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Generalized symmetries and conservation laws of (3+1)-dimensional variable coefficient Zakharov-Kuznetsov equation

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Abstract The nonlinear variable coefficient Zakharov-Kuznetsov (Vc–ZK) equation is derived using reductive perturbation technique for ion-acoustic solitary waves in magnetized three-component dusty plasma having negatively charged dust particles, isothermal ions, and electrons. The equation is investigated for generalized symmetries using a recently proposed compatibility method. Some more general symmetries are obtained and group invariant solutions are also constructed for these symmetries. Besides this, the equation is also investigated for nontrivial local conservation laws.

Keywords. Generalized symmetries, Compatibility method, Vc-ZK equation, Conservation laws.
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1. Introduction

In plasma physics, research in nonlinear wave propagation through dusty plasmas is rapidly growing. The standard equations in physics like KdV, modified KdV and Schrödinger wave equation best suited to describe wave phenomenon in plasma physics. The KdV equation was first used by Washmi and Taniuti [25] to describe ion-acoustic wave propagation through dusty plasmas and with aid of reductive perturbation technique [12, 24] several nonlinear wave equations including Zakharov–Kuznetsov equation describing ion-acoustic wave propagation through dusty plasmas have been recently derived and investigated analytically by various mathematical tools [7, 8, 16, 20, 22, 28]. In present work, we have derived (3+1)–dimensional Zakharov-Kuznetsov equation and is further constrained by putting variable coefficients to incorporate more realistic means. Although Zakharov–Kuznetsov equation and its various generalisations have been investigated for Lie symmetries [17, 26, 30], soliton solutions using Hirota's method [20, 21], exact solutions [6, 14, 29, 31], but to the best of our knowledge the Vc-ZK equation in (3+1)-dimension has never been investigated for generalized symmetries. So in this article, we tried to find new symmetries for the Vc-ZK equation using the compatibility method given by Yan and Liu [26].

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Governing equations

Consider Poisson's equations governing two-component plasma consisting of cold ions and non-extensive electrons in an external static magnetic field directed along z-axes

$$n_t + \nabla \cdot (nV) = 0, \tag{1.1a}$$

$$V_t + (V \cdot \nabla)V = -\nabla\phi + \frac{\Omega_i}{\omega_{pi}}(V \times z), \tag{1.1b}$$

$$\nabla^2\phi = n_e - n, \tag{1.1c}$$

with normalized electron density $n_e = [1 + (q-1)\,\phi]^{(q+1)/2(q-1)}$, where n and n_e are normalized unperturbed number densities of ions and electrons respectively. $V(u,v,w)$ and ϕ are ion velocity $(x,y,z$ directions) and electrostatic potential which are normalized by ion acoustic speed $C_s = \sqrt{T_e/m_i}$ and T_e/e respectively. Ω_i is the ion-cyclotron frequency. The space and time variables are in units of $\lambda_D = \sqrt{T_e/4\pi e^2 n_{e0}}$, the electrons Debye length and reciprocal of ion-plasma frequency $\omega_{pi} = \sqrt{4\pi e^2 n_{i0}/m_i}$, and q being strength of nonextensivity such that for $q < -1$, the nonextensive electron distribution can not be normalized.
To obtain (3+1)-dimensional Zakharov-Kuznetsov equation using standard reductive perturbation theory [12, 24], we chose new independent variables

$$\xi = \sqrt{\epsilon}\,x,\ \ \eta = \sqrt{\epsilon}\,y,\ \ \zeta = \sqrt{\epsilon}\,(z - \lambda_0 t),\ \ \tau = \epsilon\sqrt{\epsilon}\,t, \tag{1.2}$$

where ϵ being small and dimensionless perturbation parameter, which also characterizes the strength non-linearity of the system and λ is the phase velocity of the wave along x-axes. Let us consider generalized expressions for state variables as follow:

$$X = \sum_{m=0}^{\infty} \epsilon^m X_m,\ \ Y = \sum_{m=1}^{\infty} \epsilon^{1+\frac{m}{2}} Y_m, \tag{1.3}$$

where $X(= n, w, \phi)$ and $Y(= u, v)$ describe state of system and equilibrium state is corresponding to $m = 0$, such that $\phi_0 = 0, w_0 = 0$ and $n_0 = 1$. Substituting (1.3) along with stretching variables (1.2) into Poisson's equations (1.1) and from lowest powers of ϵ, we obtain

$$n_1 = \frac{w_1}{\lambda_0},\ \ u_1 = -\frac{\omega_{pi}}{\Omega_i}\phi_\eta,\ \ v_1 = -\frac{\omega_{pi}}{\Omega_i}\phi_\xi,\ \ w_1 = \frac{\phi_1}{\lambda_0},\ \ \phi_1 = \frac{n_1}{c_1}. \tag{1.4}$$

Subsequently, from next higher orders of ϵ we obtain following

$$u_2 = \frac{\omega_{pi}\lambda_0}{\Omega_i}v_{1,\zeta},\ \ v_2 = -\frac{\omega_{pi}\lambda_0}{\Omega_i}u_{1,\zeta}. \tag{1.5}$$

Finally, by reductive perturbation method, the following (3+1)-dimensional Zakharov-Kuznetsov equation is obtained

$$\phi_{1,\tau} + P\phi_1\phi_{1,\zeta} + Q\phi_{1,\zeta\zeta\zeta} + R\,(\phi_{1,\zeta\eta\eta} + \phi_{1,\zeta\xi\xi}) = 0, \tag{1.6}$$

where the coefficients are given by the following formulae

$$P = \frac{\lambda_0}{2c_1}\left[\frac{3c_1}{\lambda_0^2} - 2c_2\right], \quad Q = \frac{\lambda_0}{2c_1}, \quad R = \frac{\lambda_0}{2c_1}\left[1 + \frac{c_1\lambda_0^2\omega_{pi}^2}{\Omega_i^2}\right],$$

$$\text{for } c_1 = \frac{q+1}{2}, \quad c_2 = \frac{(q+1)(3-q)}{8}.$$

In following, the various versions of equation (1.6) have been listed.

(1) The constant coefficient version:

$$u_t + auu_x + bu_{xxx} + u_{xyy} + u_{xzz} = 0. \tag{1.7}$$

is investigated by Lie group analysis [30] wherein multi-parameter optimal system along with several exact solutions are also reported. The fractional variant of equation (1.7) has also been studied using the first integral method and the functional variable method to establish exact traveling wave solutions[9]. The equation (1.7) has physical importance in plasma laboratories as it helps in describing the evolution of solitary waves in plasma and other properties of solitary waves and double layers [5, 7].

(2) The Zakharov-Kuznetsov equation with power law non-linearity:

$$u_t + au^n u_x + b(u_{xx} + u_{yy})_x = 0, \tag{1.8}$$

and dual power law non-linearity

$$u_t + (au^n + bu^{2n})u_x + c(u_{xx} + u_{yy})_x = 0, \tag{1.9}$$

both equations (1.8) and (1.9) describe the passage of the ion-acoustic wave through cold plasma where electrons behave in a non-isothermic way [1, 2]. The fractional variant of equation (1.9) has been studied for soliton solutions using Riccati sub equation method and new exact solutions involving parameters, expressed by generalized hyperbolic functions are also obtained [13]. The non-linearity is included in order to incorporate the change in electron number density in reductive perturbation method. Besides this, the (3+1)-dimensional Zakharov-Kuznetsov equation with power law non-linearity:

$$u_t + au^n u_x + b(u_{xx} + u_{yy} + u_{zz})_x = 0, \tag{1.10}$$

is investigated via Lie group analysis [15] and its fractional variant has been exploited in [19].

(3) The generalized variable coefficient Zakharov-Kuznetsov equation:

$$u_t + a(t)uu_x + b(t)u^2u_x + u_{xxx} + c(t)u_{xyy} = 0, \tag{1.11}$$

and slightly more general form

$$u_t + \alpha(t)uu_x + \beta(t)u^2u_x + \rho(t)u_{xx} + \lambda(t)u_{xxx} + \gamma(t)u_{xyy} = 0. \tag{1.12}$$

The equations (1.11) and (1.12) have been investigated via similarity analysis [17, 26] and several exact solutions have also been obtained using Riccati equation mapping method [14].

In present work, we would like to analyze the variable coefficient version of (1.6) derived using reductive perturbation technique, it reads as follow:

$$u_t + a(t)uu_x + u_{xxx} + b(t)u_{xyy} + c(t)u_{xzz} = 0. \tag{1.13}$$

This is (3+1)-dimensional variable coefficient Zakharov-Kuznetsov (Vc-ZK) equation. Although, the Lie symmetry analysis of equations similar to (1.13) in lower dimensions can be seen in the literature, but to best of our knowledge the equation is not explored for generalized symmetries.

2. Classical Lie point symmetries of Vc-ZK equation

The Lie point symmetries for (1.13) can be obtained by a standard procedure given in [3, 18] and for recent applications the Refs. [10, 11, 23] can be seen. Consider one-parameter local Lie group of point transformations in the following manner:

$$\begin{aligned} &x^* = x + \epsilon X(x,y,z,t,u) + O(\epsilon^2), \quad y^* = y + \epsilon Y(x,y,z,t,u) + O(\epsilon^2),\\ &z^* = t + \epsilon Z(x,y,z,t,u) + O(\epsilon^2), \quad t^* = t + \epsilon T(x,y,z,t,u) + O(\epsilon^2),\\ &u^* = u + \epsilon \Phi(x,y,z,t,u) + O(\epsilon^2), \end{aligned} \tag{2.1}$$

where ϵ is a group parameter. The invariance of (1.13) under infinitesimal transformations (2.1) provides an overdetermined system of linear partial differential equations in X, Y, Z, T and Φ. Such an overdetermined system can be derived by considering the following associated vector field:

$$V = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z} + T\frac{\partial}{\partial t} + \Phi\frac{\partial}{\partial u}. \tag{2.2}$$

This forms Lie algebra on the space of independent and dependent variables and group transformations can be recovered from (2.2) by Lie's first theorem. Since V acts on the solution space of (1.13), so this can be prolonged to act on space of all partial derivatives in (1.13). However, the prolonged formula for (2.2) is rather complicated yet algorithmic and is successfully implemented in several symbolic manipulation programs including Maple and Mathematica. For symmetry determination, the equation (1.13) have to be written in the following symbolic form

$$\Delta \equiv u_t + a(t)uu_x + u_{xxx} + b(t)u_{xyy} + c(t)u_{xzz} = 0. \tag{2.3}$$

The determining system of PDEs for X, Y, Z, T and Φ is given by the invariance criterion

$$V^{(3)}(\Delta)\big|_{\Delta=0} = 0, \tag{2.4}$$

where the third order prolongation $V^{(3)}$ of vector field (2.2) is given by

$$V^{(3)} = V + \Phi^{[x]}\frac{\partial}{\partial u_x} + \Phi^{[xxx]}\frac{\partial}{\partial u_{xxx}} + \Phi^{[xyy]}\frac{\partial}{\partial u_{xyy}} + \Phi^{[xzz]}\frac{\partial}{\partial u_{xzz}}. \tag{2.5}$$

The extended components $\Phi^{[x]}, \Phi^{[xxx]}, \Phi^{[xyy]}$ and $\Phi^{[xzz]}$ explicitly depend on X, Y, Z, T and Φ and their derivatives, for details see Ref. [3]. The simplification of equation (2.4) with the aid of Maple gives a system of 130 PDEs for X, Y, Z, T and Φ, which further reduce to system of 70 PDEs when the direct procedure of determining system is adopted. However, if the option of integrability condition is imposed, the invariance criteria (2.4) reduce to just 20 PDEs which are listed below

$$\begin{aligned}
&X = X\left(x,t\right), \quad Y = Y\left(y,z\right), \quad Z = Z\left(y,z\right), \quad T = T\left(t\right), \quad X_{x,x} = 0,\\
&\Phi_{u,u} = 0, \quad \Phi_{u,x} = 0, \quad cY_z + bZ_y = 0, \quad -Tc_t + 2cZ_z - 2cX_x = 0,\\
&2Z_z bc - 2Y_y bc - bTc_t + Tcb_t = 0, \quad -bY_{yy} + 2b\Phi_{uy} - cY_{zz} = 0,\\
&2c\Phi_{uz} - bZ_{yy} - cZ_{zz} = 0,\\
&a\Phi_x u + b\Phi_{xyy} + c\Phi_{xzz} + \Phi_t + \Phi_{xxx} = 0, \quad -Tc_t - cT_t + 2cZ_z + cX_x = 0,\\
&bc\Phi_{uyy} + c^2\Phi_{uzz} - cX_t + 2acZ_z u - u(ac_t - ca_t)T + ac\,\Phi = 0.
\end{aligned} \tag{2.6}$$

Solving the determining equations (2.6), following solution may be obtained:

$$\begin{aligned}
X =& c_6 x + c_7, \quad Y = c_1 y + c_3 z, \quad Z = c_4 y + c_2 z\\
T =& 3\,c_6 t + c_5, \quad \Phi = c_8\, u,
\end{aligned} \tag{2.7}$$

along with additional constraints given as follow:

$$\begin{aligned}
&-Tc'(t) + 2c_2 c(t) - 2c(t)c_6 = 0,\\
&2b(t)c(t)(c_2 - c_1) + T(c(t)b'(t) - b(t)c'(t)) = 0,\\
&c_3 c(t) + c_4 b(t) = 0,\\
&2c_2 a(t)c(t) - T(a(t)c'(t) - c(t)a'(t)) + a(t)c(t)c_8 = 0.
\end{aligned} \tag{2.8}$$

In the next section, we shall investigate equation (1.13) for generalized symmetries using compatibility method.

3. Generalized symmetries of Vc-ZK equation

In following, main aim is to obtain generalized symmetries of the Vc-ZK equation (1.13) using the compatibility method [26, 27]. The generalized symmetries may be assumed to be in the following form:

$$\alpha u_x + \beta u_y + \gamma u_z - u_t + \delta u + e \equiv 0. \tag{3.1}$$

The unknown functions $\alpha, \beta, \gamma, \delta$ and e of (x, y, z, t) can be determined from set of determining equations when compatibility between (1.13) and (3.1) is established. To obtain compatibility condition, the time derivative u_t in the equations (1.13) and (3.1) must be equated, and result reads as follows:

$$a(t)uu_x + u_{xxx} + b(t)u_{xyy} + c(t)u_{xzz} + \alpha u_x + \beta u_y + \gamma u_z + \delta u + e = 0. \tag{3.2}$$

Further, on equating $u_{t,t}$ in equations (1.13) and (3.1), following relation may be obtained

$$\begin{aligned}
&a_t uu_x + au_t u_x + auu_{tx} + u_{txxx} + b_t u_{xyy} + bu_{txyy} + c_t u_{xzz} + cu_{txzz}\\
&+ \alpha_t u_x + \alpha u_{tx} + \beta_t u_y + \beta u_{ty} + \gamma_t u_z + \gamma u_{tz} + \delta_t u + \delta u_t + e_t = 0.
\end{aligned} \tag{3.3}$$

With the aid of Maple, on substituting $u_t, u_{tx}, u_{ty}, u_{tz}, u_{txxx}, u_{txyy}$ and u_{txzz} from equation (3.1) in equation (3.3) a very large equation is obtained which for sake of brevity we have omitted writing here. In this equation, using equation (3.2) we can further eliminate $u_{xxx}, u_{xxxx}, u_{xxxy}$ and u_{xxxz}. On equating to zero the coefficients of various partial derivatives of u, the following set of determining equations have been obtained:

$$\begin{aligned}
&\alpha_y = \alpha_z = 0, \beta_x = \gamma_x = \delta_x = 0, \quad -2c\alpha_x + 2c\gamma_z + c_t = 0, 2c\beta_z + 2b\gamma_y = 0,\\
&b_t - 2b\alpha_x + 2b\beta_y = 0, 2c\delta_z + c\gamma_{zz} + b\gamma_{yy} = 0, -3\alpha_x\gamma + \gamma_t = 0,\\
&2b\delta_y + c\beta_{zz} + b\beta_{yy} = 0, -3\alpha_x\beta + \beta_t = 0, \alpha_{xx} = 0, a\delta - 2a\alpha_x + a_t = 0,\\
&\alpha_t + b\delta_{yy} + c\delta_{z,z} - 3\alpha\alpha_x + ae = 0, \delta_t - 3\alpha_x\delta + ae_x = 0,\\
&-3\alpha_x e + be_{xyy} + ce_{xzz} + e_t + e_{xxx} = 0.
\end{aligned} \tag{3.4}$$

Solving the determining equations (3.4), we get:

$$\begin{aligned}
&\alpha = f(t)x + g(t), \beta = (c_2 y + c_3 z)\mathrm{e}^{3\int f(t)dt}, \gamma = (c_4 y + c_1 z)\mathrm{e}^{3\int f(t)dt},\\
&\delta = \left(-c_5\int a(t)dt + c_7\right)\mathrm{e}^{3\int f(t)dt}, e = (c_5 x + c_6)\mathrm{e}^{3\int f(t)dt},
\end{aligned} \tag{3.5}$$

along with the additional constraint conditions on coefficient functions which are given as follow:

$$\begin{aligned}
&f'(t) - 3f^2(t) + c_5 a(t)\mathrm{e}^{3\int f(t)dt} = 0, g'(t) - 3f(t)g(t) + c_6 a(t)\mathrm{e}^{3\int f(t)dt} = 0,\\
&a(t)\left(-c_5\int a(t)dt + c_7\right)\mathrm{e}^{3\int f(t)dt} - 2a(t)f(t) + a'(t) = 0,\\
&c_3 c(t) + c_4 b(t) = 0, -2c(t)f(t) + 2c_1 c(t)\mathrm{e}^{3\int f(t)dt} + c'(t) = 0,\\
&b'(t) - 2b(t)f(t) + 2c_2\mathrm{e}^{3\int f(t)dt} = 0,
\end{aligned} \tag{3.6}$$

with $' := \frac{d}{dt}$ and $f(t), g(t)$ are functions constrained by relations (3.6).

Remark 3.1. From the structure of generalized symmetries (3.5) and on comparing with symmetries (2.7) obtained using Lie classical method, it is evident that we able to obtain more general symmetries for (1.13). Moreover, as a particular case for $c_5 = c_6 = 0$, from generalized symmetries (3.5) classical symmetries (2.7) can be recovered.

4. Reduction of Vc-ZK equation and some explicit solutions

In this section, the similarity reduction of Vc-ZK (1.13) by generalized symmetries (3.4) is given. The similarity transformations for reduction can be obtained by solving the following characteristics equations:

$$\frac{dx}{\alpha} = \frac{dy}{\beta} = \frac{dz}{\gamma} = \frac{dt}{-1} = \frac{du}{-\delta u - e}. \tag{4.1}$$

As an alternative measure it is quite obvious to solve generalized determining equations (3.4) for $e_x = 0$ with the aid of Maple and the results read as follow:

$$\begin{aligned}
&\alpha = -\frac{x}{-3\,q + 3\,t} - \frac{k_1\lambda\,(q-t)^{-s-1}}{s} + \frac{\kappa}{q-t}, \beta = \frac{ry - 6\,\nu}{-6\,q + 6\,t},\\
&\gamma = \frac{pz - 6\,\mu}{-6\,q + 6\,t}, \delta = \frac{3\,s + 1}{-3\,q + 3\,t}, \quad e = \frac{\lambda}{q-t},
\end{aligned} \tag{4.2}$$

along with the additional constraint conditions for coefficient functions

$$a(t) = \frac{k_1}{(q-t)^{s+1}}, \quad b(t) = \frac{k_2}{(q-t)^{\frac{r}{3}+\frac{2}{3}}}, \quad c(t) = \frac{k_3}{(q-t)^{\frac{p}{3}+\frac{2}{3}}}, \tag{4.3}$$

where $p, q, r, s, \lambda, \mu, \nu$ and κ are integral constants, k_1, k_2 and k_3 are arbitrary constants.
Thus, the corresponding generalized symmetry (3.1) can be written as follows:

$$\begin{aligned} \psi = & \left(-\frac{x}{-3\,q+3\,t} - \frac{k_1\lambda\,(q-t)^{-s-1}}{s} + \frac{\kappa}{q-t}\right)u_x + \frac{ry-6\,\nu}{-6\,q+6\,t}u_y \\ & + \frac{pz-6\,\mu}{-6\,q+6\,t}u_z - u_t + \frac{3\,s+1}{-3\,q+3\,t}u + \frac{\lambda}{q-t}. \end{aligned} \tag{4.4}$$

Next, is to proceed for reduction of Vc-ZK equation (1.13) under action of generalized symmetry (4.4). Solving characteristics equations (4.1), the following invariant transformations are obtained:

$$\begin{aligned} & \xi_1 = \frac{x}{(q-t)^{\frac{1}{3}}} - \frac{3k_1\lambda\,(q-t)^{-s-\frac{1}{3}}}{s\,(3\,s+1)} + \frac{3\kappa}{(q-t)^{\frac{1}{3}}}, \quad \xi_2 = \frac{1}{(q-t)^{-\frac{r}{6}}}\left(y - \frac{6\nu}{r}\right), \\ & \xi_3 = \frac{1}{(q-t)^{-\frac{p}{6}}}\left(z - \frac{6\mu}{p}\right), \end{aligned} \tag{4.5}$$

and reduction field

$$u = \frac{3\lambda}{3\,s+1} + (q-t)^{s+\frac{1}{3}}\,F(\xi_1, \xi_2, \xi_3)\,. \tag{4.6}$$

The unknown function $F(\xi_1, \xi_2, \xi_3)$ satisfies following relation:

$$\begin{aligned} & -\left(s+\frac{1}{3}\right)F + \frac{1}{3}\xi_1 F_{\xi_1} - \frac{1}{6}r\xi_2 F_{\xi_2} - \frac{1}{6}p\xi_3 F_{\xi_3} \\ & + k_1\,FF_{\xi_1} + F_{\xi_1\xi_1\xi_1} + k_2\,F_{\xi_1\xi_2\xi_2} + k_3\,F_{\xi_1\xi_3\xi_3} = 0. \end{aligned} \tag{4.7}$$

It is obvious to see that the reduced (4.7) admits following solutions:

$$F = \frac{s}{k_1}(\xi_1 + \xi_2 + \xi_3) \text{ for } r = p = -2, \tag{4.8a}$$

$$F = \frac{1}{\xi_1^2} \text{ for } s = -1,\ k_1 = -12, \tag{4.8b}$$

$$F = \frac{1}{\xi_2^2} \text{ for } r = 3s+1 \tag{4.8c}$$

$$F = \frac{1}{\xi_3^2} \text{ for } p = 3s+1 \tag{4.8d}$$

with the aid of (4.5) back substituting (4.8) into (4.6), we get exact solution for (1.13) as follow:

$$u = \frac{3\lambda}{3\,s+1} + (q-t)^{s+\frac{1}{3}} \times \left[\frac{x}{(q-t)^{\frac{1}{3}}} - \frac{3k_1\lambda\,(q-t)^{-s-\frac{1}{3}}}{s\,(3\,s+1)} \right. \\ \left. + \frac{3\kappa}{(q-t)^{\frac{1}{3}}} + \frac{(y+3\nu)}{(q-t)^{-\frac{r}{6}}} + \frac{(z+3\mu)}{(q-t)^{-\frac{p}{6}}} \right], \tag{4.9a}$$

$$u = -\frac{3\lambda}{2} + (q-t)^{-\frac{2}{3}} \left[\frac{x}{(q-t)^{\frac{1}{3}}} + \frac{48\lambda\,(q-t)^{\frac{2}{3}}}{2} + \frac{3\kappa}{(q-t)^{\frac{1}{3}}} \right]^{-2}, \tag{4.9b}$$

$$u = \frac{3\lambda}{3\,s+1} + (q-t)^{s+\frac{1}{3}} \left[\frac{1}{(q-t)^{-\frac{r}{6}}} \left(y - \frac{6\nu}{r} \right) \right]^{-2}, \text{ for } r = 3s+1 \tag{4.9c}$$

$$u = \frac{3\lambda}{3\,s+1} + (q-t)^{s+\frac{1}{3}} \left[\frac{1}{(q-t)^{-\frac{p}{6}}} \left(z - \frac{6\mu}{p} \right) \right]^{-2}, \text{ for } p = 3s+1, \tag{4.9d}$$

along with coefficient constraints (4.3) solutions (4.9) actually satisfies the Vc-ZK (1.13).

5. Nontrivial local conservation laws

In general, for a given PDE or a system of PDEs, nontrivial local conservation laws arise from the product of PDE with suitable function(also known as multiplier, factor or characteristics). Such a combination of the product always yields nontrivial divergence expression [4]. In actual practice, the PDE is multiplied by a multiplier and then expressed in some sort of exact expression(or divergence expression) by the use of certain operator called Euler's operator. In such construction, the divergence expression always vanishes on the solution space of PDE. Below, a step-wise procedure is described for the construction of nontrivial local conservation laws.

The procedure starts with considering the following Euler operator concerning variable u^j

$$E_{u^j} = \frac{\partial}{\partial u^j} - D_i \frac{\partial}{\partial u^j_i} + \cdots + (-1)^s D_{i_1} \ldots D_{i_s} \frac{\partial}{\partial u^j_{i_1 \ldots i_s}}, \tag{5.1}$$

for each $j = 1, \ldots, m$, and D_i is the total derivative operator with respect to ith independent variable and is defined as $D_i = \frac{\partial}{\partial x^i} + u^\mu_i \frac{\partial}{\partial u^\mu} + u^\mu_{ii_1} \frac{\partial}{\partial u^\mu_{i_1}} + u^\mu_{ii_1 i_2} \frac{\partial}{\partial u^\mu_{i_1 i_2}} + \ldots$ for $i = 1, \ldots, n$. The interesting property of Euler operator (5.1) is that, it can annihilate any divergence expression of the type $D_i \Phi^i(u)$. In particular, following is always true:

$$E_{u^j}(D_i \Phi^i(x, u, \partial u, \ldots, \partial^r u)) \equiv 0,\ j = 1, \ldots, m. \tag{5.2}$$

The converse of (5.2) is also true, that is, if $E_{u^j} F(x, u, \partial u \ldots, \partial^r u) \equiv 0$, then following also holds true:

$$F(x, u, \partial u \ldots, \partial^r u) \equiv D_i \Psi^i(x, u, \partial u, \ldots, \partial^{r-1} u),\ i = 1, \ldots, n, \tag{5.3}$$

holds for functions $\Psi^i(x, u, \partial u, \ldots, \partial^{r-1}u)$. The relation (5.3) leads to the construction of conservation laws.

Example 5.1. As an illustrative example, consider KdV equation

$$\Delta_{KdV} = u_t + uu_x + u_{xxx} = 0. \tag{5.4}$$

The Euler's operator (5.1) takes the following form

$$E_u = \frac{\partial}{\partial u} - \left(D_t \frac{\partial}{\partial u_t} + D_x \frac{\partial}{\partial u_x}\right) + D_x^2 \frac{\partial}{\partial u_{xx}} - D_x^3 \frac{\partial}{\partial u_{xxx}}, \tag{5.5}$$

which truncates after 3rd order x-derivative of u. The total derivatives D_x and D_t are defined as follow:

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + \ldots,$$
$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \ldots.$$

It may be proved that $E_u(\Delta_{KdV}) \neq 0$. However, on multiplying Δ_{KdV} with suitable a factor $\Lambda(x, t, u)$ (also called as zero order multiplier) and under the following assumption

$$E_u(\Lambda(x, t, u)\Delta_{KdV}) = E_u(\Lambda(x, t, u)(u_t + uu_x + u_{xxx})) \equiv 0. \tag{5.6}$$

The unknown multipliers $\Lambda(x, t, u)$ can be obtained. The action of Euler's operator in (5.6) yields the following

$$\begin{aligned}&(\Lambda_t + u\Lambda_x + \Lambda_{xxx}) + 3\,\Lambda_{xxu}u_x + 3\,\Lambda_{xuu}u_x^2 + \Lambda_{uuu}u_x^3 \\ &+ 3\,\Lambda_{xu}u_{xx} + 3\,\Lambda_{uu}u_x u_{xx} = 0.\end{aligned} \tag{5.7}$$

Due to independence of u_x and u_{xx}, the equation (5.7) splits into following three equations

$$\Lambda_t + u\Lambda_x + \Lambda_{xxx} = 0, \Lambda_{xu} = 0, \Lambda_{uu} = 0. \tag{5.8}$$

The solution of (5.8) yields following set of multipliers

$$\Lambda_1 = 1, \Lambda_2 = u, \Lambda_3 = tu - x. \tag{5.9}$$

Then from properties (5.2) and (5.3), following divergence expression for KdV (5.4) may be written

$$\Lambda(x, t, u)(u_t + uu_x + u_{xxx}) = D_x\psi^1 + D_t\psi^2, \tag{5.10}$$

which further gives local conservation in the following form

$$D_x\psi^1 + D_t\psi^2 = 0. \tag{5.11}$$

The fluxes ψ^1, ψ^2 can be obtained by direct matching of derivatives in (5.10). The divergence expression (5.11) vanishes over the solution space of KdV (5.4) implying the non-triviality of conservation laws.

Based on procedure described above, the determining equations for zero oder multipliers $\Lambda(x,y,z,t,u)$ are given as follows:

$$E_u(\Lambda(x,y,z,t,u)(u_t + a(t)uu_x + u_{xxx} + b(t)u_{xyy} + c(t)u_{xzz})) = 0. \tag{5.12}$$

Under the action of Euler's operator (5.12) splits into following overdetermined system of equations in $\Lambda(x,y,z,t,u)$

$$\begin{aligned} &\Lambda_{uu} = 0, \Lambda_{ux} = 0, \Lambda_{uy} = 0, \Lambda_{uz} = 0,\\ &a(t)u\Lambda_x + b(t)\Lambda_{xyy} + c(t)\Lambda_{xzz} + \Lambda_{xxx} + \Lambda_t = 0. \end{aligned} \tag{5.13}$$

The immediate solution for (5.13) may be obtained as follows:

$$\Lambda_1 = x - \int a(t)u\,dt,\ \Lambda_2 = u,\ \Lambda_3 = F(y,z). \tag{5.14}$$

The determination of multipliers immediately followed by following identity:

$$\begin{aligned} &\Lambda(x,y,z,t,u)(u_t + a(t)uu_x + u_{xxx} + b(t)u_{xyy} + c(t)u_{xzz})\\ &\qquad\qquad = D_x\psi^1 + D_y\psi^2 + D_z\psi^3 + D_t\psi^4. \end{aligned} \tag{5.15}$$

Substituting Λ's from (5.14) and direct matching in (5.15), yields the following set of conservation laws:

- $\Lambda_1 = x - \int a(t)u\,dt$.

$$\begin{aligned} \psi^1 =& -\frac{1}{3}a(t)u^3\int a(t)\ \mathrm{d}t + \frac{1}{2}a(t)xu^2 - \frac{1}{3}b(t)uu_{y,y}\int a(t)\ \mathrm{d}t\\ &+\frac{1}{6}b(t){u_y}^2\int a(t)\ \mathrm{d}t - \frac{1}{3}c(t)uu_{z,z}\int a(t)\ \mathrm{d}t\\ &+\frac{1}{6}c(t){u_z}^2\int a(t)\ \mathrm{d}t - uu_{x,x}\int a(t)\ \mathrm{d}t + \frac{1}{2}{u_x}^2\int a(t)\ \mathrm{d}t\\ &+\frac{1}{3}b(t)xu_{y,y} + \frac{1}{3}c(t)xu_{z,z} + xu_{x,x} - u_x,\\ \psi^2 =& -\frac{1}{3}b(t)\left(2\,uu_{x,y}\int a(t)\ \mathrm{d}t - u_xu_y\int a(t)\ \mathrm{d}t - 2\,xu_{x,y} + u_y\right),\\ \psi^3 =& -\frac{1}{3}c(t)\left(2\,uu_{x,z}\int a(t)\ \mathrm{d}t - u_xu_z\int a(t)\ \mathrm{d}t - 2\,xu_{x,z} + u_z\right),\\ \psi^4 =& -\frac{1}{2}u\left(u\int a(t)\ \mathrm{d}t - 2\,x\right). \end{aligned}$$

The divergence expression

$$\begin{aligned} &D_x\psi^1 + D_y\psi^2 + D_z\psi^3 + D_t\psi^4 =\\ &(u_t + a(t)uu_x + u_{x,x,x} + b(t)u_{x,y,y} + c(t)u_{x,z,z})\,x\\ &+\left(-u_xa(t)u^2 - u_{x,z,z}uc(t) - u_{x,y,y}ub(t) - uu_t - uu_{x,x,x}\right)\int a(t)\ \mathrm{d}t, \end{aligned} \tag{5.16}$$

vanishes over the solution space of (1.13) implying the non-triviality of conservation laws.

- $\Lambda_1 = u$.

$$\psi^1 = \frac{1}{3}\,a\,(t)\,u^3 + \frac{1}{3}\,c\,(t)\,uu_{z,z} - \frac{1}{6}\,c\,(t)\,{u_z}^2 + \frac{1}{3}\,b\,(t)\,uu_{y,y} - \frac{1}{6}\,b\,(t)\,{u_y}^2 + uu_{x,x} - \frac{1}{2}\,{u_x}^2,$$
$$\psi^2 = \frac{1}{3}\,b\,(t)\,(2\,uu_{x,y} - u_x u_y)\,,$$
$$\psi^3 = \frac{1}{3}\,c\,(t)\,(2\,uu_{x,z} - u_x u_z)\,,$$
$$\psi^4 = \frac{1}{2}\,u^2.$$

The divergence expression

$$D_x\psi^1 + D_y\psi^2 + D_z\psi^3 + D_t\psi^4 = u\,(u_t + a\,(t)\,uu_x + u_{x,x,x} + b\,(t)\,u_{x,y,y} + c\,(t)\,u_{x,z,z})\,. \tag{5.17}$$

vanishes over the solution space of (1.13) implying the non-triviality of conservation laws.

- $\Lambda_1 = F(y, z)$.

$$\psi^1 = \frac{1}{2}\,a\,(t)\,Fu^2 - \frac{1}{3}\,c\,(t)\,F_z u_z - \frac{1}{3}\,b\,(t)\,F_y u_y + \frac{1}{3}\,c\,(t)\,Fu_{z,z} + \frac{1}{3}\,b\,(t)\,Fu_{y,y} + \frac{1}{3}\,c\,(t)\,uF_{zz} + \frac{1}{3}\,b\,(t)\,uF_{yy} + Fu_{x,x},$$
$$\psi^2 = -\frac{1}{3}\,b\,(t)\,(F_y u_x - 2\,Fu_{x,y})\,,$$
$$\psi^3 = -\frac{1}{3}\,c\,(t)\,(F_z u_x - 2\,Fu_{x,z})\,,$$
$$\psi^4 = Fu.$$

The divergence expression

$$D_x\psi^1 + D_y\psi^2 + D_z\psi^3 + D_t\psi^4 = F\,(y,z)\,(u_t + a\,(t)\,uu_x + u_{x,x,x} + b\,(t)\,u_{x,y,y} + c\,(t)\,u_{x,z,z})\,. \tag{5.18}$$

vanishes over the solution space of (1.13) implying the non-triviality of conservation laws.

6. Conclusion

In this work, the Zakharov-Kuznetsov equation in magnetized dusty plasma is derived by the reductive perturbation technique. This equation has been further constrained by plugging variable coefficients to incorporate more realistic means of the physical phenomenon in plasma physics. In this investigation, the classical symmetries using the Lie method and new generalized symmetries using the compatibility method are obtained and compared. In the comparison, it has been proved that the new generalized symmetries (3.5) reduce to classical symmetries (2.7) for $c_5 = c_6 = 0$. Some simple traveling wave solutions for Vc-ZK are also presented. In addition to this, the multiplier method is used to construct nontrivial local conservation laws. To prove the non-triviality of conservation laws, it has been proved at (5.16), (5.17) and (5.18) that the divergence expression for fluxes vanishes over the solution space of equation (1.13).

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