) &= \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} x(\tau)\mathrm{d}\tau.
\end{aligned}
$$

Definition 2.2. The left and right Riemann-Liouville fractional derivatives of order $\alpha > 0$ of the function x are defined as, respectively

$$
\begin{aligned}
{}_aD_t^\alpha x(t) &= \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left(\int_a^t (t-\tau)^{n-\alpha-1} x(\tau)\mathrm{d}\tau \right),\\
{}_tD_b^\alpha x(t) &= \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left(\int_t^b (\tau-t)^{n-\alpha-1} x(\tau)\mathrm{d}\tau \right),
\end{aligned}
$$

where $n-1 < \alpha \leq n$, and $n \in \mathbb{N}$.

Definition 2.3. The left and right Caputo fractional derivatives of order $\alpha > 0$ of the function x are defined as follows, respectively

$$
\begin{aligned}
{}_a^CD_t^\alpha x(t) &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} \frac{\mathrm{d}^n}{\mathrm{d}\tau^n} x(\tau)\mathrm{d}\tau,\\
{}_t^CD_b^\alpha x(t) &= \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (\tau-t)^{n-\alpha-1} \frac{\mathrm{d}^n}{\mathrm{d}\tau^n} x(\tau)\mathrm{d}\tau,
\end{aligned}
$$

where $n-1 < \alpha \leq n$, and $n \in \mathbb{N}$.

2.2. **The Bernoulli Basis.** The sequence of Bernoulli polynomials $\{B_m\}, m \in \mathbb{N} \cup \{0\}$ is defined on the interval $[0,1]$ by the following formula

$$B_m(t) = \sum_{k=0}^{m} \binom{m}{k} \beta_{m-k} t^k, \tag{2.1}$$

where $\beta_k = B_k(0) = (-1)^k B_k(1)$ are the known Bernoulli numbers and they can be obtained from the generating function as

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \beta_n \frac{z^n}{n!}. \tag{2.2}$$

The Bernoulli polynomials can also be extended to be defined on the interval $[a,b]$. Therefore they are defined as [24]

$$B_m^{[a,b]}(t) = (b-a)^m B_m\left(\frac{t-a}{b-a}\right). \tag{2.3}$$

The first few number of these polynomials on the interval $[0,1]$ are

$$B_0(t) = 1,\ B_1(t) = t - \frac{1}{2},\ B_2(t) = t^2 - t + \frac{1}{6}.$$

Its worth to note that the Bernoulli polynomials are dense in the complete space $L^2[0,1]$ [17, 19].

3. Bernoulli Fractional Operational Matrix of Caputo Derivative

In this section, without loss of generality, we assume that the interval be $[0,1]$. In order to compute the fractional and integer order derivatives of the Bernoulli polynomials and to simplify computational complexity of the equations, a new modified Bernoulli operational matrix is introduced as

$${}_0^C D_t^{\alpha}\left(t^p \mathfrak{B}_m(t)\right) \simeq \mathbf{A}_{m,\tilde{m}}^{\alpha,p} \mathfrak{B}_{\tilde{m}}(t), \tag{3.1}$$

where $p \geq 0$ is a real number and $\mathfrak{B}_m(t)$ is a vector of Bernoulli polynomials as

$$\mathfrak{B}_m(t) = \begin{bmatrix} B_0(t) \\ B_1(t) \\ \vdots \\ B_m(t) \end{bmatrix}. \tag{3.2}$$

Obviously, referring to the fractional computations, one may compute ${}_0^C D_t^{\alpha} t^p = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} t^{p-\alpha}$. Hence, using the linearity of fractional Caputo derivative and its definition we get

$$\begin{aligned} {}_0^C D_t^{\alpha}\left(t^p B_i(t)\right) &= \sum_{k=0}^{i} \binom{i}{k} \beta_{i-k}\, {}_0^C D_t^{\alpha} t^{k+p} \\ &= \sum_{k=0}^{i} \binom{i}{k} \beta_{i-k} \frac{\Gamma(k+p+1)}{\Gamma(k+p+1-\alpha)} t^{k+p-\alpha}. \end{aligned} \tag{3.3}$$

Expanding the expression $t^{k+p-\alpha}$ in terms of the Bernoulli polynomials implies

$$t^{k+p-\alpha} \simeq \sum_{j=0}^{\tilde{m}} b_j B_j(t), \tag{3.4}$$

where b_j is the Bernoulli coefficient and can be computed according to Lemma 3.1 as

$$b_j = \frac{1}{j!}\int_0^1 \frac{\mathrm{d}^j}{\mathrm{d}t^j}(t^{k+p-\alpha})\mathrm{d}t = \frac{\Gamma(k+p-\alpha+1)}{j!\Gamma(k+p-\alpha-j+2)}.$$

Hence the fractional operational matrix of Bernoulli basis can be determined as

$$\mathbf{A}^{\alpha,p}_{m,\tilde{m}} = \begin{bmatrix} \mathrm{a}^{\alpha,p}_{0,0} & \mathrm{a}^{\alpha,p}_{0,1} & \cdots & \mathrm{a}^{\alpha,p}_{0,\tilde{m}} \\ \mathrm{a}^{\alpha,p}_{1,0} & \mathrm{a}^{\alpha,p}_{1,1} & \cdots & \mathrm{a}^{\alpha,p}_{1,\tilde{m}} \\ \vdots & \vdots & & \vdots \\ \mathrm{a}^{\alpha,p}_{m,0} & \mathrm{a}^{\alpha,p}_{m,1} & \cdots & \mathrm{a}^{\alpha,p}_{m,\tilde{m}} \end{bmatrix}_{(m+1)\times(\tilde{m}+1)}, \tag{3.5}$$

where

$$\mathrm{a}^{\alpha,p}_{i,j} = \sum_{k=0}^{i} \binom{i}{k} \beta_{i-k} \frac{\Gamma(k+p+1)}{j!\Gamma(k+p-\alpha-j+2)}, \quad 0 \le i \le m, \ 0 \le j \le \tilde{m}. \tag{3.6}$$

3.1. **Function approximation.** Consider $H_m = \text{span} < B_0(t), B_1(t), ..., B_m(t) >$ be a finite-dimensional closed subspace of the Hilbert space $L^2[0,1]$ spanned by the Bernoulli polynomials. The subspace H_m is complete [19]. Therefore an arbitrary element $g \in L^2[0,1]$ can be uniquely approximated by the subspace H_m, that is the unique g_m exists as

$$\|g - g_m\| \le \|g - f\|, \quad \forall f \in H_m. \tag{3.7}$$

Now let the function g is expressed in terms of the subspace H_m as

$$g(t) \simeq \sum_{k=0}^{m} c_k B_k(t), \tag{3.8}$$

where c_k are the Bernoulli coefficients. The following lemma indicates the coefficients of an approximation by the Bernoulli polynomials.

Lemma 3.1. *Let $f \in L^2[0,1]$ be an arbitrary function and is estimated by the truncated Bernoulli series as $f(t) \simeq \sum_{k=0}^{m} c_k B_k(t)$, then the coefficients c_k for all $k = 0, 1, ..., m$ can be computed from the following relation*

$$c_k = \frac{1}{k!}\int_0^1 f^{(k)}(t)\mathrm{d}t.$$

Proof. See [7]. □

C M
D E

4. Problem statement and approximation method

The following nonlinear vector component fractional OCP is considered:

$$\min \quad J[\mathbf{x},\mathbf{u}] = \int_0^1 f\left(t, \mathbf{x}(t), {}_0^C D_t^{\alpha}\mathbf{x}(t), \mathbf{u}(t)\right) \mathrm{d}t, \tag{4.1}$$

subject to dynamical system constraints

$$\sum_{i=1}^{I} M_i \mathbf{x}^{(i)}(t) + \sum_{j=1}^{J} N_j \, {}_0^C D_t^{\alpha_j}\mathbf{x}(t) = F(t,\mathbf{x}(t)) + h(t)\mathbf{u}(t), \tag{4.2a}$$

$$\mathbf{x}^{(i)}(0) = \mathbf{x}_0^i, \quad \mathbf{x}^{(i)}(1) = \mathbf{x}_1^i, \quad 0 \le i \le n-1, \tag{4.2b}$$

where $n_j - 1 < \alpha_j \le n_j$ and $n_j \in \mathbb{Z}$. The parameters M_i and N_j are real matrices with appropriate dimensions which have columns same as the size of the state vector. It is also assumed that $n = \max_j\{I, n_j\}$, f is a continuous function and h is a nonzero continuous function. Note that, since the approach is based on the direct solution of FOCP rather than indirect one, the control input is assumed on the right hand side of dynamical system with a nonzero coefficient function.

To solve the problem numerically, the spectral Ritz method is applied. This scheme is to approximate the state and control functions in such a way that all the initial and boundary conditions of the problem are initially met. Accordingly, by considering the basis (3.2), the state function is approximated as

$$\mathbf{x}(t) \simeq \phi_1(t) Y^m \mathfrak{B}_m(t) + \phi_2(t), \tag{4.3}$$

where $\phi_2(t)$ is a function depending to initial and boundary conditions, namely, $\phi_2(t, \mathbf{x}_0^0, \mathbf{x}_0^1, ..., \mathbf{x}_0^{n-1}, \mathbf{x}_1^0, \mathbf{x}_1^1, ..., \mathbf{x}_1^{n-1})$ and $Y^m = [\mathbf{y}^0, \mathbf{y}^1, ..., \mathbf{y}^m]$ is the real unknown matrix to be determined with each $\mathbf{y}^i$ is a vector by the same size as $\mathbf{x}$. The function $\phi_1(t)$ shall be chosen such that the homogeneous initial-boundary conditions are imposed, namely $\mathbf{x}^{(i)}(0) = \mathbf{x}^{(i)}(1) = \mathbf{0}$ and $\phi_2(t)$ must be chosen such that the given initial-boundary conditions of (4.2b) are met. So the trial function $\phi_1(t)$ may be selected as follows

$$\phi_1(t) = \begin{cases} t^n & \text{if } \mathbf{x}(0) \text{ is known,} \\ (1-t)^n & \text{if } \mathbf{x}(1) \text{ is known,} \\ t^n(1-t)^n & \text{if both } \mathbf{x}(0), \mathbf{x}(1) \text{ are known.} \end{cases} \tag{4.4}$$

On the other hand, our suggestion for the second trial function $\phi_2(t)$ is the Hermite polynomial interpolation acting on vectorized conditions.

Remark 4.1. It is remarkable that if the initial and boundary conditions consist of just $\mathbf{x}(0)$ and $\mathbf{x}(1)$ without their known derivatives, the best choice for $\phi_2(t)$ is stated as: $\phi_2(t) = \mathbf{x}(0)$ if just $\mathbf{x}(0)$ is known, and the convex combination of the conditions as $\phi_2(t) = (1-t)\mathbf{x}(0) + t\mathbf{x}(1)$ if both of the $\mathbf{x}(0)$ and $\mathbf{x}(1)$ are known.

Therefore, the estimated state function is computed by

$$\begin{aligned}\mathbf{x}_m(t) &= t^n(t-1)^n Y^m \mathfrak{B}_m(t) + \phi_2(t)\\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k t^{n+k} Y^m \mathfrak{B}_m(t) + \phi_2(t).\end{aligned} \tag{4.5}$$

By considering the system dynamics (4.2), the control input can also be approximated by

$$\mathbf{u}_m(t) = \frac{1}{h(t)} \Big(\sum_{i=1}^{I} M_i \mathbf{x}_m^{(i)}(t) + \sum_{j=1}^{J} N_j \, {}_0^C D_t^{\alpha_j} \mathbf{x}_m(t) - F(t, \mathbf{x}_m(t)) \Big). \tag{4.6}$$

Now we can substitute the approximated state and control function into the performance index and then solve the resultant optimization problem. But since the fractional and integer order derivatives of the basis appear in the control function, it may cause some computational complexity. To overcome this issue, we apply a new constructed fractional operational matrix (3.5). Hence, utilizing the operational matrix on the control function (4.6) implies

$$\begin{aligned}\bar{\mathbf{u}}_m(t) =& \frac{1}{h(t)} \Big(\sum_{i=1}^{I} M_i (\sum_{k=0}^{n} \binom{n}{k} (-1)^k Y^m \mathbf{A}_{m,\tilde{m}}^{i,n+k} \mathfrak{B}_{\tilde{m}}(t) + \phi_2^{(i)}(t))\\ &+ \sum_{j=0}^{J} N_j (\sum_{k=0}^{n} \binom{n}{k} (-1)^k Y^m \mathbf{A}_{m,\tilde{m}}^{\alpha_j,n+k} \mathfrak{B}_{\tilde{m}}(t) + {}_0^C D_t^{\alpha_j} \phi_2(t))\\ &- F(t, \mathbf{x}_m(t)) \Big).\end{aligned} \tag{4.7}$$

By computing the fractional derivative of the basis in the approximate functions via operational matrix and substituting Eqs. (4.5) and (4.7) in the performance index (4.1), the following nonlinear optimization problem is achieved:

$$\begin{aligned}\min \bar{J}[Y^m] = \int_0^1 f\Big(&t, \sum_{k=0}^{n} \binom{n}{k} (-1)^k t^{n+k} Y^m \mathfrak{B}_m(t) + \phi_2(t), \sum_{k=0}^{n} \binom{n}{k}\\ &\times (-1)^k Y^m \mathbf{A}_{m,\tilde{m}}^{\alpha,n+k} \mathfrak{B}_{\tilde{m}}(t)) + {}_0^C D_t^{\alpha}(\phi_2(t)),\\ &\frac{1}{h(t)} \Big[\sum_{i=1}^{I} M_i (\sum_{k=0}^{n} \binom{n}{k} (-1)^k Y^m \mathbf{A}_{m,\tilde{m}}^{i,n+k} \mathfrak{B}_{\tilde{m}}(t) + \phi_2^{(i)}(t))\\ &+ \sum_{j=0}^{J} N_j (\sum_{k=0}^{n} \binom{n}{k} (-1)^k Y^m \mathbf{A}_{m,\tilde{m}}^{\alpha_j,n+k} \mathfrak{B}_{\tilde{m}}(t) + {}_0^C D_t^{\alpha_j} \phi_2(t))\\ &- F(t, \mathbf{x}_m(t)) \Big] \Big) \mathrm{d}t.\end{aligned} \tag{4.8}$$

The function on (4.8) is a nonlinear optimization problem of unconstrained one and the following necessary conditions of optimality should be held by optimizing the Y^m

as

$$\frac{\partial \bar{J}}{\partial y_j^i}[Y^m] = 0, \qquad 0 \leq i \leq m. \tag{4.9}$$

5. Theoretical analysis of the method

We define the error vector for Caputo fractional operational matrix of Bernoulli polynomials as follows

$$\begin{aligned}\mathbf{E}^{\alpha,p}_{m,\tilde{m}}(t) &= {}_0^C D_t^{\alpha}(t^p \mathfrak{B}_m(t)) - \mathbf{A}^{\alpha,p}_{m,\tilde{m}} \mathfrak{B}_{\tilde{m}}(t) \\ &= \begin{bmatrix} {}_0^C D_t^{\alpha}(t^p B_0(t)) - \sum_{j=0}^{\tilde{m}} \mathrm{a}^{\alpha,p}_{0,j} B_j(t) \\ {}_0^C D_t^{\alpha}(t^p B_1(t)) - \sum_{j=0}^{\tilde{m}} \mathrm{a}^{\alpha,p}_{1,j} B_j(t) \\ \vdots \\ {}_0^C D_t^{\alpha}(t^p B_m(t)) - \sum_{j=0}^{\tilde{m}} \mathrm{a}^{\alpha,p}_{m,j} B_j(t) \end{bmatrix} = \begin{bmatrix} e^{\alpha,p}_{0,\tilde{m}}(t) \\ e^{\alpha,p}_{1,\tilde{m}}(t) \\ \vdots \\ e^{\alpha,p}_{m,\tilde{m}}(t) \end{bmatrix}.\end{aligned} \tag{5.1}$$

On the other hand, by considering relation (3.4) we get

$$\left\| t^{k+p-\alpha} - \sum_{j=0}^{\tilde{m}} b_j B_j(t) \right\|_2^2 = \frac{G\left(t^{k+p-\alpha}, B_0(t), B_1(t), ..., B_{\tilde{m}}(t)\right)}{G\left(B_0(t), B_1(t), ..., B_{\tilde{m}}(t)\right)}, \tag{5.2}$$

where G is the Gram determinant [19]. Hence using Eqs. (5.2) and (3.3) we have

$$\begin{aligned}\|e^{\alpha,p}_{i,\tilde{m}}\|_2 &= \left\| {}_0^C D_t^{\alpha}(t^p B_i(t)) - \sum_{j=0}^{\tilde{m}} \mathrm{a}^{\alpha,p}_{i,j} B_j(t) \right\|_2 \\ &\leq \sum_{k=0}^{i} \binom{i}{k} |\beta_{i-k}| \frac{\Gamma(k+p+1)}{\Gamma(k+p+1-\alpha)} \\ &\quad \times \left(\frac{G\left(t^{k+p-\alpha}, B_0(t), B_1(t), ..., B_{\tilde{m}}(t)\right)}{G\left(B_0(t), B_1(t), ..., B_{\tilde{m}}(t)\right)} \right)^{1/2}, \ 0 \leq i \leq m.\end{aligned} \tag{5.3}$$

Therefore an error upper bound for the fractional Caputo operational matrix is determined. By increasing the number of Bernoulli basis, the fractional operational matrix converges to fractional derivative of the basis.

Now Consider the main problem (4.1). By computing the control function from the system dynamics, the problem is converted into equivalent problem as

$$\begin{aligned}\min \quad J[\mathbf{x}] &= \int_0^1 f\Bigg(t, \mathbf{x}(t), {}_0^C D_t^{\alpha}\mathbf{x}(t), \frac{1}{h(t)}\Bigg(\sum_{i=1}^{I} M_i \mathbf{x}^{(i)}(t) \\ &\qquad + \sum_{j=1}^{J} N_j \, {}_0^C D_t^{\alpha_j}\mathbf{x}(t) - F(t, \mathbf{x}(t))\Bigg)\Bigg)\mathrm{d}t \\ &:= \int_0^1 W\big(t, \mathbf{x}(t), ..., \mathbf{x}^{(n)}(t), {}_0^C D_t^{\alpha_1}\mathbf{x}(t), ..., {}_0^C D_t^{\alpha_J}\mathbf{x}(t)\big)\mathrm{d}t.\end{aligned} \tag{5.4}$$

Let $n = \max_j \{I, \alpha_j\}$. We define the space $C^n[0,1]$ as

$$C^n[0,1] = \{\mathbf{x} \mid \mathbf{x} \text{ is continuous up to } n\text{th derivative}\} \tag{5.5}$$

equipped with the following norm

$$\|\mathbf{x}\| = \sum_{j=0}^{n} \|\mathbf{x}^{(j)}\|_\infty. \tag{5.6}$$

Theorem 5.1. *Let $\boldsymbol{x} \in C^n[0,1]$. The defined functional on (5.4) is uniformly continuous.*

Proof. Considering the fractional derivative and norm definition for $\alpha_j > 0$ we get

$$\begin{aligned} \|{}_0^C D_t^{\alpha_j} \mathbf{x}\|_\infty &\leq \frac{1}{\Gamma(n_j - \alpha_j)} \int_0^t (t-\tau)^{n_j - \alpha_j - 1} \|\mathbf{x}^{(n_j)}\|_\infty \mathrm{d}\tau \\ &\leq \frac{\|\mathbf{x}^{(n_j)}\|_\infty}{\Gamma(n_j - \alpha_j + 1)}. \end{aligned} \tag{5.7}$$

Let $\epsilon > 0$ and $\delta > 0$ be arbitrary. Assume $\mathbf{y} \in C^n[0,1]$ such that

$$\|\mathbf{x} - \mathbf{y}\| = \sum_{j=0}^{n} \|\mathbf{x}^{(j)} - \mathbf{y}^{(j)}\|_\infty < \delta. \tag{5.8}$$

Regarding (5.7) we get

$$\|{}_0^C D_t^{\alpha_j} \mathbf{x} - {}_0^C D_t^{\alpha_j} \mathbf{y}\|_\infty \leq \frac{\|\mathbf{x}^{(n_j)} - \mathbf{y}^{(n_j)}\|_\infty}{\Gamma(n_j - \alpha_j + 1)}. \tag{5.9}$$

According to the assumption of fractional OCP (4.1), the function W defined on (5.4) is continuous, hence for sufficiently small values of $\delta > 0$, we get

$$\begin{aligned} &\|W\big(t, \mathbf{x}(t), ..., \mathbf{x}^{(I)}(t), {}_0^C D_t^{\alpha_1} \mathbf{x}(t), ..., {}_0^C D_t^{\alpha_J} \mathbf{x}(t)\big) \\ &\quad - W\big(t, \mathbf{y}(t), ..., \mathbf{y}^{(I)}(t), {}_0^C D_t^{\alpha_1} \mathbf{y}(t), ..., {}_0^C D_t^{\alpha_J} \mathbf{y}(t)\big)\|_\infty < \epsilon, \end{aligned} \tag{5.10}$$

provided that $\|\mathbf{x} - \mathbf{y}\| < \delta$. Therefore we have

$$|J[\mathbf{x}] - J[\mathbf{y}]| < \epsilon.$$

□

Now we state the main theorem about the convergence of functional to its optimum value by increasing the Bernoulli basis order.

Theorem 5.2. *Let $P_m[0,1]$ be the subspace of $C^n[0,1]$ spanned by the first $m+1$ Bernoulli polynomials and ν be the minimum of the functional J on $C^n[0,1]$. If ν_m is the optimum of J on $C^n[0,1] \cap P_m[0,1]$, then*

$$\lim_{m \to \infty} \nu_m = \nu.$$

Proof. $\epsilon > 0$ is given. Let $\mathbf{x}^*(t) \in C^n[0,1]$ be a function satisfying the following relation

$$J[\mathbf{x}^*] < \nu + \epsilon.$$

Because of infimum properties, the existence of such function is obvious. Considering Theorem 5.1, for $\mathbf{x} \in C^n[0,1]$, the relation $\|\mathbf{x} - \mathbf{x}^*\| < \delta$ implies

$$|J[\mathbf{x}] - J[\mathbf{x}^*]| < \epsilon$$

Since the set of the Bernoulli polynomials $\{B_k(t)\}_{k=0}^{\infty}$ are dense in the complete space $L^2[0,1]$ [45], so for sufficiently large $m \in \mathbb{N}$, there is a function $\mathbf{x}_m$ such that $\|\mathbf{x}_m - \mathbf{x}\| < \delta$. Hence referring to continuity of the functional J, we get $|J[\mathbf{x}^*] - J[\mathbf{x}_m]| < \epsilon$. Denoted $J[\mathbf{x}_m]$ by ν_m, we get

$$\nu \leq \nu_m = |J[\mathbf{x}_m] - J[\mathbf{x}^*] + J[\mathbf{x}^*]| \leq |J[\mathbf{x}^*]| + |J[\mathbf{x}_m] - J[\mathbf{x}^*]| \leq \nu + 2\epsilon.$$

Because ϵ is arbitrary, as a result we conclude

$$\lim_{m\to\infty} \nu_m = \nu.$$

□

6. Illustrative test problems

In this section, to show the application of the proposed method, several illustrative test problems are provided.

Example 6.1. Assume the following fractional optimal control problem given in [38]

$$\min \quad J[x,u] = \frac{1}{2}\int_0^1 (x^2(t) + u^2(t))\mathrm{d}t,$$

subject to

$${}_0^C D_t^{\alpha} x(t) = -x(t) + u(t), \quad 0 \leq t \leq 1,$$

and the initial and boundary conditions as

$$x(0) = 0, \quad x(1) = 2.$$

The exact solution of this problem for $\alpha = 1$ is as

$$x^*(t) = \frac{2\sinh 2t}{\sinh 2}, \ u^*(t) = \frac{2(\sinh 2t + 2\cosh 2t)}{\sinh 2}.$$

Considering the proposed numerical scheme, the state function is approximated as

$$x_m(t) = t(t-1)Y^m \mathfrak{B}_m(t) + 2t,$$

and by regarding the system dynamics and taking the fractional operational matrix into account, the estimated control input is computed by

$$\bar{u}_m(t) =$$
$$Y^m(\mathbf{A}_{m,\tilde{m}}^{\alpha,2} - \mathbf{A}_{m,\tilde{m}}^{\alpha,1})\mathfrak{B}_{\tilde{m}}(t) + \frac{2}{\Gamma(2-\alpha)} t^{2-\alpha} + (t^2 - t)Y^m \mathfrak{B}_m(t) + 2t.$$

By substituting the approximate functions into the cost function, a nonlinear optimization problem is obtained. In Table 1, the optimal cost function for the present method and another method in the literature are displayed. In comparison with that method, by selecting a small number of the basis, the optimal value is obtained. Figures 1 and 2 depicts the exact and the numerical solutions of the problem for integer order and two different values of the fractional order α.

TABLE 1. The optimal cost function for different values of $m, \tilde{m}$ with different methods in Example 6.1

Numerical Method	Method of [38]	Present study
Optimal cost function (J^*)	6.149258977 (m=5)	6.149258977 ($m = 3, \tilde{m}$=3)

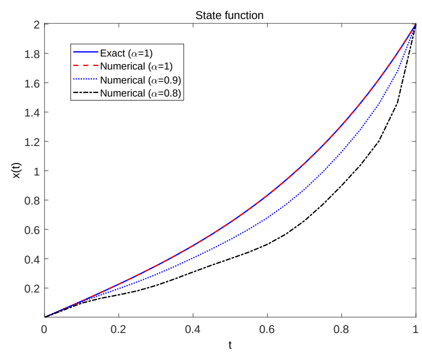

FIGURE 1. Plots of the exact and the numerical state functions for $\alpha = 1, 0.9, 0.8$ and $m = \tilde{m} = 4$ in Example 6.1

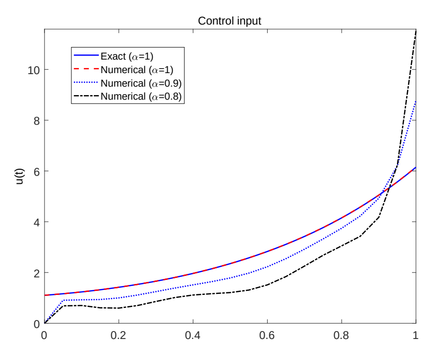

FIGURE 2. Plots of the exact and the numerical control functions for $\alpha = 1, 0.9, 0.8$ and $m = \tilde{m} = 4$ in Example 6.1

Example 6.2. Consider the following fractional optimal control problem as [3, 23]

$$\min \ J[\mathbf{x}, u] = \int_0^1 \big[(x_1(t) - t^{\frac{3}{2}} - 1)^2 + (x_2(t) - t^{\frac{5}{2}})^2 + (u(t) - \frac{3\sqrt{\pi}}{4}t + t^{\frac{5}{2}})^2\big]\mathrm{d}t,$$

subject to the given constraint

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} {}_0^C D_t^{\frac{1}{2}} \mathbf{x}(t) - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x}(t) = \begin{pmatrix} 0 \\ \frac{15\sqrt{\pi}}{16}t^2 - t^{\frac{3}{2}} - 1 \end{pmatrix} + \begin{pmatrix} u(t) \\ 0 \end{pmatrix},$$

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

The exact state and control functions of this problem are as

$$\mathbf{x}^*(t) = \begin{pmatrix} 1 + t^{\frac{3}{2}} \\ t^{\frac{5}{2}} \end{pmatrix}, \quad u^*(t) = \frac{3\sqrt{\pi}}{4}t - t^{\frac{5}{2}}.$$

To solve the problem numerically by the proposed spectral method and considering Hermite polynomial interpolation, two components of the state function are approximated as

$$\mathbf{x}_m(t) = tY^m \mathfrak{B}_m(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where $Y^m = \begin{bmatrix} Y_1^m \\ Y_2^m \end{bmatrix}$ is the unknown coefficient matrix to be determined. By substituting the estimated state function into the system dynamics, the control function is approximated as

$$u_m(t) = {}_0^C D_t^{\frac{1}{2}}(x_1(t) + x_2(t)) - x_1(t) - x_2(t) + t^{\frac{3}{2}} - \frac{15\sqrt{\pi}}{16}t^2 + 1.$$

As stated, to reduce the computational complexity on the estimated control function, we applied the fractional and integer order operational derivative matrix. Consequently the control function is simplified as

$$\begin{aligned} \bar{u}_m(t) = &(Y_1^m + Y_2^m)\mathbf{A}_{m,\tilde{m}}^{0.5,1}\mathfrak{B}_{\tilde{m}}(t) - t(Y_1^m + Y_2^m)\mathfrak{B}_m(t) \\ &+ t^{\frac{3}{2}} - \frac{15\sqrt{\pi}}{16}t^2 + 1. \end{aligned}$$

By taking $m = \tilde{m} = 5$ and the zero matrix of dimension 2×6 as starting point for the Newton's method, after 7 iteration we get

$$Y^5(7) = \begin{bmatrix} 0.6698 & 0.8389 & -0.6582 & 0.6333 & -0.5888 & 0.1391 \\ 0.3987 & 1.0307 & 0.4872 & 0.2566 & -0.1722 & 0.6564 \end{bmatrix}.$$

By substituting this matrix into the estimated function, the approximate state and control functions can be identified. Figures 3–6 depict the exact and estimated state and control functions which demonstrate coincidence of the exact solution and the numerical solutions to each other. In Table 2, the optimal cost function for the Epsilon-Ritz method and present study are presented and compared.

Consequently, the superiority of the proposed method is clarified. The effects of increasing the number of basis functions, m, over the optimal cost function is also depicted in the results of Table 2. It is remarkable that many real-world optimal control problems have no exact solution at hand analytically. To apply the proposed method on such problems, the effect of cost function improvement as well as the solutions convergence with different basis order is an alternative approach to test. Therefore, we take the logarithm of the two subsequent solutions to verify the method for such a condition. Figure 6 depicts the logarithmic improvement of the two subsequent solutions.

TABLE 2. The optimal cost function for different values of m and $\tilde{m} = 6$ with different methods for Example 6.2

Method	Epsilon-Ritz [23]	Present study
Optimal cost function	1.01378×10^{-4} (k=3, ϵ=0.01)	$1.82793964 \times 10^{-5}$ (m=2)
	5.3424×10^{-5} (k=5, ϵ=0.001)	$7.90498552 \times 10^{-7}$ (m=5)
	8.0027×10^{-6} (k=8, ϵ=0.0001)	$9.41573213 \times 10^{-8}$ (m=8)

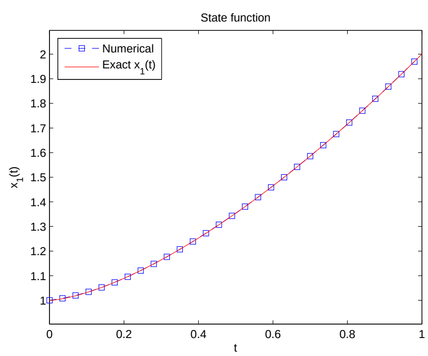

FIGURE 3. The exact solution and the numerical state function ($x_1(t)$) with $m = 6, \tilde{m} = 8$ in Example 6.2

Example 6.3. Assume the following fractional optimal control problem with variable fractional order given in [48]

$$\min \; J[\mathbf{x}, u] = \int_0^1 \left(x_1(t) + x_2(t) - \frac{2t}{\alpha + 3} u(t) \right)^2 \mathrm{d}t,$$

subject to the dynamical system constraint as

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \dot{\mathbf{x}}(t) + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} {}_0^C D_t^{\alpha} \mathbf{x}(t) = \begin{pmatrix} t^3 \\ t^3 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} u(t),$$

with the given initial and boundary conditions as $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; \mathbf{x}(1) = \begin{pmatrix} \frac{6}{\Gamma(\alpha+4)} \\ \frac{6}{\Gamma(\alpha+4)} \end{pmatrix}$.

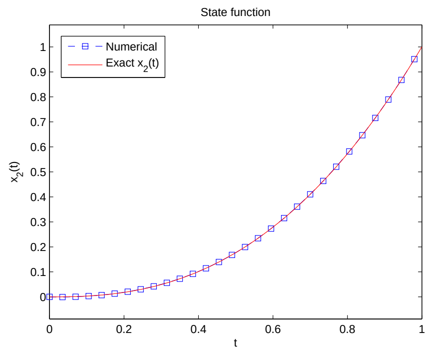

FIGURE 4. The exact solution and the numerical state function $(x_2(t))$ with $m = 6, \tilde{m} = 8$ in Example 6.2

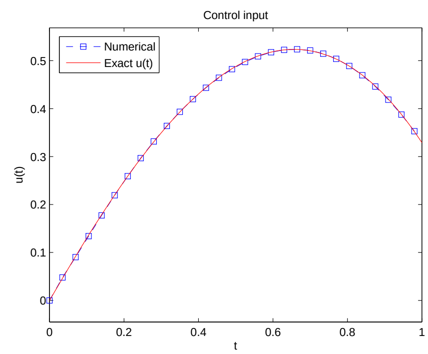

FIGURE 5. The exact solution and the numerical control input with $m = 6, \tilde{m} = 8$ in Example 6.2

The exact state and control functions minimizing the performance index are as follows

$$\mathbf{x}^*(t) = \begin{pmatrix} \frac{6t^{\alpha+3}}{\Gamma(\alpha+4)} \\ \frac{6t^{\alpha+3}}{\Gamma(\alpha+4)} \end{pmatrix}, \quad \mathbf{u}^*(t) = \frac{6t^{\alpha+2}}{\Gamma(\alpha+3)}.$$

To solve the problem numerically by the proposed spectral method and considering Hermite polynomial interpolation, two components of the state function are approximated as

$$\mathbf{x}_m(t) = t(t-1)Y^m\mathfrak{B}_m(t) + \begin{pmatrix} \frac{6t}{\Gamma(\alpha+4)} \\ \frac{6t}{\Gamma(\alpha+4)} \end{pmatrix},$$

where $Y^m = \begin{bmatrix} Y_1^m \\ Y_2^m \end{bmatrix}$ are the unknown coefficient matrix to be determined. By substituting the estimated state function into the system dynamics, the control function is

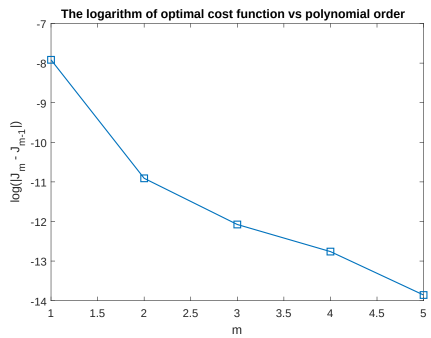

FIGURE 6. The logarithmic improvement of the cost function versus polynomial order with $m = \tilde{m}$ in Example 6.2

approximated as

$$u_m(t) = \dot{x}_1(t) + {}_0^C D_t^\alpha \dot{x}_1(t) - {}_0^C D_t^\alpha \dot{x}_2(t).$$

By applying the fractional operational derivative matrix, the control function is simplified as

$$\begin{aligned}\bar{u}_m(t) =& Y_1^m(\mathbf{A}_{m,\tilde{m}}^{1,2} - \mathbf{A}_{m,\tilde{m}}^{1,1})\mathfrak{B}_{\tilde{m}}(t) + Y_1^m(\mathbf{A}_{m,\tilde{m}}^{\alpha,2} - \mathbf{A}_{m,\tilde{m}}^{\alpha,1})\mathfrak{B}_{\tilde{m}}(t) \\ &- Y_2^m(\mathbf{A}_{m,\tilde{m}}^{\alpha,2} - \mathbf{A}_{m,\tilde{m}}^{\alpha,1})\mathfrak{B}_{\tilde{m}}(t) + \frac{6}{\Gamma(\alpha+4)}.\end{aligned} \tag{6.1}$$

By manipulating Eq. (6.1), we get

$$\begin{aligned}\bar{u}_m(t) =& Y_1^m(\mathbf{A}_{m,\tilde{m}}^{1,2} - \mathbf{A}_{m,\tilde{m}}^{1,1})\mathfrak{B}_{\tilde{m}}(t) \\ &+ (Y_1^m - Y_2^m)(\mathbf{A}_{m,\tilde{m}}^{\alpha,2} - \mathbf{A}_{m,\tilde{m}}^{\alpha,1})\mathfrak{B}_{\tilde{m}}(t) + \frac{6}{\Gamma(\alpha+4)}.\end{aligned}$$

by substituting the estimated functions into the cost function, the unknown coefficient matrix are obtained. Table 3 presents the absolute error for the state vector by increasing the approximation order where the approximate solution converges to exact solution of the problem. Figures 7 and 8 depict the exact and numerical state functions for different values of fractional order $\alpha = 0.7, 1, 1.8$. Furthermore, Figure 9 depicts the optimal control input for different choices of fractional order. As evident, the exact and numerical solutions are coincide each other for different values of fractional order with high accurate decision. By increasing the number of basis order m, the absolute error of the solution is also altered.

Example 6.4. Consider the following fractional optimal control problem

$$\begin{aligned}&\min\ J[\mathbf{x}, \mathbf{u}] = \\ &\int_0^1 \left((t^{\frac{3}{2}} x_1(t) - x_2(t) - 1)^2 + (2tu_1(t) + \frac{35}{2} t^{\frac{3}{2}} - 7u_2(t) - \frac{15\sqrt{\pi}}{8} t^3)^2 \right) \mathrm{d}t,\end{aligned}$$

Table 3. The absolute error $\|\mathbf{x}^* - \mathbf{x}_m\|_\infty$ and $\|\mathbf{u}^* - \mathbf{u}_m\|_\infty$ with different values of $m = \tilde{m}$ in Example 6.3

	$\|\mathbf{x}^* - \mathbf{x}_m\|_\infty$				$\|\mathbf{u}^* - \mathbf{u}_m\|_\infty$
t	$m = 1$	$m = 3$	$m = 8$	$m = 11$	$m = 11$
0	0	0	0	0	3.59017×10^{-6}
0.1	3.28346×10^{-3}	5.12710×10^{-5}	4.55978×10^{-8}	1.01628×10^{-8}	4.05691×10^{-8}
0.2	7.06115×10^{-3}	4.39807×10^{-5}	4.63290×10^{-8}	1.00189×10^{-8}	1.96347×10^{-8}
0.3	9.68796×10^{-3}	5.66568×10^{-6}	4.65423×10^{-8}	6.47133×10^{-8}	1.11622×10^{-8}
0.4	1.02825×10^{-2}	1.89133×10^{-5}	9.06090×10^{-8}	9.05742×10^{-9}	1.06974×10^{-8}
0.5	8.63186×10^{-3}	1.04993×10^{-5}	4.62760×10^{-8}	3.18470×10^{-9}	1.54917×10^{-8}
0.6	5.13418×10^{-3}	1.98290×10^{-5}	8.46870×10^{-9}	9.40608×10^{-9}	1.72286×10^{-8}
0.7	7.58144×10^{-4}	4.27898×10^{-5}	9.08595×10^{-8}	2.38176×10^{-9}	1.97583×10^{-8}
0.8	2.98775×10^{-3}	3.26659×10^{-5}	3.94343×10^{-8}	8.36313×10^{-9}	1.81963×10^{-8}
0.9	4.07992×10^{-3}	4.75873×10^{-6}	1.00529×10^{-7}	1.02494×10^{-9}	7.25227×10^{-9}
1	0	0	0	0	8.16531×10^{-8}

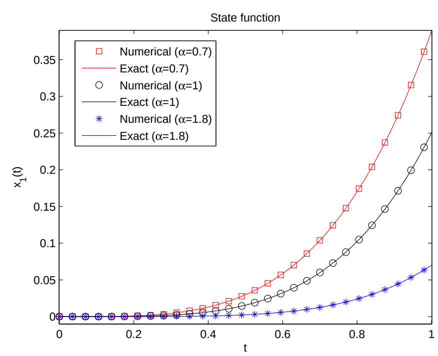

Figure 7. Plots of the exact and numerical state function $(x_1(t))$ for $\alpha = 0.7, 1, 1.8$ and $m = \tilde{m} = 3$ in Example 6.3

subject to the dynamical system constraint as

$$M\ \dot{\mathbf{x}}(t) + N_1\ {}^C_0D_t^{\frac{3}{2}}\mathbf{x}(t) + N_2\ {}^C_0D_t^{\frac{1}{2}}\mathbf{x}(t) = \begin{pmatrix} 0 \\ \frac{15\sqrt{\pi}}{8}t \end{pmatrix} + \mathbf{u}(t),$$

with the given initial and boundary conditions as

$$\mathbf{x}(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix},\ \mathbf{x}(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$, $M = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $N_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $N_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

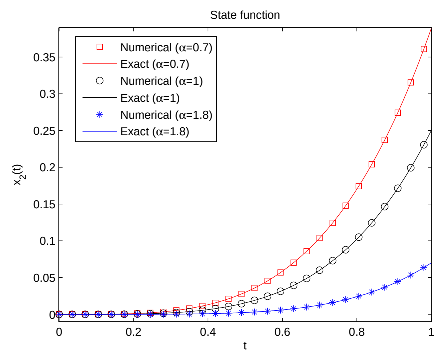

FIGURE 8. Plots of exact and numerical state function ($x_2(t)$) for $\alpha = 0.7, 1, 1.8$ and $m = \tilde{m} = 3$ in Example 6.3

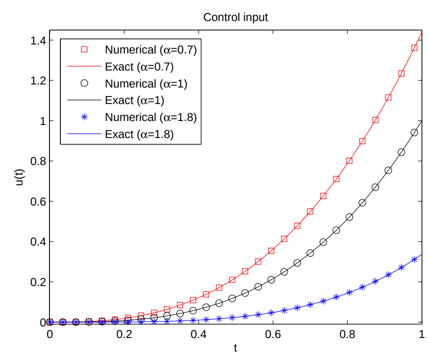

FIGURE 9. Plots of the exact and numerical control input function for $\alpha = 0.7, 1, 1.8$ and $m = \tilde{m} = 3$ in Example 6.3

For the above problem, the optimizer functions satisfying the initial and boundary functions are as

$$\mathbf{x}^*(t) = \begin{pmatrix} t^{\frac{5}{2}} \\ t^4 - 1 \end{pmatrix}, \ \mathbf{u}^*(t) = \begin{pmatrix} \frac{64}{5\sqrt{\pi}} t^{\frac{5}{2}} + \frac{15\sqrt{\pi}}{16} t^2 \\ \frac{128}{35\sqrt{\pi}} t^{\frac{7}{2}} + \frac{5}{2} t^{\frac{3}{2}} \end{pmatrix}.$$

Using the presented method, the approximate functions are determined as:

$$\mathbf{x}_m(t) = t(t-1)Y^m \mathfrak{B}_m(t) + \begin{pmatrix} t \\ t-1 \end{pmatrix}.$$

The approximate control input by regarding the system dynamic is

$$\begin{aligned} \bar{\mathbf{u}}_m(t) =& MY^m(\mathbf{A}^{1,2}_{m,\tilde{m}} - \mathbf{A}^{1,1}_{m,\tilde{m}})\mathfrak{B}_{\tilde{m}}(t) + N_1 Y^m(\mathbf{A}^{1.5,2}_{m,\tilde{m}} - \mathbf{A}^{1.5,1}_{m,\tilde{m}})\mathfrak{B}_{\tilde{m}}(t) \\ &+ N_2 Y^m[(\mathbf{A}^{0.5,2}_{m,\tilde{m}} - \mathbf{A}^{0.5,1}_{m,\tilde{m}})\mathfrak{B}_{\tilde{m}}(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{2}\sqrt{\frac{t}{\pi}}] - \begin{pmatrix} 0 \\ \frac{15\sqrt{\pi}}{8} t \end{pmatrix}. \end{aligned}$$

Table 4 displays the optimal performance index (J^*) in terms of the polynomial order.

TABLE 4. The cost function with different values of $m, \tilde{m}$ in Example 6.4

Polynomial Order	$m=2,\tilde{m}=3$	$m=5,\tilde{m}=5$	$m=8,\tilde{m}=9$	$m=11,\tilde{m}=13$
Optimal Cost function	0.0183940	0.0142326	4.70911048×10^{-4}	2.0854212×10^{-6}

Obviously, by increasing the order of Bernoulli polynomials m, the cost function is consequently improved. The state function and control input are also depicted in Figures 10–12, respectively.

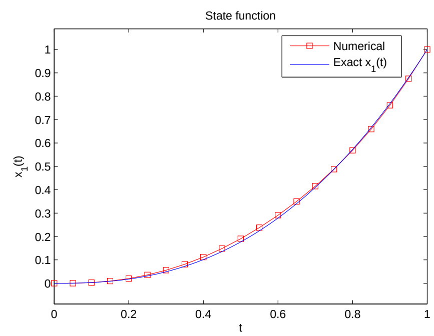

FIGURE 10. The exact and the numerical state function ($x_1(t)$) with $m=6,\tilde{m}=7$ in Example 6.4

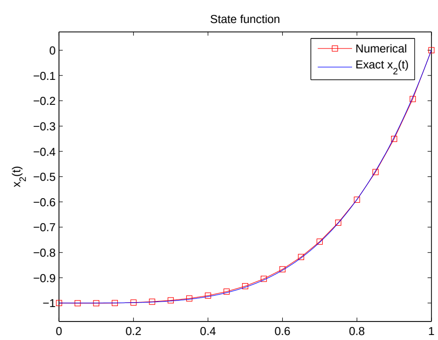

FIGURE 11. The exact and the numerical state function ($x_2(t)$) with $m=6,\tilde{m}=7$ in Example 6.4

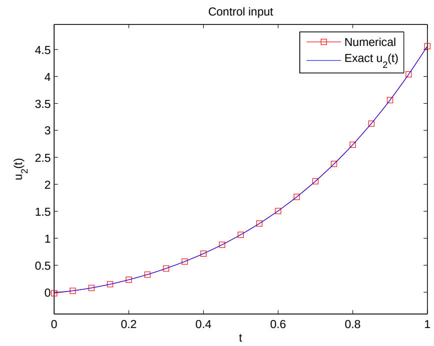

Figure 12. The exact and numerical control input $(u_2(t))$ with $m = 6, \tilde{m} = 7$ in Example 6.4

7. Conclusion

In this article, a numerical solution of FOCP based on Caputo fractional derivative was established. The method is to convert the problem to its equivalent problem then approximate it using Bernoulli polynomial basis and its fractional operational matrix. The approximate spectral method has a good property satisfying all the initial and boundary conditions. Next, the derived problem was solved by the Newton's iterative method. Moreover, the convergence analysis of the new method is investigated. Finally, to show the effectiveness and applicability of the proposed scheme, four test problems were included and compared with the other methods of the literature to emphasis the fast and efficient convergence of the proposed methodology.

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