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Hyperbolic Ricci-Bourguignon flow

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Abstract In this paper, we consider the hyperbolic Ricci-Bourguignon flow on a compact manifold M and show that this flow has a unique solution on short-time with imposing on initial conditions. After then, we find evolution equations for Riemannian curvature tensor, Ricci curvature tensor and scalar curvature of M under this flow. In the final section, we give some examples of this flow on some compact manifolds.

Keywords. Geometric flow, Hyperbolic equation, Strictly hyperbolicity.

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1. Introduction

Geometric flows are important tools in differential geometry and physic, because by these flows we can find canonical metrics on manifolds. A geometric flow is an evolution of a geometric structure under a differential equation with a functional on a manifold. Let M be an n-dimensional complete Riemannian manifold with Riemannian metric $g = (g_{ij})$. The first important geometric flow is Ricci flow which defined as follows:

$$\frac{\partial}{\partial t} g = -2Ric, \qquad g(0) = g_0, \tag{1.1}$$

where Ric denotes the Ricci curvature of g. For the first time the Ricci flow was introduced by R. Hamilton in 1982 [7] and evolves a Riemannian metric by its Ricci curvature. The short-time existence and uniqueness for solution of Ricci flow studied by R. Hamilton (see [7]) and D. DeTurck (see [6]) on compact Riemannian manifolds. Also evolution equations for geometric structures dependent to metric were investigated by some researchers (see [3]).
The second important geometric flow is the Ricci-Bourguignon flow which is defined as follows

$$\frac{\partial}{\partial t} g = -2Ric + 2\rho R g = -2(Ric - \rho R g), \qquad g(0) = g_0. \tag{1.2}$$

where R is the scalar curvature of g and ρ is a real constant. The Ricci-Bourguignon flow was introduced by Bourguignon for the first time in 1981 (see [1]). Short-time existence and uniqueness for solution to the Ricci-Bourguignon flow on $[0, T)$

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have been shown by Catino et al. in [2] for $\rho < \frac{1}{2(n-1)}$. When $\rho = 0$, the Ricci-Bourguignon flow is the Ricci flow.
The other important geometric flow was introduced by Kong and Liu [9] which is the generalized hyperbolic geometric flow and defined as follows

$$\frac{\partial^2 g}{\partial t^2} + 2Ric + \mathcal{F}(g, \frac{\partial g}{\partial t}) = 0, \tag{1.3}$$

where $\mathcal{F}$ is a smooth function of the Riemann metric g an its first derivative with respect to t. In [5, 10] Dai et al. introduced a new kind of hyperbolic geometric flow as follows

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij} + 2g^{pq}\frac{\partial g_{ip}}{\partial t}\frac{\partial g_{jq}}{\partial t} - (d + 2g^{pq}\frac{\partial g_{pq}}{\partial t})\frac{\partial g_{ij}}{\partial t} \tag{1.4}$$

$$+\frac{1}{n-1}\left[(g^{pq}\frac{\partial g_{pq}}{\partial t})^2 + \frac{\partial g^{pq}}{\partial t}\frac{\partial g_{pq}}{\partial t}\right] g_{ij}, \tag{1.5}$$

that is called dissipative hyperbolic geometric flow and they established the short-time existence and uniqueness theorem for this flow. The hyperbolic geometric flow is a system of nonlinear evolution partial differential equations of second order, it is very similar to wave equation flow metrics, and as follows

$$\frac{\partial^2}{\partial t^2} g = -2Ric, \qquad g(0) = g_0, \qquad \frac{\partial g}{\partial t}(0) = k_0, \tag{1.6}$$

where k_0 is a $(0,2)$-type symmetric tensor field on M and this flow is similar to Einstein equation

$$\frac{\partial^2}{\partial t^2} g_{ij} = -2R_{ij} - \frac{1}{2}g^{pq}\frac{\partial g_{ij}}{\partial t}\frac{\partial g_{pq}}{\partial t} + g^{pq}\frac{\partial g_{ip}}{\partial t}\frac{\partial g_{jq}}{\partial t}. \tag{1.7}$$

The existences and uniqueness of (1.6) studied in [4] on compact Riemannian manifold and in [12] funded some some evolution of geometric structures under the hyperbolic geometric flow. Also, the existences and uniqueness of (1.7) and some other properties of (1.7) studied in [11].
Motivated by the above works in this paper, we consider an n-dimensional compact smooth Riemannian manifold M and introduce the hyperbolic Ricci-Bourguignon flow on M as

$$\begin{cases} \frac{\partial^2 g}{\partial t^2} = -2Ric + 2\rho R g, \\ g(0) = g_0(x), \qquad \frac{\partial g}{\partial t}|_{t=0} = k(x), \end{cases} \tag{1.8}$$

where $k(x)$ is a symmetric tensor on M and ρ is real constant. After this in short we will display the hyperbolic Ricci-Bourguignon flow with the HRB flow. Then, we show the short-time existence and uniqueness for solution to the HRB flow for $\rho < \frac{1}{2(n-1)}$. Next, we find evolution equation for some geometric structures dependent to g along the HRB flow. Finally, we give some examples of this flow on some compact Riemannian manifolds.

2. Short-time existence and uniqueness the HRB flow

In this section, by a similar argument with the existence and uniqueness of geometric flow such as Ricci flow, Ricci-Bourguignon flow or hyperbolic geometric flow, we establish the short-time existence and uniqueness for the HRB flow on a compact n-dimensional Riemannian manifold.

Theorem 2.1. *Let (M, g_0) be a compact n-dimensional Riemannian manifold and $k(x)$ be a symmetric tensor on M. Then, there exists a constant $T > 0$ such that the initial value problem (1.8) has a unique smooth solution metric g on $M \times [0, T)$.*

Proof. Let $\hat{g}_{ij}(t)$ be a solution of hyperbolic Ricci-Bourguignon flow (1.8), and $\phi_t : M \to M$ be a family of diffeomorphisms of M. We consider the pull-back of metrics as $g_{ij}(x,t) = \phi_t^* \hat{g}_{ij}(x,t)$. Suppose that $y(x,t) = \phi_t(x) = (y^1(x,t), y^2(x,t), ..., y^n(x,t))$ is the representation of $y(x,t)$ in a local coordinates. Then

$$g_{ij}(x,t) = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\hat{g}_{\alpha\beta}(y,t), \tag{2.1}$$

and

$$\begin{aligned}\frac{\partial^2 g_{ij}}{\partial t^2}(x,t) \quad = \quad & \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{d^2\hat{g}_{\alpha\beta}}{dt^2}(y(x,t),t) + \frac{\partial}{\partial x^i}(\frac{\partial^2 y^\alpha}{\partial t^2})\frac{\partial y^\beta}{\partial x^j}\hat{g}_{\alpha\beta} \\ & + \frac{\partial}{\partial x^j}(\frac{\partial^2 y^\beta}{\partial t^2})\frac{\partial y^\alpha}{\partial x^i}\hat{g}_{\alpha\beta} + 2\frac{\partial}{\partial x^i}(\frac{\partial y^\alpha}{\partial t})\frac{\partial y^\beta}{\partial x^j}\frac{d\hat{g}_{\alpha\beta}}{dt} \\ & + 2\frac{\partial}{\partial x^j}(\frac{\partial y^\beta}{\partial t})\frac{\partial y^\alpha}{\partial x^i}\frac{d\hat{g}_{\alpha\beta}}{dt} + 2\frac{\partial}{\partial x^i}(\frac{\partial y^\alpha}{\partial t})\frac{\partial}{\partial x^j}(\frac{\partial y^\beta}{\partial t})\hat{g}_{\alpha\beta}.\end{aligned} \tag{2.2}$$

On the other hand

$$\frac{d\hat{g}_{\alpha\beta}}{dt}(y(x,t),t) = \frac{\partial \hat{g}_{\alpha\beta}}{\partial y^\gamma}\frac{\partial y^\gamma}{\partial t} + \frac{\partial \hat{g}_{\alpha\beta}}{\partial t}, \tag{2.3}$$

$$\frac{d^2\hat{g}_{\alpha\beta}}{dt^2}(y(x,t),t) = \frac{\partial^2 \hat{g}_{\alpha\beta}}{\partial y^\gamma \partial y^\lambda}\frac{\partial y^\gamma}{\partial t}\frac{\partial y^\lambda}{\partial t} + 2\frac{\partial^2 \hat{g}_{\alpha\beta}}{\partial y^\gamma \partial t}\frac{\partial y^\gamma}{\partial t} + \frac{\partial^2 \hat{g}_{\alpha\beta}}{\partial t^2} + \frac{\partial \hat{g}_{\alpha\beta}}{\partial y^\gamma}\frac{\partial^2 y^\gamma}{\partial t^2}, \tag{2.4}$$

and

$$\frac{\partial^2 \hat{g}_{\alpha\beta}}{\partial t^2}(y,t) = -2\hat{R}_{\alpha\beta}(y,t) + 2\rho\hat{R}(y,t)\hat{g}_{\alpha\beta}(y,t), \tag{2.5}$$

hence

$$\begin{aligned}
\frac{\partial^2 g_{ij}}{\partial t^2}(x,t) \quad = \quad & -2\hat{R}_{\alpha\beta}(y,t)\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} + 2\rho\hat{R}(y,t)\hat{g}_{\alpha\beta}(y,t)\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} \\
& +\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial^2 \hat{g}_{\alpha\beta}}{\partial y^\gamma \partial y^\lambda}\frac{\partial y^\gamma}{\partial t}\frac{\partial y^\lambda}{\partial t} + 2\frac{\partial^2 \hat{g}_{\alpha\beta}}{\partial y^\gamma \partial t}\frac{\partial y^\gamma}{\partial t}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} \\
& +\frac{\partial}{\partial x^i}\left(\hat{g}_{\alpha\beta}\frac{\partial y^\beta}{\partial x^j}\frac{\partial^2 y^\alpha}{\partial t^2}\right) + \frac{\partial}{\partial x^j}\left(\hat{g}_{\alpha\beta}\frac{\partial y^\beta}{\partial x^i}\frac{\partial^2 y^\alpha}{\partial t^2}\right) \\
& +\left[\frac{\partial \hat{g}_{\alpha\beta}}{\partial y^\gamma}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} - \frac{\partial}{\partial x^i}\left(\frac{\partial y^\beta}{\partial x^j}\hat{g}_{\alpha\gamma}\right) - \frac{\partial}{\partial x^j}\left(\frac{\partial y^\beta}{\partial x^i}\hat{g}_{\alpha\gamma}\right)\right]\frac{\partial^2 y^\gamma}{\partial t^2} \\
& +2\frac{\partial}{\partial x^i}(\frac{\partial y^\alpha}{\partial t})\frac{\partial y^\beta}{\partial x^j}\left(\frac{\partial \hat{g}_{\alpha\beta}}{\partial y^\gamma}\frac{\partial y^\gamma}{\partial t} + \frac{\partial \hat{g}_{\alpha\beta}}{\partial t}\right) \qquad (2.6) \\
& +2\frac{\partial}{\partial x^j}(\frac{\partial y^\beta}{\partial t})\frac{\partial y^\alpha}{\partial x^i}\left(\frac{\partial \hat{g}_{\alpha\beta}}{\partial y^\gamma}\frac{\partial y^\gamma}{\partial t} + \frac{\partial \hat{g}_{\alpha\beta}}{\partial t}\right) + 2\frac{\partial}{\partial x^i}(\frac{\partial y^\alpha}{\partial t})\frac{\partial}{\partial x^j}(\frac{\partial y^\beta}{\partial t})\hat{g}_{\alpha\beta}.
\end{aligned}$$

Now, using the normal coordinates $\{x^i\}$ around a fixed point $p \in M$ we have $\frac{\partial g_{ij}}{\partial x^k} = 0$ and

$$\frac{\partial \hat{g}_{\alpha\beta}}{\partial y^\gamma}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} - \frac{\partial}{\partial x^i}\left(\frac{\partial y^\beta}{\partial x^j}\hat{g}_{\alpha\gamma}\right) - \frac{\partial}{\partial x^j}\left(\frac{\partial y^\beta}{\partial x^i}\hat{g}_{\alpha\gamma}\right) = 0, \ \forall i,j,\gamma = 1,2,...,n. \tag{2.7}$$

Therefore

$$\begin{aligned}
\frac{\partial^2 g_{ij}}{\partial t^2}(x,t) \quad = \quad & -2R_{ij}(x,t) + 2\rho R(x,t)g_{ij}(x,t) \\
& +\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial^2 \hat{g}_{\alpha\beta}}{\partial y^\gamma \partial y^\lambda}\frac{\partial y^\gamma}{\partial t}\frac{\partial y^\lambda}{\partial t} + 2\frac{\partial^2 \hat{g}_{\alpha\beta}}{\partial y^\gamma \partial t}\frac{\partial y^\gamma}{\partial t}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} \\
& +\frac{\partial}{\partial x^i}\left(g_{mj}\frac{\partial x^m}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial t^2}\right) + \frac{\partial}{\partial x^j}\left(g_{mi}\frac{\partial x^m}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial t^2}\right) \\
& +2\frac{\partial}{\partial x^i}(\frac{\partial y^\alpha}{\partial t})\frac{\partial y^\beta}{\partial x^j}\left(\frac{\partial \hat{g}_{\alpha\beta}}{\partial y^\gamma}\frac{\partial y^\gamma}{\partial t} + \frac{\partial \hat{g}_{\alpha\beta}}{\partial t}\right) \qquad (2.8) \\
& +2\frac{\partial}{\partial x^j}(\frac{\partial y^\beta}{\partial t})\frac{\partial y^\alpha}{\partial x^i}\left(\frac{\partial \hat{g}_{\alpha\beta}}{\partial y^\gamma}\frac{\partial y^\gamma}{\partial t} + \frac{\partial \hat{g}_{\alpha\beta}}{\partial t}\right) + 2\frac{\partial}{\partial x^i}(\frac{\partial y^\alpha}{\partial t})\frac{\partial}{\partial x^j}(\frac{\partial y^\beta}{\partial t})\hat{g}_{\alpha\beta}.
\end{aligned}$$

Now, we define $y(x,t) = \phi_t(x)$ by the following initial value problem

$$\begin{cases} \frac{\partial^2 y^\alpha}{\partial t^2} = \frac{\partial y^\alpha}{\partial x^k} g^{jl}(\Gamma^k_{jl} - \bar{\Gamma}^k_{jl}), \\ y^\alpha(x,0) = x^\alpha, \qquad \frac{\partial}{\partial t} y^\alpha(x,0) = y^\alpha_1(x), \end{cases} \tag{2.9}$$

and define the vector field $V_i = g_{ik}g^{jl}(\Gamma^k_{jl} - \bar{\Gamma}^k_{jl})$, where Γ^k_{jl} and $\bar{\Gamma}^k_{jl}$ are the connection coefficients corresponding to the metrics $g(t)$ and g_0, respectively and $y^\alpha_1(x)$ for $\alpha = 1,2,...,n$, are arbitrary smooth functions on the manifold M. We get

$$\frac{\partial^2 g_{ij}}{\partial t^2}(x,t) = -2R_{ij}(x,t) + 2\rho R(x,t)g_{ij}(x,t) + \nabla_i V_j + \nabla_j V_i + F(Dy, D_t D_x y),$$

(2.10)

where

$$Dy = (\frac{\partial y^\alpha}{\partial t}, \frac{\partial y^\alpha}{\partial x^i}), \quad D_t D_x y = (\frac{\partial^2 y^\alpha}{\partial x^i \partial t}), \quad \alpha, i = 1, 2, ..., n.$$

Hence

$$\frac{\partial^2 g_{ij}}{\partial t^2}(x,t) = g^{kl}\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}(x,t) - 2\rho g_{ij} g^{pq} g^{kl}\frac{\partial^2 g_{kl}}{\partial x^p \partial x^q}(x,t) + 2\rho g_{ij} g^{pq} g^{kl}\frac{\partial^2 g_{ql}}{\partial x^p \partial x^k}(x,t) + G(g, D_x g) + F(Dy, D_t D_x y), \tag{2.11}$$

where $g = (g_{ij})$ and $D_x g = \frac{\partial g_{ij}}{\partial t}$ for $i, j, k = 1, 2, ..., n$. Let

$$L_g(h_{ij}) = g^{kl}\frac{\partial^2 h_{ij}}{\partial x^k \partial x^l} - 2\rho g_{ij} g^{pq} g^{kl}\frac{\partial^2 h_{kl}}{\partial x^p \partial x^q} + 2\rho g_{ij} g^{pq} g^{kl}\frac{\partial^2 h_{ql}}{\partial x^p \partial x^k}.$$

We now compute the symbol of the differential operator L_g. This is done by taking the highest order derivatives and replacing $\frac{\partial}{\partial x^i}$ by the Fourier transform variable ζ_i. The symbol of the linear differential operator L_g in the direction $\zeta = (\zeta_1, ..., \zeta_n)$ is

$$\sigma L_g(\zeta) h_{ij} = g^{kl}\zeta_k \zeta_l h_{ij} - 2\rho g_{ij} g^{pq} g^{kl} \zeta_p \zeta_q h_{kl} + 2\rho g_{ij} g^{pq} g^{kl} \zeta_p \zeta_k h_{ql}. \tag{2.12}$$

To see what the symbol does, since the symbol is homogeneous we can always assume ζ has length 1 and we perform all the computing in an orthonormal basis $\{e_i\}_{i=1}^n$ of T_pM such that $\zeta = g(e_1, .)$ that is $\zeta_i = 0$ for $i \neq 1$, then

$$\begin{cases} g_{ij} = \delta_{ij}, \\ \zeta = (1, 0, ..., 0). \end{cases}$$

Hence, we obtain

$$\sigma L_g(\zeta) h_{ij} = h_{ij} - 2\rho \delta_{ij} \delta_{kl} h_{kl} + 2\rho \delta_{ij} h_{11}, \tag{2.13}$$

which can be represented in the coordinate system

$$(h_{11}, h_{22}, ..., h_{nn}, h_{12}, ..., h_{1n}, h_{23}, h_{24}, ..., h_{n-1,n}),$$

for any $h \in \Gamma(S^2 M)$, by the following matrix

$$\sigma L_g = \left(\begin{array}{c|c|c} \begin{matrix} 1 & -2\rho & \cdots & -2\rho \\ \vdots & & A[n-1] & \\ 0 & & & \end{matrix} & O & O \\ \hline O & I_{(n-1)} & O \\ \hline O & O & I_{\frac{(n-1)(n-2)}{2}} \end{array} \right)$$

where O is the zero matrix and $A[n-1]$ is the $(n-1)\times(n-1)$ matrix given by

$$A[n-1]=\begin{pmatrix} 1-2\rho & -2\rho & \cdots & -2\rho \\ -2\rho & 1-2\rho & \cdots & -2\rho \\ \vdots & \vdots & \ddots & \vdots \\ -2\rho & -2\rho & \cdots & 1-2\rho \end{pmatrix}.$$

The matrix σL_g has $\frac{1}{2}n(n+1)-1$ eigenvalues equal to 1 and 1 eigenvalue equal to $1-2(n-1)\rho$ (see [2]). Therefore for $\rho<\frac{1}{2(n-1)}$ the equation (2.11) is hyperbolic. Since manifold M is compact, then by the standard theory of hyperbolic equation the system (2.11) has a unique smooth solution for a short time. From the solution of (2.11) we can obtain a solution of the HRB flow (1.8) and now, we show the uniqueness of the solution. Since

$$\Gamma^k_{jl}=\frac{\partial y^\alpha}{\partial x^j}\frac{\partial y^\beta}{\partial x^i}\frac{\partial x^k}{\partial y^\gamma}\hat{\Gamma}^\gamma_{\alpha\beta}+\frac{\partial x^k}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^j\partial x^i},$$

the initial value problem (2.9) can be rewritten as

$$\begin{cases}\frac{\partial^2 y^\alpha}{\partial t^2}=g^{jl}\left(\frac{\partial^2 y^\alpha}{\partial x^j\partial x^l}-\bar{\Gamma}^k_{jl}\frac{\partial y^\alpha}{\partial x^k}+\hat{\Gamma}^\alpha_{\beta\gamma}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^i}\right),\\ y^\alpha(x,0)=x^\alpha,\quad \frac{\partial}{\partial t}y^\alpha(x,0)=y_1^\alpha(x).\end{cases}\tag{2.14}$$

Since the equation (2.14) is a strictly hyperbolic system and manifold M is compact, it follows from the standard theory of hyperbolic equation (see [8]) that the system (2.14) has a unique smooth solution for a short time. For any two solution $\hat{g}^{(1)}_{ij}(x,t)$ and $\hat{g}^{(2)}_{ij}(x,t)$ of the HRB flow (1.8) with the same initial data, we solve the initial value problem (2.14) and find two families $\phi^{(1)}_t$ and $\phi^{(2)}_t$ of diffeomorphisms of M. Therefore we get two solutions, $g^{(1)}_{ij}(x,t)=(\phi^{(1)}_t)^*\hat{g}^{(1)}_{ij}(x,t)$ and $g^{(2)}_{ij}(x,t)=(\phi^{(2)}_t)^*\hat{g}^{(2)}_{ij}(x,t)$, to the modified evolution equation (2.11) with same initial data. The uniqueness result for the strictly hyperbolic equation (2.11) implies that $g^{(1)}_{ij}(x,t)=g^{(2)}_{ij}(x,t)$ and then by system (2.14) and the standard uniqueness result of PDE system, the corresponding solutions $\phi^{(1)}_t$ and $\phi^{(2)}_t$ of (2.14) must agree. Consequently the metrics $\hat{g}^{(1)}_{ij}(x,t)$ and $\hat{g}^{(2)}_{ij}(x,t)$ must agree also. Hence we have proved the uniqueness for solution of the HRB flow (1.8). □

3. Evolution equations of curvature tensor along the HRB flow

The HRB flow is an evolution equation on the metric. In the following, we use the techniques and ideas to find the evolution equations for geometric structures under the Ricic flow, the Ricci-Bourguignon flow (see [2, 3]) and the evolution equation along the hyperbolic geometric flow by W. R. Dai et al (see [4]) to give the evolution formula for Riemannian curvature tensor, Ricci curvature tensor and scalar curvature of (M,g) under the HRB flow.

Theorem 3.1. *Under the HRB flow the Riemannian curvature tensor R_{ijkl} of (M, g) satisfies the following evolution equation*

$$\begin{aligned}\frac{\partial^2}{\partial t^2}R_{ijkl} &= \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) \\ &-g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} + R_{ijpl}R_{qk} + R_{ijkp}R_{ql}) \\ &+2g_{pq}\Big(\frac{\partial}{\partial t}\Gamma^p_{il}.\frac{\partial}{\partial t}\Gamma^q_{jk} - \frac{\partial}{\partial t}\Gamma^p_{jl}.\frac{\partial}{\partial t}\Gamma^q_{ik}\Big) \qquad (3.1) \\ &-\rho\left[\nabla_i\nabla_k Rg_{jl} - \nabla_i\nabla_l Rg_{jk} - \nabla_j\nabla_k Rg_{il} + \nabla_j\nabla_l Rg_{ik}\right] + 2\rho RR_{ijkl},\end{aligned}$$

where $B_{ijkl} = g^{pr}g^{qs}R_{piqj}R_{rksl}$ and Δ is the Beltrami-Laplace operator with respect to the evolving metric g.

Proof. For an evolution metric g along the HRB flow the Christoffel symbol of metric g is $\Gamma^h_{jl} = \frac{1}{2}g^{hm}(\frac{\partial g_{mj}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^j} - \frac{\partial g_{jl}}{\partial x^m})$, therefore by direct computing, we obtain the second variation of Γ^h_{jl} as follows

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\Gamma^h_{jl} &= \frac{1}{2}\frac{\partial^2 g^{hm}}{\partial t^2}\Big(\frac{\partial g_{mj}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^j} - \frac{\partial g_{jl}}{\partial x^m}\Big) + \frac{\partial g^{hm}}{\partial t}\Big(\frac{\partial^2 g_{mj}}{\partial x^l\partial t} + \frac{\partial^2 g_{ml}}{\partial x^j\partial t} - \frac{\partial^2 g_{jl}}{\partial x^m\partial t}\Big) \\ &+\frac{1}{2}g^{hm}\left[\frac{\partial}{\partial x^l}(\frac{\partial^2 g_{mj}}{\partial t^2}) + \frac{\partial}{\partial x^j}(\frac{\partial^2 g_{ml}}{\partial t^2}) - \frac{\partial}{\partial x^m}(\frac{\partial^2 g_{jl}}{\partial t^2})\right]. \qquad (3.2)\end{aligned}$$

On the other hand, the Riemannian curvature tensor of (M, g) is

$$R^h_{ijl} = \frac{\partial\Gamma^h_{jl}}{\partial x^i} - \frac{\partial\Gamma^h_{il}}{\partial x^j} + \Gamma^h_{ip}\Gamma^p_{jl} - \Gamma^h_{jp}\Gamma^p_{il}, \quad R_{ijkl} = g_{hk}R^h_{ijl}, \qquad (3.3)$$

hence with a double differentiation respect to t we have

$$\frac{\partial^2}{\partial t^2}R^h_{ijl} = \frac{\partial}{\partial x^i}(\frac{\partial^2}{\partial t^2}\Gamma^h_{jl}) - \frac{\partial}{\partial x^j}(\frac{\partial^2}{\partial t^2}\Gamma^h_{il}) + \frac{\partial^2}{\partial t^2}(\Gamma^h_{ip}\Gamma^p_{jl} - \Gamma^h_{jp}\Gamma^p_{il}), \qquad (3.4)$$

and

$$\begin{aligned}\frac{\partial^2}{\partial t^2}R_{ijkl} &= g_{hk}\left[\frac{\partial}{\partial x^i}(\frac{\partial^2\Gamma^h_{jl}}{\partial t^2}) - \frac{\partial}{\partial x^j}(\frac{\partial^2\Gamma^h_{il}}{\partial t^2}) + \frac{\partial^2}{\partial t^2}(\Gamma^h_{ip}\Gamma^p_{jl} - \Gamma^h_{jp}\Gamma^p_{il})\right] \\ &+2\frac{\partial g_{hk}}{\partial t}\left[\frac{\partial}{\partial x^i}(\frac{\partial\Gamma^h_{jl}}{\partial t}) - \frac{\partial}{\partial x^j}(\frac{\partial\Gamma^h_{il}}{\partial t}) + \frac{\partial}{\partial t}(\Gamma^h_{ip}\Gamma^p_{jl} - \Gamma^h_{jp}\Gamma^p_{il})\right] \\ &+R^h_{ijl}\frac{\partial^2 g_{hk}}{\partial t^2}. \qquad (3.5)\end{aligned}$$

We choose the normal coordinates $\{x^1, x^2, ..., x^n\}$ around a fixed point p on M, then $\frac{\partial g_{ij}}{\partial x^k}(p) = 0$ and $\Gamma^k_{ij}(p) = 0$. Then at this point we have

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}R_{ijkl} &= \frac{1}{2}\left[\frac{\partial^2}{\partial x^i\partial x^l}(\frac{\partial^2 g_{kj}}{\partial t^2}) - \frac{\partial^2}{\partial x^i\partial x^k}(\frac{\partial^2 g_{jl}}{\partial t^2})\right] \\
&\quad -\frac{1}{2}\left[\frac{\partial^2}{\partial x^j\partial x^l}(\frac{\partial^2 g_{kj}}{\partial t^2}) - \frac{\partial^2}{\partial x^j\partial x^k}(\frac{\partial^2 g_{il}}{\partial t^2})\right] \\
&\quad -g^{pm}\frac{\partial^2 g_{kp}}{\partial x^i\partial t}\Big(\frac{\partial^2 g_{mj}}{\partial x^l\partial t} + \frac{\partial^2 g_{ml}}{\partial x^j\partial t} - \frac{\partial^2 g_{jl}}{\partial x^m\partial t}\Big) \\
&\quad +g^{pm}\frac{\partial^2 g_{kp}}{\partial x^j\partial t}\Big(\frac{\partial^2 g_{mi}}{\partial x^l\partial t} + \frac{\partial^2 g_{ml}}{\partial x^i\partial t} - \frac{\partial^2 g_{il}}{\partial x^m\partial t}\Big) \\
&\quad +2g_{hk}\Big(\frac{\partial}{\partial t}\Gamma^h_{ip}.\frac{\partial}{\partial t}\Gamma^p_{jl} - \frac{\partial}{\partial t}\Gamma^h_{jp}.\frac{\partial}{\partial t}\Gamma^p_{il}\Big).
\end{aligned} \tag{3.6}$$

Since $\frac{\partial^2}{\partial t^2}g = -2Ric + 2\rho Rg$, so we can rewrite (3.5) as follows:

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}R_{ijkl} &= \frac{1}{2}\left[\frac{\partial^2}{\partial x^i\partial x^l}(-2R_{kj} + 2\rho Rg_{kj}) - \frac{\partial^2}{\partial x^i\partial x^k}(-2R_{jl} + 2\rho Rg_{jl})\right] \\
&\quad -\frac{1}{2}\left[\frac{\partial^2}{\partial x^j\partial x^l}(-2R_{ki} + 2\rho Rg_{ki}) - \frac{\partial^2}{\partial x^j\partial x^k}(-2R_{il} + 2\rho Rg_{il})\right] \\
&\quad -g^{pm}\frac{\partial^2 g_{kp}}{\partial x^i\partial t}\Big(\frac{\partial^2 g_{mj}}{\partial x^l\partial t} + \frac{\partial^2 g_{ml}}{\partial x^j\partial t} - \frac{\partial^2 g_{jl}}{\partial x^m\partial t}\Big) \\
&\quad +g^{pm}\frac{\partial^2 g_{kp}}{\partial x^j\partial t}\Big(\frac{\partial^2 g_{mi}}{\partial x^l\partial t} + \frac{\partial^2 g_{ml}}{\partial x^i\partial t} - \frac{\partial^2 g_{il}}{\partial x^m\partial t}\Big) \\
&\quad +2g_{hk}\Big(\frac{\partial}{\partial t}\Gamma^h_{ip}.\frac{\partial}{\partial t}\Gamma^p_{jl} - \frac{\partial}{\partial t}\Gamma^h_{jp}.\frac{\partial}{\partial t}\Gamma^p_{il}\Big).
\end{aligned} \tag{3.7}$$

On the other hand, we have

$$\frac{\partial^2}{\partial x^i\partial x^l}R_{jk} = \nabla_i\nabla_l R_{jk} + R_{jp}\nabla_i\Gamma^p_{lk} + R_{kp}\nabla_i\Gamma^p_{lj}, \tag{3.8}$$

and

$$\begin{aligned}
&-g^{pm}\frac{\partial^2 g_{kp}}{\partial x^i\partial t}\Big(\frac{\partial^2 g_{mj}}{\partial x^l\partial t} + \frac{\partial^2 g_{ml}}{\partial x^j\partial t} - \frac{\partial^2 g_{jl}}{\partial x^m\partial t}\Big) \\
&+g^{pm}\frac{\partial^2 g_{kp}}{\partial x^j\partial t}\Big(\frac{\partial^2 g_{mi}}{\partial x^l\partial t} + \frac{\partial^2 g_{ml}}{\partial x^i\partial t} - \frac{\partial^2 g_{il}}{\partial x^m\partial t}\Big) \\
&+2g_{hk}\Big(\frac{\partial}{\partial t}\Gamma^h_{ip}.\frac{\partial}{\partial t}\Gamma^p_{jl} - \frac{\partial}{\partial t}\Gamma^h_{jp}.\frac{\partial}{\partial t}\Gamma^p_{il}\Big) \\
= \;& 2g_{pq}\Big(\frac{\partial}{\partial t}\Gamma^p_{il}.\frac{\partial}{\partial t}\Gamma^q_{jk} - \frac{\partial}{\partial t}\Gamma^p_{jl}.\frac{\partial}{\partial t}\Gamma^q_{ik}\Big).
\end{aligned} \tag{3.9}$$

Therefore, plug in (3.8) and (3.9) in (3.7) lead to

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}R_{ijkl} &= -\nabla_i\nabla_l R_{jk} + \nabla_i\nabla_k R_{jl} + \nabla_j\nabla_l R_{ki} - \nabla_j\nabla_k R_{il} \\
&\quad -g^{pq}(R_{ijql}R_{kp} + R_{ijkq}R_{kp}) + 2g_{pq}\Big(\frac{\partial}{\partial t}\Gamma^p_{il}.\frac{\partial}{\partial t}\Gamma^q_{jk} - \frac{\partial}{\partial t}\Gamma^p_{jl}.\frac{\partial}{\partial t}\Gamma^q_{ik}\Big) \\
&\quad +\rho\left[\frac{\partial^2 R}{\partial x^i\partial x^l}g_{kj} - \frac{\partial^2 R}{\partial x^i\partial x^k}g_{jl} - \frac{\partial^2 R}{\partial x^j\partial x^l}g_{ki} + \frac{\partial^2 R}{\partial x^j\partial x^k}g_{il}\right] \\
&\quad +\rho\left[\frac{\partial^2 g_{kj}}{\partial x^i\partial x^l}R - \frac{\partial^2 g_{jl}}{\partial x^i\partial x^k}R - \frac{\partial^2 g_{ki}}{\partial x^j\partial x^l}R + \frac{\partial^2 g_{il}}{\partial x^j\partial x^k}R\right] \\
&= \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) \\
&\quad -g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} + R_{ijpl}R_{qk} + R_{ijkp}R_{ql}) \\
&\quad +2g_{pq}\Big(\frac{\partial}{\partial t}\Gamma^p_{il}.\frac{\partial}{\partial t}\Gamma^q_{jk} - \frac{\partial}{\partial t}\Gamma^p_{jl}.\frac{\partial}{\partial t}\Gamma^q_{ik}\Big) \qquad (3.10) \\
&\quad -\rho\left[\nabla_i\nabla_k R g_{jl} - \nabla_i\nabla_l R g_{jk} - \nabla_j\nabla_k R g_{il} + \nabla_j\nabla_l R g_{ik}\right] + 2\rho R R_{ijkl},
\end{aligned}$$

where $B_{ijkl} = g^{pr}g^{qs}R_{piqj}R_{rksl}$, so the proof is complete. □

Theorem 3.2. *Evolution equation for Ricci curvature tensor under the HRB flow is as follow:*

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}R_{ij} &= \Delta R_{ij} - (n-2)\rho\nabla_i\nabla_j R - \rho\Delta R g_{ij} + 2g^{pr}g^{qs}R_{piqj}R_{rs} - 2g^{pq}R_{pi}R_{qj} \\
&\quad +2g^{kl}g_{pq}\Big(\frac{\partial}{\partial t}\Gamma^p_{il}.\frac{\partial}{\partial t}\Gamma^q_{kj} - \frac{\partial}{\partial t}\Gamma^p_{kl}.\frac{\partial}{\partial t}\Gamma^q_{ij}\Big) - 2g^{kp}g^{lq}\frac{\partial g_{pq}}{\partial t}\frac{\partial R_{ikjl}}{\partial t} \qquad (3.11) \\
&\quad +2g^{kp}g^{rq}g^{sl}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}R_{ikjl}.
\end{aligned}$$

Proof. Choose $\{x^1, ..., x^n\}$ to be a normal coordinate system at a fixed point. At this point, we compute

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}R_{ij} &= \frac{\partial^2}{\partial t^2}(g^{kl}R_{ikjl}) \\
&= g^{kl}\frac{\partial^2}{\partial t^2}R_{ikjl} + 2\frac{\partial g^{kl}}{\partial t}\frac{\partial R_{ikjl}}{\partial t} + R_{ikjl}\frac{\partial^2 g^{kl}}{\partial t^2}. \qquad (3.12)
\end{aligned}$$

Since $\frac{\partial g^{kl}}{\partial t} = -g^{kp}g^{lq}\frac{\partial g_{pq}}{\partial t}$ and $\frac{\partial^2 g^{kl}}{\partial t^2} = -g^{kp}g^{lq}\frac{\partial^2 g_{pq}}{\partial t^2} + 2g^{kp}g^{rq}g^{sl}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}$, we get

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}R_{ij} &= g^{kl}\frac{\partial^2}{\partial t^2}R_{ikjl} - 2g^{kp}g^{lq}\frac{\partial g_{pq}}{\partial t}\frac{\partial R_{ikjl}}{\partial t} - g^{kp}g^{lq}\frac{\partial^2 g_{pq}}{\partial t^2}R_{ikjl} \\
&\quad +2g^{kp}g^{rq}g^{sl}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}R_{ikjl}, \qquad (3.13)
\end{aligned}$$

by replacing (3.1) and $\frac{\partial^2}{\partial t^2} g_{ij} = -2R_{ij} + 2\rho R g_{ij}$ in (3.13) we have

$$\begin{aligned} \frac{\partial^2}{\partial t^2} R_{ij} &= \Delta R_{ij} + 2g^{kl}(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) \\ &\quad - g^{kl} g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} + R_{ijpl}R_{qk} + R_{ijkp}R_{ql}) \\ &\quad + 2g^{kl} g_{pq}\Big(\frac{\partial}{\partial t}\Gamma^p_{il} \cdot \frac{\partial}{\partial t}\Gamma^q_{jk} - \frac{\partial}{\partial t}\Gamma^p_{jl} \cdot \frac{\partial}{\partial t}\Gamma^q_{ik}\Big) \\ &\quad - (n-2)\rho \nabla_i \nabla_j R - \rho \Delta R g_{ij} - 2g^{kp} g^{lq} \frac{\partial g_{pq}}{\partial t} \frac{\partial R_{ikjl}}{\partial t} \\ &\quad - g^{kp} g^{lq}(-2R_{pq} + 2\rho R g_{pq}) R_{ikjl} + 2g^{kp} g^{rq} g^{sl} \frac{\partial g_{pq}}{\partial t} \frac{\partial g_{rs}}{\partial t} R_{ikjl}, \end{aligned} \tag{3.14}$$

where

$$2g^{kl}(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) = 2g^{kl}(B_{ikjl} - 2B_{iklj}) + 2g^{pr} g^{qs} R_{piqj} R_{rs},$$

and

$$g^{kl} g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} + R_{ijpl}R_{qk} + R_{ijkp}R_{ql}) = 2g^{pq} R_{pi} R_{qj} + 2g^{pr} g^{qs} R_{piqj} R_{rs},$$

but $g^{kl}(B_{ikjl} - 2B_{iklj}) = 0$. Hence by replaceing last equations in (3.14) the proof of the theorem is complete. □

From $R = g^{ij} R_{ij}$ and using (3.11) we have the following result:

Corollary 3.3. *Under the HRB flow, the evolution equation of the scalar curvature satisfies*

$$\begin{aligned} \frac{\partial^2}{\partial t^2} R &= (1 - 2(n-1)\rho)\Delta R + 2|Ric|^2 - 2\rho R^2 \\ &\quad + 2g^{ij} g^{kl} g_{pq}\Big(\frac{\partial}{\partial t}\Gamma^p_{il} \cdot \frac{\partial}{\partial t}\Gamma^q_{kj} - \frac{\partial}{\partial t}\Gamma^p_{kl} \cdot \frac{\partial}{\partial t}\Gamma^q_{ij}\Big) \\ &\quad - 2g^{ij} g^{kp} g^{lq} \frac{\partial g_{pq}}{\partial t} \frac{\partial R_{ikjl}}{\partial t} + 4g^{kp} g^{rq} g^{sl} \frac{\partial g_{pq}}{\partial t} \frac{\partial g_{rs}}{\partial t} R_{kl} \\ &\quad - 2g^{ip} g^{jq} \frac{\partial g_{pq}}{\partial t} \frac{\partial R_{ij}}{\partial t}. \end{aligned}$$

4. Examples

In this section, we give some examples of HRB flow.

Example 4.1. Let $(M^4, g(t)) = (S^2 \times L, c(t) g_{S^2} \oplus d(t) g_L)$ where (S^2, g_{S^2}) is a round sphere with Gauss curvature 1 and (L, G_L) is a surface with constant Gauss curvature -1. The HRB flow results that

$$\begin{cases} \frac{\partial^2}{\partial t^2} c(t) = -2 + 4\rho(1 - c^2), & c(0) = 1, \ c'(0) = 0, \\ \frac{\partial^2}{\partial t^2} d(t) = 2 + 4\rho(d^2 - 1), & d(0) = 1, \ d'(0) = 0. \end{cases} \tag{4.1}$$

If $0 < \rho < \frac{1}{4}$, then $\frac{\partial^2}{\partial t^2} c(t) < 0$ implies that $c(t)$ is decreasing and $\frac{\partial^2}{\partial t^2} d(t) > 0$ results that $d(t)$ is increasing.

Example 4.2. Let $(M, g(0))$ be an arbitrary compact Riemannian manifold. If the initial metric $g_{ij}(0,x)$ is Ricci flat, i.e. $R_{ij}(0,x) = 0$, then $g_{ij}(t,x) = g_{ij}(0,x)$ is obviously a solution to the evolution equation HRB flow, therefore any Ricci flat metric is a stationary solution of the HRB flow.

Example 4.3. Let $(M, g(0))$ be a closed Riemannian manifold and the initial metric $g(0)$ is Einstein that is for some constant λ it holds

$$R_{ij}(0) = \lambda g_{ij}(0). \tag{4.2}$$

Since, the initial metric is Einstein for some constant λ, let $g_{ij}(t,x) = c(t)g_{ij}(0)$. By the definition of the Ricci tensor, we have

$$R_{ij}(t) = R_{ij}(0) = \lambda g_{ij}(0), \quad R_{g(t)} = \frac{n\lambda}{c}. \tag{4.3}$$

Therefore the equation (1.8) becomes

$$\frac{\partial^2 c(t)g_{ij}(0))}{\partial t^2} = -2\lambda g_{ij}(0) + 2\rho n\lambda g_{ij}(0), \tag{4.4}$$

this implies

$$\frac{d^2 c(t)}{\partial t^2} = -2\lambda + 2\rho n\lambda, \quad c(0) = 1, \ \ c'(0) = \nu, \tag{4.5}$$

which solution of it is given by

$$\rho(t) = (-2\lambda + 2\rho n\lambda)t^2 + \nu t + 1. \tag{4.6}$$

Hence, the solution of the HRB flow remains Einstein.

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