Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 9, No. 2, 2021, pp. 480-492
DOI:10.22034/cmde.2020.35875.1620

Solving a class of fractional optimal control problems via a new efficient and accurate method

Samaneh Soradi-Zeid*
Faculty of Industry and Mining (Khash),
University of Sistan and Baluchestan, Zahedan, Iran.
E-mail: soradizeid@eng.usb.ac.ir

Abstract The present paper aims to get through a class of fractional optimal control problems (FOCPs). Furthermore, the fractional derivative portrayed in the Caputo sense through the dynamics of the system as fractional differential equation (FDE). Getting through the solution, firstly the FOCP is transformed into a functional optimization problem. Then, by using known formulas for computing fractional derivatives of Legendre wavelets (LWs), this problem has been reduce to an equivalent system of algebraic equations. In the next step, we can simply solved this algebraic system. In the end, some examples are given to bring about the validity and applicability of this technique and the convergence accuracy.

Keywords. Fractional optimal control problem, Fractional integrals, Fractional derivatives, Legendre wavelets, Lagrange multipliers method.

2010 Mathematics Subject Classification. 49K05, 49M30.

1. Introduction

According to variability of arbitrary value for the order of derivatives and integrals, we can consider an order which generates an extension of the classical calculus, namely fractional calculus. More precisely, this topic is a development of classical calculus where the order of derivative operators are estimated by any arbitrary value. This subject applied through many physical applications evidently, when the memory effects is sensitive to the time and place of the event. Several ways are existing that define fractional derivatives but the Riemann-Liouville and the Caputo fractional derivatives have been used more than the other definitions. Overwhelming research on developing applications of fractional calculus has been done to prove the diversity of scientific, engineering fields and physical models [5, 24, 31]. More specifically, fractional order formulations used to exemplified the mechanics and dynamic systems governed by FDEs. It must be considered that due to the high complexity of fractional order derivatives, the analytically handling equations described by this derivatives is extremely difficult and even impossible. To overcome this challenge, practical numerical/approximate methods have been presenting to solve them. An up-to-date

Received: 15 October 2019 ; Accepted: 5 February 2020.
∗ corresponding.

bibliography on numerical methods for solving FDEs was recently reported by Zeid in [35].

A generalization of classic optimal control problems when the dynamics system is described by FDEs is considered as FOCPs. The reason to formulate and solve FOCPs has recently been answered affirmatively turn into significant increasing of these problems in control systems. On the other hand, all the methods for solving FOCPs divided mainly in two category namely as direct and indirect in which the first methods describe the continuous FOCPs to a finite-dimensional nonlinear programming problem and others are based on the necessary optimality conditions of a FOCP. Although, many computational methods have been proposed for solving FOCPs, [33], but the recent study in FOCP are referred to [1, 2, 3, 4, 11, 14, 21, 23, 27, 26, 29, 30, 32, 34]. The overall aim of this paper is to interoducing a class of FOCPs and use the interpolate approximate basis functions to reformulated the FOCPs as an equivalent system of algebraic equations that makes the problem significantly simpler. For this purpose, we consider the following FOCP:

$$\min J(t,x,u) = \int_{t_0}^{t_f} L(t,x(t),u(t))dt \tag{1.1}$$

with the fractional system

$$D^{\alpha}x(t) = f(t,x(t),u(t)),\ \alpha \in (0,1),\ t \in [t_0,t_f] \tag{1.2}$$

and boundary conditions

$$x(t_0) = x_0,\ \ u(t) \in U, \tag{1.3}$$

where $J \in C^1[t_0,t_f]$, t_0 and t_f are two positive constant, $x_0 \in \mathbb{R}$, L and f are continuous functions, the set $U \subset \mathbb{R}^m$ represents the allowable inputs, which are considered to be continuous functions and D^{α} denote the fractional derivative of order α which will be introduced in the next.

Overwhelming research on developing applications of using wavelets for solving fractional equations has been done to prove the applicability of this approximation. Here, in the suggested technique, the FOCP (1.1)-(1.3)are reformulated into an optimization problem with discrete parameters in a way that the main computational cost of this FOCP comes from solving the discrete optimization problem. To improve the quality of numerical solutions we applied the necessary and sufficient condition for obtaining the optimal extremum to obtain the unknown coefficients. We can observe that the solutions obtained by this method are in good agreement with the exact solution. Moreover, for the proposed method we requires less computational work of the other proposed methods. In short, the main contributions of this work listed as follows:

- The main motivation of this work that is the FOCP is recast to an optimization functional. Then, by utilizing the LWs, this optimization problem is rtransformed into a system of algebraic equations in which solves the FOCP very easy.

- Direct method based on LWs is presentedd and its error estimation is also investigated.

The outline of the paper is formed as follows: Section 2 includes the necessary definitions and mathematical preliminaries of the fractional calculus. The properties of LWs and the corresponding fractional operators of these functions are described in section 3. In section 4, with the underlying FOCP we derive the optimal solutions under consideration with direct method. Convergence analysis are also presented in this section. Some illustrative numerical examples which are solved by the proposed method are provided in section 5. Also, we compare our results with previous methods available in the literatures in this section. A brief summary is stated in the last section.

2. Basic Definitions and Notations of Fractional Calculus

Here we briefly propose some definitions regarding fractional operators permitting us to formulate FOCP. For more details, see [15, 18, 20].

Definition 2.1. The left and right Riemann-Liouville fractional integrals operator of order $\alpha > 0$ of a given function $f \in L_1([t_0, t_f], \mathbb{R}^n)$, are defined by

$$ {}_{t_0}I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau) d\tau \tag{2.1}$$

and

$$ {}_{t}I_{t_f}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t}^{t_f} (\tau-t)^{\alpha-1} f(\tau) d\tau, \tag{2.2}$$

where $\Gamma(.)$ is the Euler-Gamma function. It is identity that ${}_{t_0}I_t^0 f(t) = {}_tI_{t_f}^0 f(t) = f(t)$.

Definition 2.2. The left RLFD, and the right RLFD, of order $\alpha > 0$ of a given function $f(t)$ are defined by [15, 19, 22]:

$$ {}_{t_0}D_t^{\alpha} f(t) = D^n({}_{t_0}I_t^{n-\alpha}) f(t) = \frac{1}{\Gamma(n-\alpha)} \Big(\frac{d}{dt}\Big)^n \int_{t_0}^{t} (t-\tau)^{n-\alpha-1} f(\tau) d\tau \tag{2.3}$$

and

$$ {}_{t}D_{t_f}^{\alpha} f(t) = (-1)^n D^n({}_{t}I_{t_f}^{n-\alpha}) f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \Big(\frac{d}{dt}\Big)^n \int_{t}^{t_f} (\tau-t)^{n-\alpha-1} f(\tau) d\tau, \tag{2.4}$$

respectively, where $n-1 < \alpha \leq n$, $n \in \mathbb{N}$. When α is an integer, the usual definitions of the derivatives are considered.

From this definition we have $(\alpha, \beta \geq 0)$:

$$ {}_{t_0}D_t^{\alpha}\, {}_{t_0}D_t^{\beta} f =_{t_0} D_t^{\alpha+\beta} f, \qquad {}_{t_0}D_t^{\alpha}\, {}_{t_0}D_t^{\beta} =_{t_0} D_t^{\beta}\, {}_{t_0}D_t^{\alpha}, \tag{2.5}$$

$$ {}_{t_0}D_t^{\alpha} k = \frac{k\Gamma(t-t_0)^{\alpha}}{\Gamma(1-\alpha)} \tag{2.6}$$

and

$$ {}_{t_0}D_t^{\alpha} (t-t_0)^n = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} (t-t_0)^{n-\alpha},\ t > t_0. \tag{2.7}$$

Definition 2.3. The left and the right Caputo fractional derivatives of function $f(t)$ of order $\alpha > 0$ are defined by [7]:

$$ {}^{C}_{t_0}D^{\alpha}_{t} f(t) = ({}_{t_0}I^{n-\alpha}_{t})D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau \tag{2.8} $$

and

$$ {}^{C}_{t}D^{\alpha}_{t_f} f(t) = (-1)^n ({}_{t}I^{n-\alpha}_{t_f})D^n f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_{t}^{t_f} (\tau - t)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \tag{2.9} $$

respectively, where $n-1 < \alpha \leq n$, $n \in \mathbb{N}$.

Also we have:

$$ {}^{C}_{t_0}D^{\alpha}_{t} {}^{C}_{t_0}D^{\beta}_{t} f = {}^{C}_{t_0} D^{\alpha+\beta}_{t} f, \qquad {}^{C}_{t_0}D^{\alpha}_{t} {}^{C}_{t_0}D^{\beta}_{t} f = {}^{C}_{t_0} D^{\beta}_{t} {}^{C}_{t_0}D^{\alpha}_{t} f, \tag{2.10} $$

$$ {}^{C}_{t_0}D^{\alpha}_{t} k = 0 \tag{2.11} $$

and

$$ {}^{C}_{t_0}D^{\alpha}_{t} (t-t_0) = \begin{cases} \dfrac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}(t-t_0)^{n-\alpha}, & n \geq [\alpha], \\ 0, & n < [\alpha]. \end{cases} \tag{2.12} $$

3. The LWs and their properties

For a square integrable function $f(t)$ on $[0,1]$ we have the following expanded based on the LWs :

$$ f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(t), \tag{3.1} $$

where $c_{nm} =< f(t), \psi_{nm}(t) >, < .,. >$ denotes the inner produc in $L^2[0,1]$ and $\psi_{nm}(t)$ are the LWs that defined as:

$$ \psi_{nm}(t) = \begin{cases} \sqrt{2m+1} 2^{\frac{k}{2}} P_m(2^{k+1}t - 2n + 1), & t \in \left[\frac{n-1}{2^k}, \frac{n}{2^k}\right], \\ 0, & 0.w. \end{cases} \tag{3.2} $$

where $n = 1, 2, \cdots, 2^k$, k is any arbitrary positive integer and $\{P_m(t)\}$ denote the Legender polynomials of degree m. . So, we can rewrite Eq. (3.1) as:

$$ f(t) = \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}(t) = C^T \Psi(t). \tag{3.3} $$

For simplicity, we can write:

$$ f(t) \simeq \sum_{i=1}^{m^*} c_i \psi_i(t) = C^T \Psi(t), \tag{3.4} $$

in which C and $\Psi(t)$ are $m^* = 2^k M$ column vectors, $c_i = c_{nm}$ and $\psi_i(t) = \psi_{nm}(t)$ where i is determined by the relation $i = M(n-1) + m + 1$.

Theorem 3.1. *Providnig that $f(t) \in L^2[0,1]$. Also, presume that there exixts a constant $\hat{M}$ in which $|f\varepsilon(t)| \leq \hat{M}$. Then, the expanision (3.4) is uniformly convergens to the function $f(t)$ and c_{nm} fulfills the condition [16]:*

$$c_{nm} \leq \frac{2\sqrt{3}\hat{M}}{(2n)^{\frac{5}{2}}(2m-3)^2}. \tag{3.5}$$

Theorem 3.2. *Assuming the conditions of the preceding theorem are satisfied. Then, we have [12]:*

$$\Big(\int_0^1 (f(t) - C^T\Psi(t))^2 dt\Big)^{\frac{1}{2}} \leq \Big(\sum_{n=1}^{\infty}\sum_{m=M}^{\infty} c_{nm}^2 + \sum_{n=2^k+1}^{\infty}\sum_{m=0}^{M-1} c_{nm}^2\Big)^{\frac{1}{2}}. \tag{3.6}$$

Now, by taking the collocation points $t_i = \frac{i}{m^*-1}$, $i = 0, 1, \cdots, m^*-1$, we can get the LWs matrix $\Phi_{m^*\times m^*}$ as follows:

$$\Phi_{m^*\times m^*}\Big[\Psi(0), \Psi(\frac{1}{m^*-1}), \cdots, \Psi(1)\Big]. \tag{3.7}$$

Also, for the fractional integration of $\Psi(t)$ we have:

$$\Big(I^\alpha\Psi\Big)(t) \simeq P^\alpha\Psi(t) \simeq \Big(\Phi_{m^*\times m^*}Q^\alpha\Phi^{-1}_{m^*\times m^*}\Big)\Psi(t), \tag{3.8}$$

where $\Phi_{m^*\times m^*}$ is defined in (3.7) and Q^α is defined as follows [28]:

$$Q^\alpha = \frac{h^\alpha}{\Gamma(\alpha+2)}\begin{pmatrix} 0 & \varrho_1 & \varrho_2 & \cdots & \varrho_{m^*-2} & \varrho_{m^*-1} \\ 0 & 1 & \mu_1 & \cdots & \mu_{m^*-3} & \mu_{m^*-2} \\ 0 & 0 & 1 & \cdots & \mu_{m^*-4} & \mu_{m^*-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \mu_1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{m^*\times m^*} \tag{3.9}$$

in which

$$\begin{cases} \varrho_i = i^\alpha(\alpha - i + 1) + (i-1)^{\alpha+1}, & i = 1, 2, \cdots, m^*-1, \\ \mu_i = (i+1)^{\alpha+1} - 2i^{\alpha+1} + (i-1)^{\alpha+1}, & i = 1, 2, \cdots, m^*-2. \end{cases} \tag{3.10}$$

4. Numerical Simulation of FOCP

The FOCP consists of finding the optimal control $u(t)$ and the corresponding state variable $x(t)$ which minimize the functional (1.1) and that applies the conditions (1.2)-(1.3) with $0 < \alpha \leq 1$ and $t \in [0,1]$. It is worth noting that for reducing computations, here we restrict our attention to those FOCPs that the FDE (1.2) can be solve with respect to u such that:

$$u(t) = g(t, x(t), D^\alpha x(t)). \tag{4.1}$$

So, we get through the Eq. (1.1) as follows:

$$\min J(x) = \int_{t_0}^{t_f} L\Big(t, x(t), g(t, x(t), D^\alpha x(t))\Big)dt, \tag{4.2}$$

with the boundary conditions (1.3). To solve this problem, we consider (4.2) involving a left fractional operator. To find the function $x(t)$ that solves problem LW7, $D^\alpha x(t)$, is approximated by LWs as:

$$D^\alpha x(t) \simeq C^T\Psi(t), \tag{4.3}$$

where $C^T = [c_1, c_2, \cdots, c_{m^*}]$ is an unknown vector which should be computed. Now, according to the relation between the fractional operators, Eq. (4.3) is transformed into the following equivalent equation:

$$x(t) \simeq C^T P^\alpha \Psi(t) + x_0 = (C^T P^\alpha + x_0 d^T)\Psi(t), \tag{4.4}$$

where $d = [1, 0, \cdots, 0]$. By substituting (4.3)-(4.4) into (4.1), we have:

$$g\Big(t, (C^T P^\alpha + x_0 d^T)\Psi(t), C^T\Psi(t)\Big) - u(t) \cong R(t) \simeq 0. \tag{4.5}$$

Now, we proceed by the following residual function:

$$M(C) = \int_{t_0}^{t_f} R(t)^2 dt, \tag{4.6}$$

which can be utilized for the following optimization problem:

$$\min J(C) = M(C), \tag{4.7}$$

subject to

$$C^T\Psi(0) - x_0 = 0, \tag{4.8}$$

where J is the goal function which should be minimized on C with equality constraints (4.8). According to the constrained extremum method, we assign

$$J^*[C, \lambda] = J(C) + \lambda(C^T\Psi(0) - x_0), \tag{4.9}$$

where λ is the Lagrange multiplier vector. Finally, the solution of the following system of algebraic equations yields an extremum for the optimization problem:

$$\frac{\partial J^*}{\partial C} = 0, \quad \frac{\partial J^*}{\partial \lambda} = 0. \tag{4.10}$$

Furthermore, we choose the value of m^* such that the required accuracy be provided.

4.1. **Convergence of The Method.** For the convergence behavior of the proposed method, we give a set $X = \{x_1, x_2, \cdots, x_N\}$ of pairwise distinct points in $\Omega = [t_0, t_f]$ with the fill distance $h_{X,\Omega} = \sup_{x\in\Omega} \inf_{x_i\in X} \|x - x_i\|$. Indeed, we used the collocation points $x_i \in X$, that for them $J(x_i) = 0$, $i = 1, 2, \cdots, N$. In this case, we get at a completely nonlinear equation which we will discuss in the following about its residual error. Let $\{\phi_1, \phi_2, \cdots, \phi_N\}$ be a collocation set of functions on Ω such that $\phi_i(x_j) = \delta_{i,j}$, in which $\delta_{i,j}$ recalled the cardinal functions. Also, assume that $f \in C^m(\Omega)$. Then, for every $\epsilon > 0$ and for any $a \in \Omega$, there exists $c_i(a) \in \mathbb{R}$, $i = 1, 2, \cdots, N$, such that $f(x) - \sum_{i=1}^N c_i(a)\phi_i(x) = h_{f,m}(x, a)$, in which $x \in B_\epsilon(a)$ (a neighborhood of a with the radius ϵ), $|h_{f,m}(x, a)| \le H_{f,m}(a, h_{X,\Omega})$ and $\lim_{h\to 0} H_{f,m}(a, h) = 0$ [6]. An immediate result from this discussion is that $|f(x) - \sum_{i=1}^{N} c_i\phi_i(x)| \le H_{f,m}(h_{X,\Omega})$ for all $x \in \Omega$ where Ω is a compact subset of $\mathbb{R}$. So, we can define all previous continuous functions with this approximation as we denote by LWs approximation in (3.4). We continue the discussion by estimate the error bound.

Theorem 4.1. *Assume that $X = \{x_1, x_2, \cdots, x_N\}$ be a set of pairwise distinct points in compact set Ω and $\{\phi_1, \phi_2, \cdots, \phi_N\}$ as before. If there exists $H_{f,m}(h_X, \Omega)$ such that for any $f \in C^m(\Omega)$ have*

$$|f(x) - \sum_{i=1}^{N} f(x_i)\phi_i(x)| \le H_{f,m}(h_{X,\Omega}), \tag{4.11}$$

then, there exist w_i, $i = 1, 2, \cdots, N$, such that:

$$|\int_{\Omega} f(x)dx - \sum_{i=1}^{N} w_i f(x_i)| \le H_{f,m}(h_{X,\Omega}). \tag{4.12}$$

Proof. It is directly provided from the above context (interested readers can refer to [8, 9, 10]). □

To continue, let $H_{u,m}(h_{X,\Omega}) = Ch_{X,\Omega}^{m}\|u\|_{H^m(\Omega)}$, $C > 0$. Consider a general nonlinear equation as follows:

$$Lu = f(u), \tag{4.13}$$

where L is a linear differential operator (such as D^α) from $H^m(\Omega)$ to χ that is the Banach space of functions and f is a nonlinear operator. Using the proposed methed in this paper, the approximation solution of equation (4.13) will be satisfied in the following system:

$$\sum_{i=1}^{N} c_i \int_{\Omega} L\phi_i(x)\phi_j(x)dx = \int_{\Omega} f\Big(\sum_{i=1}^{N} c_i\phi_i(x)\phi_j(x)\Big)dx. \tag{4.14}$$

By Theorem 4.1, we obtain

$$\sum_{i=1}^{N} c_i \sum_{k=1}^{N} w_k L\phi_i(x_k)\phi_j(x_k) = \sum_{k=1}^{N} w_k f\Big(\sum_{i=1}^{N} c_i\phi_i(x_k)\phi_j(x_k)\Big), \tag{4.15}$$

in whcih $x_k \in X$, $k = 1, 2, \cdots, N$. Furthermore we know $\phi_j(x_i) = \delta_{i,j}$. So, if $w_j \neq 0$, we have:

$$\sum_{i=1}^{N} c_i L\phi_i(x_j) = f(c_j), \quad j = 1, 2, \cdots, N. \tag{4.16}$$

Now, we can determine the unknown coefficients from (4.16) and then approximate $u(x_i)$, $i = 1, 2, \cdots, N$. In generally, it can be concluded that the proposed method has consistency with the following errore estimate:

$$\|Lu_N - f(u_N)\|_{\chi} \le Ch_{X,\Omega}^{m}\|u\|_{H^m(\Omega)}, \tag{4.17}$$

where u_N is the approximate solution of (4.13) with LWs.

5. Illustrative test problems

In this section, we solve three FOCPs that were investigated before in [3, 17, 21, 25]. All the computations were carried out using the Matlab software. The results are provided for various values of α and the calculations are performed using the Matlab software. For all examples, the error between the exact solution and the approximate solution, found using our method, is computed as follows:

$$Error\{x, \bar{x}\} = \int_{t_0}^{t_f} (x(t) - \bar{x}(t))^2 dt, \tag{5.1}$$

in which $x(t)$ and $\bar{x}(t)$ are the exact and approximated solutions of our problem, respectively.

Example 1. Consider the following time invariant FOCP:

$$\min J(u(.)) = \frac{1}{2} \int_0^1 \Big(x^2(t) + u^2(t) \Big) dt \tag{5.2}$$

subject to the dynamical system:

$${}_0D_t^{\alpha} x(t) = -x(t) + u(t) \tag{5.3}$$

with the initial condition $x(0) = 1$. The exact solution of this problem for $\alpha = 1$ is given in [3, 17] as:

$$x(t) = cosh(\sqrt{2}t) + \beta sinh(\sqrt{2}t) \tag{5.4}$$

and

$$u(t) = (1 + \beta\sqrt{2})cosh(\sqrt{2}t) + \beta\sqrt{2}sinh(\sqrt{2}t), \tag{5.5}$$

where:

$$\beta = -\frac{cosh(\sqrt{2}) + \sqrt{2}sinh(\sqrt{2})}{\sqrt{2}cosh(\sqrt{2}) + sinh(\sqrt{2})} \cong -0.9799$$

. Using our proposed method with $m^* = 40$ ($k = 2$ and $M = 10$), we calculate the c_i's by solving the system of equations (4.10). Figure 1 show the behavior of the numerical solutions of $x(t)$ and $u(t)$. In addition, Tables I and II show the absolute errors of $x(t)$ and $u(t)$, respectively. From these results, it can be concluded that the proposed method provides approximate solutions, accurately.

Our results that are achieved with much less computational work, are agreement with the results obtained in [3, 12, 17].

Example 2. Consider the FOCP in the form [10]:

$$\min J(x, u) = \frac{1}{2} \int_0^1 (3x^2(t) + u^2(t)) dt, \tag{5.6}$$

subject to the following dynamical system:

$${}_0D_t^{\alpha} x(t) = x(t) + u(t), \tag{5.7}$$

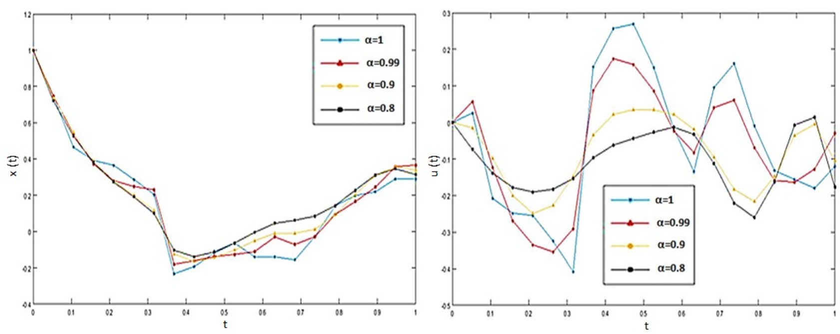

FIGURE 1. The numerical solutions of $x(t)$ and $u(t)$ for Example 1 at different values of α.

TABLE I. The error of $x(t)$ at different values of α for Example 1.

t	$\alpha = 0.5$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$
0.1	7.959×10^{-3}	7.477×10^{-3}	7.466×10^{-4}	7.450×10^{-4}
0.2	4.599×10^{-3}	4.097×10^{-3}	4.085×10^{-4}	4.069×10^{-4}
0.3	2.546×10^{-3}	2.047×10^{-3}	2.035×10^{-4}	2.019×10^{-4}
0.4	1.390×10^{-3}	9.158×10^{-4}	9.044×10^{-4}	8.893×10^{-4}
0.5	7.928×10^{-4}	3.608×10^{-4}	3.504×10^{-4}	3.366×10^{-4}
0.6	5.005×10^{-4}	1.259×10^{-4}	1.170×10^{-4}	1.050×10^{-4}
0.7	3.500×10^{-4}	4.323×10^{-5}	3.587×10^{-5}	2.605×10^{-7}
0.8	2.515×10^{-4}	1.874×10^{-5}	1.315×10^{-5}	5.704×10^{-6}
0.9	1.665×10^{-4}	1.059×10^{-5}	6.854×10^{-6}	1.860×10^{-6}
1.0	8.324×10^{-5}	5.209×10^{-6}	3.334×10^{-6}	8.348×10^{-7}

with the initial condition $x(0) = 1$. The exact solution of this problem for $\alpha = 1$ is given by:

$$x(t) = \frac{3}{3+e^4}e^{2t} + \frac{e^4}{3+e^4}e^{-2t}, \quad u(t) = \frac{3e^4}{3+e^4}e^{-2t} - \frac{3}{3+e^4}e^{2t}. \tag{5.8}$$

Figure 2 show the behavior of the numerical solutions of the proposed method with $m^* = 40$. By comparing our results with other methods, one can see that our results, which are shown in Figure 2, provides the approximate solutions obtained in [10]. The absolute error of this approximation is shown in Figure 3.

Example 3. Consider the FOCP of [13]:

$$\min J(x,u) = \int_0^1 \Big(u^2(t) - 4x(t)\Big)^2 dt, \tag{5.9}$$

Table II. The error of $u(t)$ at different values of α for Example 1.

t	$\alpha = 0.5$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$
0.1	6.7655×10^{-2}	9.9122×10^{-4}	2.4869×10^{-6}	6.1863×10^{-7}
0.2	6.5035×10^{-2}	5.7142×10^{-4}	2.8044×10^{-7}	5.3133×10^{-7}
0.3	6.934×10^{-2}	2.416×10^{-4}	2.512×10^{-7}	4.809×10^{-7}
0.4	5.502×10^{-2}	1.950×10^{-4}	2.412×10^{-7}	5.721×10^{-8}
0.5	4.927×10^{-2}	1.307×10^{-4}	2.905×10^{-8}	4.355×10^{-8}
0.6	4.214×10^{-2}	2.068×10^{-4}	2.171×10^{-7}	3.988×10^{-8}
0.7	3.380×10^{-2}	2.120×10^{-5}	2.399×10^{-7}	3.638×10^{-8}
0.8	2.662×10^{-3}	1.717×10^{-5}	1.545×10^{-7}	3.456×10^{-8}
0.9	1.715×10^{-3}	9.776×10^{-6}	3.103×10^{-7}	3.329×10^{-8}
1.0	1.700×10^{-4}	6.006×10^{-7}	5.789×10^{-8}	3.583×10^{-8}

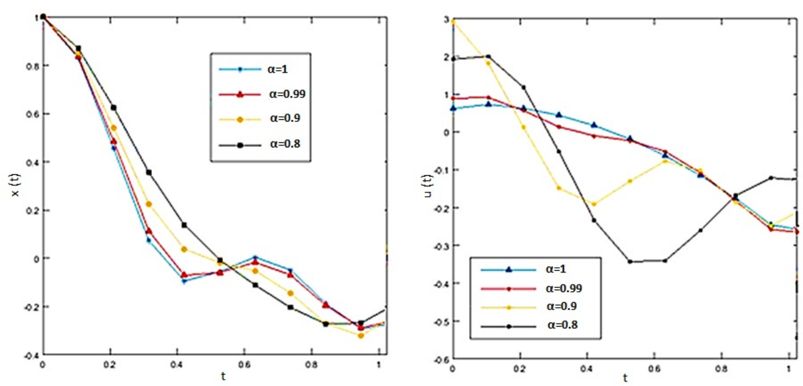

Figure 2. The numerical solutions of $x(t)$ and $u(t)$ for Example 2 with some different values of α.

subject to the control system

$$\dot{x}(t) + {}_0^C D_t^{0.5} x(t) = u(t) + \frac{2}{\Gamma(2.5)} t^{1.5}, \tag{5.10}$$

and the boundary condition

$$x(0) = 0. \tag{5.11}$$

The exact solution to (5.9)-(5.11) is given by $(x(t), u(t)) = (t^2, 2t)$.

This problem is now solved by the proposed method for $m^* = 96(k = 4$ and $M = 6)$. The absolute errors for $x(t)$ and $u(t)$ are represented in Tables III and IV, respectively. We corroborate that these accurate results have been acquired as $\alpha \to 1$ and the accuracy can be prospered by increasing the amount of α.

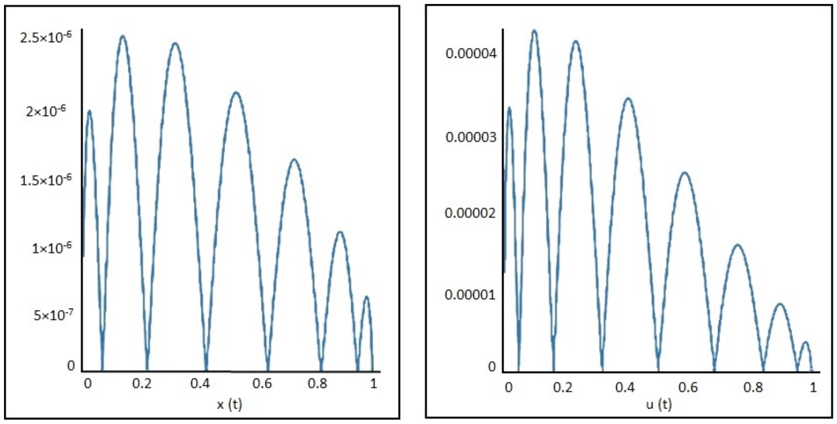

FIGURE 3. The graph of the absolute error of $x(t)$ and $u(t)$ for Example 2 with $\alpha = 1$.

TABLE III. The error of $x(t)$ at different values of α for Example 3.

t	$\alpha = 0.5$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$
0.1	7.116×10^{-4}	8.275×10^{-5}	6.810×10^{-5}	4.856×10^{-5}
0.2	7.027×10^{-4}	7.739×10^{-5}	6.240×10^{-5}	4.241×10^{-5}
0.3	7.027×10^{-4}	7.228×10^{-5}	5.712×10^{-5}	3.691×10^{-5}
0.4	7.065×10^{-4}	6.714×10^{-5}	5.204×10^{-5}	3.191×10^{-5}
0.5	7.215×10^{-4}	6.170×10^{-5}	4.695×10^{-5}	2.728×10^{-5}
0.6	5.959×10^{-4}	5.563×10^{-5}	4.161×10^{-5}	2.291×10^{-5}
0.7	5.572×10^{-4}	4.854×10^{-5}	3.573×10^{-5}	1.865×10^{-5}
0.8	5.369×10^{-4}	4.001×10^{-5}	2.902×10^{-5}	1.438×10^{-5}
0.9	3.996×10^{-4}	2.950×10^{-5}	2.112×10^{-5}	9.950×10^{-6}
1.0	1.755×10^{-4}	1.641×10^{-5}	1.161×10^{-5}	5.210×10^{-6}

6. Conclusion

The main contribution of this paper was to identify optimization functional for which optimal control problems with fractional systems can be solved numerically. Indeed, a new computational approach based on the LWs is constructed for numerically approximating the solutions of a class of FOCPs. Getting through the solution, firstly the FOCP is transformed into a functional optimization problem. Then, by using known formulas for computing fractional derivatives of Legendre wavelets (LWs), this problem has been reduce to an equivalent system of algebraic equations. In order to test our formalism, and to get a somewhat deeper understanding, we have examined three examples of FOCP. Moreover, the results are reasonably well in perfect agreement with those obtained in other literatures.

Table IV. The error of $u(t)$ at different values of α for Example 3.

t	$\alpha = 0.5$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$
0.1	4.213×10^{-3}	3.432×10^{-3}	8.858×10^{-6}	7.573×10^{-7}
0.2	2.517×10^{-3}	2.212×10^{-4}	6.186×10^{-6}	4.197×10^{-7}
0.3	1.320×10^{-3}	2.642×10^{-4}	5.772×10^{-6}	4.472×10^{-8}
0.4	3.978×10^{-3}	1.634×10^{-4}	5.714×10^{-6}	2.146×10^{-8}
0.5	1.320×10^{-3}	2.692×10^{-5}	9.717×10^{-7}	2.008×10^{-8}
0.6	3.978×10^{-3}	1.434×10^{-4}	6.006×10^{-6}	1.045×10^{-8}
0.7	1.320×10^{-3}	2.692×10^{-5}	9.612×10^{-7}	1.017×10^{-8}
0.8	3.978×10^{-3}	1.436×10^{-5}	6.093×10^{-7}	2.082×10^{-8}
0.9	1.320×10^{-3}	2.649×10^{-6}	4.385×10^{-7}	6.533×10^{-9}
1.0	3.978×10^{-4}	1.436×10^{-6}	3.929×10^{-7}	4.180×10^{-9}

References

[1] O. P. Agrawal, *A general formulation and solution scheme for fractional optimal control problems*. Nonlinear Dynamics, *38* (1-4) (2004), 323-337.

[2] O. P. Agrawal, and D. Baleanu, *A Hamiltonian formulation and a direct numerical scheme for fractional optimal control problems*. Journal of Vibration and Control, *13* (9-10) (2007), 1269-1281.

[3] O. P. Agrawal, *A quadratic numerical scheme for fractional optimal control problems*. Journal of Dynamic Systems, Measurement and Control, *130* (1) (2008).

[4] D. Baleanu, O. Defterli, and O. P. Agrawal, *A central difference numerical scheme for fractional optimal control problems*. Journal of Vibration and Control, *15* (4) (2009), 583-597.

[5] G. W. Bohannan, *Analog fractional order controller in temperature and motor control applications*. Journal of Vibration and Control, *14* (9-10) (2008), 1487-1498.

[6] J. P. Boyd, *Chebyshev and Fourier spectral methods*. Courier Corporation. (2001).

[7] M. Caputo, *Linear models of dissipation whose Q is almost frequency independent*. II, Fract. Calc. Appl. Anal. *11* (1) (2008), 4-14.

[8] M. Dehghan, J. Manafian, and A. Saadatmandi, *Solving nonlinear fractional partial differential equations using the homotopy analysis method*. Numerical Methods for Partial Differential Equations, *26* (2) (2010), 448-479.

[9] M. Dehghan, J. Manafian, and A. Saadatmandi, *The solution of the linear fractional partial differential equations using the homotopy analysis method*. Zeitschrift fr Naturforschung-A, *65* (11) (2010), 935.

[10] M. Dehghan, E. A. Hamedi, and H. Khosravian-Arab, *A numerical scheme for the solution of a class of fractional variational and optimal control problems using the modified Jacobi polynomials*. Journal of Vibration and Control, (2014).

[11] T. L. Guo, *The necessary conditions of fractional optimal control in the sense of Caputo*. Journal of Optimization Theory and Applications, *156* (1) (2013) , 115-126.

[12] M. H. Heydari, M. R. Hooshmandasl, F. M. Ghaini, and C. Cattani, *Wavelets method for solving fractional optimal control problems*. Applied Mathematics and Computation, *286* (1) (2016), 139-154.

[13] S. Jahanshahi, and D. F. Torres, *A simple accurate method for solving fractional variational and optimal control problems*. Journal of Optimization Theory and Applications, *174* (1) (2017), 156-175.

[14] R. Kamocki, *On the existence of optimal solutions to fractional optimal control problems*. Applied Mathematics and Computation, *235* (2014), 94-104.

[15] A. A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*. Elsevier Science Limited, *204* (2006).

[16] N. Liu, and E. B. Lin, *Legendre wavelet method for numerical solutions of partial differential equations*. Numerical Methods for Partial Differential Equations: An International Journal, *26* (1) (2010), 81-94.

[17] A. Lotfi, M. Dehghan, and S. A. Yousefi, *A numerical technique for solving fractional optimal control problems*. Computers & Mathematics with Applications, *62* (3) (2011), 1055-1067.

[18] R. L. Magin, *Fractional Calculus in Bioengineering*. Begell House Publisher, Inc. Connecticut, (2006).

[19] I. Podlubny, *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. Academic press, *198* (1998).

[20] I. Podlubny, *Fractional Differential Equations*. Academic Press, New York, (1999).

[21] S. Pooseh, R. Almeida, and D. F. Torres, *Fractional order optimal control problems with free terminal time*. arXiv preprint arXiv:1302.1717, (2013).

[22] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*. Gordon and Breach, Yverdon, (1993).

[23] S. Soradi-Zeid, *Efficient radial basis functions approaches for solving a class of fractional optimal control problems*. Computational and Applied Mathematics, *39* (1) (2020), 20.

[24] J. I. Suarez, B. M. Vinagre, and Y. Chen, *A fractional adaptation scheme for lateral control of an AGV*. Journal of Vibration and Control, *14* (9-10) (2008), 1499-1511.

[25] N. H. Sweilam, T. M. Al-Ajami, and R. H. Hoppe, *Numerical solution of some types of fractional optimal control problems*. The Scientific World Journal, (2013).

[26] C. Tricaud, and Y. Chen, *Time-optimal control of systems with fractional dynamics*. International Journal of Differential Equations, (2010).

[27] C. Tricaud, and Y. Chen, *An approximate method for numerically solving fractional order optimal control problems of general form*. Computers & Mathematics with Applications, *59* (5) (2010), 1644-1655.

[28] M. P. Tripathi, V. K. Baranwal, R. K. Pandey, and O. P. Singh, *A new numerical algorithm to solve fractional differential equations based on operational matrix of generalized hat functions*. Communications in Nonlinear Science and Numerical Simulation, *18* (6) (2013), 1327-1340.

[29] E. Tohidi, and H. S. Nik, *A Bessel collocation method for solving fractional optimal control problems*. Applied Mathematical Modelling, *39* (2) (2015), 455-465.

[30] S. A. Yousefi, A. Lotfi, and M. Dehghan, *The use of a Legendre multiwavelet collocation method for solving the fractional optimal control problems*. Journal of Vibration and Control, *17* (13) (2011), 2059-2065.

[31] M. Zamani, M. Karimi-Ghartemani, and N. Sadati, *FOPID controller design for robust performance using particle swarm optimization*. Fractional Calculus and Applied Analysis, *10* (2) (2007), 169-187.

[32] S. S. Zeid, A. V. Kamyad, S. Effati, S. A. Rakhshan, and S. Hosseinpour, *Numerical solutions for solving a class of fractional optimal control problems via fixed-point approach*. SeMA Journal, *74* (4) (2017), 585-603.

[33] S. S. Zeid, S. Effati, and A. V. Kamyad, *Approximation methods for solving fractional optimal control problems*. Computational and Applied Mathematics, *37* (1) (2018), 158-182.

[34] S. S. Zeid, and M. Yousefi, *Approximated solutions of linear quadratic fractional optimal control problems*. Journal of Applied Mathematics, Statistics and Informatics, *12* (2) (2016), 83-94.

[35] S. S. Zeid, *Approximation methods for solving fractional equations*. Chaos, Solitons & Fractals, *125* (2019)., 171-193.