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A mathematical analysis of Zika virus transmission with optimal control strategies

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Abstract This paper presents a mathematical model for transmission dynamics of Zika virus by considering standard incidence type interaction for the human to human transmission. The model involves the transmission through the bite of infected Aedes mosquitoes and human to human sexual transmission. The equilibria of the proposed model are found and the basic reproduction number R_0 is computed. If $R_0 < 1$, the disease-free equilibrium point is locally asymptotically stable and it is also globally asymptotically stable under certain conditions. The analysis shows that the model exhibits the occurrence of backward bifurcation, which suggests that when $R_0 < 1$ is not completely sufficient for eradicating the disease where the stable disease-free equilibrium co-exists with a stable endemic equilibrium. The endemic equilibrium point of the system exists and locally asymptotically stable under some restriction on parameters, whenever $R_0 > 1$. The sensitivity analysis is performed to identify the key parameters that affect the basic reproduction number, which can be regulated to control the transmission dynamics of the Zika. Further, this model is extended to the optimal control model and to reveals the optimal control strategies we used the Pontryagin's Maximum Principle. It has been noticed that the optimal control gives better result than without the optimal control model. Numerical simulation is presented to support our mathematical findings.

Keywords. Zika virus, Basic reproduction number, Bifurcation, Stability analysis, Sensitivity analysis, Optimal control.

2010 Mathematics Subject Classification. 34D20, 34D23, 37C75, 49Q12, 90C31, 93C15.

1. Introduction

Mosquito-borne infectious diseases are a global health problem for humans. Zika virus is one of the arboviruses which are primarily spread through the bite of infected Aedes mosquitoes [2]. Zika virus is also transmitted through sexual transmission and blood transfusions, which has not been documented formerly [1] for any other arboviruses (Dengue virus, Chikungunya, Japanese encephalitis, Yellow fever virus,

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etc.). The sexual transmission of disease from male-to-female, female-to-male, and male-to-male partners have been established but female-to-female sexual transmission has not yet been found. Recent cases of blood transfusion have been identified. In 2016, the first sexual transmission of Zika virus case was noticed in France [6]. The symptoms of the Zika virus are mild fever, loss of appetite, skin rashes, joint pain, conjunctivitis, muscle pain, headache, etc. Normally the symptoms are exhibited [21] for 2-7 days. The Centers for Disease Control and Prevention (CDC) strongly recommended that Guillain Barres Syndrome [17] and microcephaly [20] are related to Zika virus. In 2008, the potential of sexual transmission of Zika virus case was reported in Senegal [27] by a scientist in his laboratory.

The Zika virus was detected in a rhesus monkey in 1947 in the Zika Forest of Uganda and 1952, it was identified from human populations in Nigeria [8]. Since 1952, Zika virus outbreaks have appeared and infected humans in many countries of Asia, Africa, the Americas, and the Pacific. It becomes a global threat as the transmission is rampant. According to the World Health Organization (WHO), Zika virus outbreaks are reported in more than 84 countries in the world, while 13 countries [25] have been reported as the sexual transmission hub of the Zika virus. The first largest outbreaks were reported in 2007 at the Island of Yap [26]. In 2013, a large number of humans were affected by Zika in South Pacific and French Polynesia. In 2015-16, Zika virus spread rapidly in Brazil [20], in most of the American and the Caribbean countries. In 2016, the World Health Organization announced Zika as a Public Health Emergency of International Concern [27].

The Mathematical model plays an important role in understanding the transmission dynamics of the Infectious disease and in preventing the disease through treatment, vaccination, and isolation of the infected population. Several authors formulated and analyzed a number of Zika virus transmission dynamics models [1, 2, 8, 13, 18, 23]. In [1] authors constrated a model of Zika virus with vertical transmission. In [2] authors formulated a simple mathematical model on Zika virus and introduced optimal control strategies . In [8] authors studied the effect of Mosquito-borne and sexual transmission on the spread and control of the disease. In [13], authors developed a mathematical model to examine the 201314 French Polynesia outbreak on the six major archipelagos. In [18], authors proposed a Zika virus transmission model by incorporating three nonlinear forces of infection from an infected mosquito. In [23] authors presented a standard mass-action type model and included media impact for a human to reduce the transmission. In [7, 10, 16], authors worked on the causes of backward bifurcation in some epidemiological models.

This paper is organized as follows: Section 2, formulates the mathematical model; Section 3, finds the existence of equilibria and computes the basic reproduction number; Section 4, discusses the existence of the bifurcation of the model; Section 5, presents the stability analysis of the model; Section 6, illustrates the numerical simulation and results of the model; Section 7, presents sensitivity analysis of basic reproduction number ; Section 8, studies the optimal control model and its analysis; Section 9, demonstrates the numerical simulation results of the optimal control model and finally in Section 10, we conclude our paper.

2. The Model

We have formulated a deterministic model of the Zika virus by assuming standard incidence type interaction for the human to human transmission. The human population has been divided into four different compartments according to the nature of the disease such as Susceptible human population S_h at time t, Exposed human population E_h at time t, Infected human population I_h at time t and Recovered human population R_h at time t. Also, the vector population has been divided into three different compartments according to the nature of the disease such as Susceptible mosquito S_v at time t, Exposed mosquito E_v at time t and Infected mosquito I_v at time t. Here incident rate $\frac{\beta_1 I_h}{N_h}$ is the average number contacts with infectives per unit time of one susceptible [24] and $\left(\frac{\beta_1 I_h}{N_h}\right) S_h$ is the number of new cases per unit time amongst the susceptibles [24]. Zika virus is transmitted between human to human, human to vector and vector to human. Based on the above consideration, we formulate the following model:

$$\begin{aligned}
S_h' &= \Lambda_h - \left(\frac{\beta_1 I_h}{N_h}\right) S_h - \left(\frac{\beta_2 I_v}{N_h}\right) S_h - \mu_h S_h,\\
E_h' &= \left(\frac{\beta_1 I_h}{N_h}\right) S_h + \left(\frac{\beta_2 I_v}{N_h}\right) S_h - (\delta_h + \mu_h) E_h,\\
I_h' &= \delta_h E_h - (\gamma_h + \mu_h + \mu_1) I_h,\\
R_h' &= \gamma_h I_h - \mu_h R_h,\\
S_v' &= \Lambda_v - \left(\frac{\beta_v I_h}{N_h}\right) S_v - \mu_v S_v,\\
E_v' &= \left(\frac{\beta_v I_h}{N_h}\right) S_v - (\mu_v + \eta_v) E_v,\\
I_v' &= \eta_v E_v - \mu_v I_v,
\end{aligned} \tag{2.1}$$

As $N_h = S_h + E_h + I_h + R_h$ and $N_v = S_v + E_v + I_v$, we consider the following form of the system for further analysis :

$$\begin{aligned}
N_h' &= \Lambda_h - \mu_h N_h - \mu_1 I_h,\\
E_h' &= \frac{\beta_1 I_h (N_h - E_h - I_h - R_h)}{N_h} + \frac{\beta_2 (N_h - E_h - I_h - R_h) I_v}{N_h} - (\delta_h + \mu_h) E_h,\\
I_h' &= \delta_h E_h - (\gamma_h + \mu_h + \mu_1) I_h,\\
R_h' &= \gamma_h I_h - \mu_h R_h,\\
N_v' &= \Lambda_v - \mu_v N_v,\\
E_v' &= \beta_v (N_v - E_v - I_v) \frac{I_h}{N_h} - (\mu_v + \eta_v) E_v,\\
I_v' &= \eta_v E_v - \mu_v I_v.
\end{aligned} \tag{2.2}$$

2.1. **Positive Invariant.**

FIGURE 1. Flow diagram of the model.

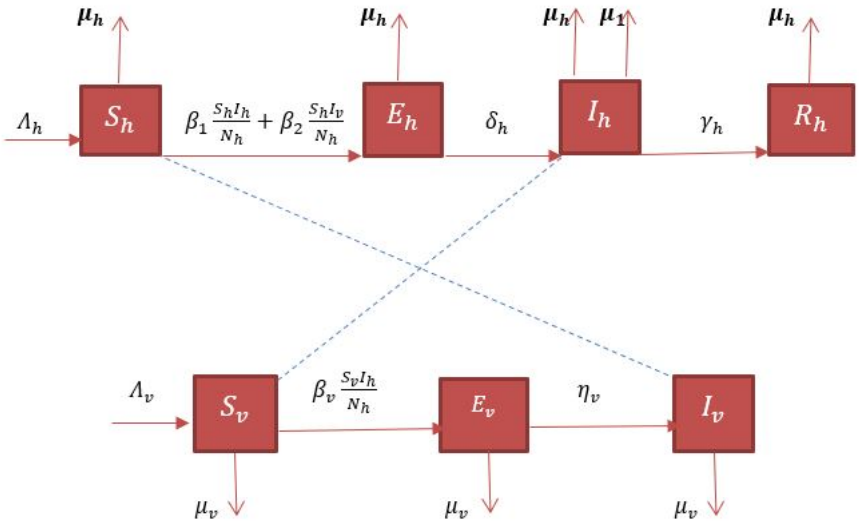


TABLE 1. Description of parameters

Parameter		Description
Λ_h	:	Rate of recruitment of human population,
Λ_v	:	Rate of recruitment of vector(mosquito) population,
β_1	:	Transmission rate between S_h and I_h,
β_2	:	Transmission rate between S_h and I_v,
β_v	:	Transmission rate between I_h and S_v,
μ_h	:	Natural mortality rate of human population,
μ_1	:	Natural mortality rate of human population due to infection,
μ_v	:	Natural mortality rate of vector(mosquito) population,
δ_h	:	Contact rate between E_h and I_h,
γ_h	:	Recovery rate of infectives(human) population,
η_v	:	Contact rate between E_v and I_v

Theorem 2.1. *If $S_h(0), E_h(0), I_h(0), R_h(0), S_v(0), E_v(0) I_v(0)$ are non-negative, the solutions of $S_h(t), E_h(t), I_h(t), R_h(t), S_v(t), E_v(t), I_v(t)$ of the systen (2.1) are positive for all $t > 0$ with the same initial non-negative condition.*

Proof. The first equation of the system (2.1), can be written as

$$\frac{dS_h}{dt} + \left(\frac{\beta_1 I_h}{N_h} + \frac{\beta_2 I_v}{N_v} + \mu_h \right) S_h \quad = \quad \Lambda_h \geq 0,$$

So that,

$$\frac{d}{dt}\left[S_h(t)\exp\left(\int_0^t\left(\frac{\beta_1 I_h(\tau)}{N_h(\tau)}+\frac{\beta_2 I_v(\tau)}{N_v(\tau)}+\mu_h\right)d\tau\right)\right]\geq 0, \tag{2.3}$$

Integrating (3) yields

$$S_h(t)\geq S_h(0)\exp\left[-\int_0^t\left(\frac{\beta_1 I_h(\tau)}{N_h(\tau)}+\frac{\beta_2 I_v(\tau)}{N_v(\tau)}+\mu_h\right)d\tau\right]>0,$$

Where,

$$N_h(\tau)=S_h(\tau)+E_h(\tau)+I_h(\tau)+R_h(\tau), N_v(\tau)=S_v(\tau)+E_v(\tau)+I_v(\tau),$$

In similarly way we can prove that $E_h(t), I_h(t), R_h(t), S_v(t), E_v(t), I_v(t)$ are non-negative for all $t>0$.
Hence proved of the theorem
Therefore, the biological feasible region attraction of the system (2.2) as follows:

$$\{\Omega=(S_h,E_h,I_h,R_h,S_v,E_v,I_v)\in R_+^7 : N_h\leq\frac{\Lambda_h}{\mu_h}, N_v\leq\frac{\Lambda_v}{\mu_v}\}$$

3. Existence of Equilibria and the Basic Reproduction Number

3.1. **Disease-free equilibrium point E_0.** We consider the system (2.1) and find the disease-free equilibrium point . For our model we have disease free equilibrium point as E_0=$(N_h^0,E_h^0,I_h^0,R_h^0,N_v^0,E_h^0,I_v^0)=\left(\frac{\Lambda_h}{\mu_h},0,0,0,\frac{\Lambda_v}{\mu_v},0,0\right)$.

3.2. **The basic reproduction number R_0.** We find the basic reproduction number R_0 by following the next generation matrix method as described in [3, 5]. Same notation we use as in [3, 5]. We find the matrix $\mathcal{F}$ and $\mathcal{V}$ as follows:

$$\mathcal{F}=\begin{pmatrix}\beta_1(N_h-E_h-I_h-R_h)\frac{I_h}{N_h}+\beta_2(N_h-E_h-I_h-R_h)\frac{I_v}{N_h}\\0\\0\\\beta_v(N_v-E_v-I_v)\frac{I_v}{N_h}\end{pmatrix},$$

and

$$\mathcal{V}=\begin{pmatrix}(\delta_h+\mu_h)E_h\\-\delta_h E_h+(\gamma_h+\mu_h+\mu_1)I_{sh}\\(\eta_v+\mu_v)E_v\\-\eta_v E_v+\mu_v I_v\end{pmatrix},$$

F= Jacobian of $\mathcal{F}$ at

$$E_0=\begin{pmatrix}0&\beta_1&0&\beta_2\\0&0&0&0\\0&\frac{\beta_v N_v^0}{N_h^0}&0&0\\0&0&0&0\end{pmatrix},$$

and V= Jacobian of $\mathcal{V}$ at

$$E_0 = \begin{pmatrix} \delta_h + \mu_h & 0 & 0 & 0 \\ -\delta_h & \gamma_h + \mu_h + \mu_1 & 0 & 0 \\ 0 & 0 & \mu_v + \eta_v & 0 \\ 0 & 0 & -\eta_v & \mu_v \end{pmatrix},$$

and it follows that

$$FV^{-1} = \begin{pmatrix} \frac{\delta_h \beta_1}{D_1 D_2} & \frac{\beta_1}{D_2} & \frac{\beta_2 \eta_v}{D_3 \mu_v} & \frac{\beta_2}{\mu_v} \\ 0 & 0 & 0 & 0 \\ \frac{\delta_h \beta_v}{D_1 D_2} & \frac{\beta_v N_v^0}{D_2 N_h^0} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

The largest eigenvalue of FV^{-1} is called the basic reproduction number R_0 and is obtained as follows:

$$R_0 = \frac{\delta_h \beta_1}{2 D_1 D_2} + \sqrt{\left(\frac{\delta_h \beta_1}{2 D_1 D_2}\right)^2 + \frac{\beta_2 \delta_h \beta_v \eta_v \mu_h \Lambda_v}{\mu_h^2 \Lambda_h D_1 D_2 D_3}} = R_1 + \sqrt{R_1^2 + 4R_2},$$

where,

$$D_1 = \delta_h + \mu_h; D_2 = \gamma_h + \mu_h + \mu_1; D_3 = \eta_v + \mu_v,$$
$$R_1 = \frac{\delta_h \beta_1}{2 D_1 D_2}; R_2 = \frac{\beta_2 \delta_h \beta_v \eta_v \mu_h \Lambda_v}{\mu_h^2 \Lambda_h D_1 D_2 D_3},$$

Here R_1 represents the basic reproduction due to human to human transmission by ignoring the transmission of vectors. Similarly, R_2 represents the basic reproduction due to interactions with vectors in the absence of human to human transmission. The reproduction number R_0 gives the average number of infected individuals generated by the one infected in a fully susceptible population and for our model it is given by above expression of R_0 .

3.3. **Existence of Endemic Equilibrium.** For the system (2.2), we get the endemic equilibrium point as $E_1 = (N_h^*, E_h^*, I_h^*, R_h^*, N_v^*, E_v^*, I_v^*)$

where

$$
\begin{aligned}
N_h^* &= \frac{\Lambda_h - \mu_1 I_h^*}{\mu_h},\\
E_h^* &= \frac{(\gamma_h + \mu_h + \mu_1) I_h^*}{\delta_h} = d_1 I_h^*,\\
R_h^* &= \frac{\gamma_h I_h^*}{\mu_h} = d_2 I_h^*,\\
N_v^* &= \frac{\Lambda_v}{\mu_v},\\
E_v^* &= \frac{\mu_v}{\eta_v} I_v^* = \frac{\Lambda_v \beta_v}{(\mu_v + \eta_v)[\mu_v \beta_v + (\Lambda_h - \mu_1 I_h^*)\mu_v I_h^*]},\\
I_v^* &= \frac{\Lambda_v \eta_v \beta_v}{\mu_v(\mu_v + \eta_v)[\mu_v \beta_v + (\Lambda_h - \mu_1 I_h^*)\mu_v I_h^*]},\ \textit{provided } \Lambda_h > \mu_1 I_h^*,
\end{aligned}
$$

Substituting the value of $N_h^*, I_h^*, R_h^*, N_v^*, E_v^*, I_v^*$ in the equilibrium $\frac{dI_h^*}{dt}$ and I_h^* is the positive root of the following non-linear equation, we get

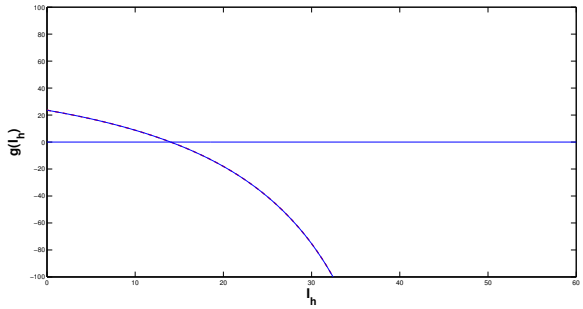
$$
\begin{aligned}
g(I_h) &= -(\delta_h + \mu_h)\left(\frac{\Lambda_h - d_1\mu_1 I_h}{\mu_h}\right) + \frac{1}{\mu_h}\left[\Lambda_h - D_4 I_h\right],\\
&\quad \left[\beta_1 d_1 + \frac{\beta_2 \Lambda_h \mu_h \beta_v \eta_v d_1}{(\mu_v + \eta_v)\{\beta_v \mu_h d_1 I_h + \mu_v(\Lambda_h - d_1\mu_1 I_h)\}}\right] = 0,\\
g(0) &= \frac{\Lambda_h}{\mu_h}\left[\beta_2 d_1 + \frac{\beta_2 \Lambda_h \mu_h \beta_v \eta_v d_1}{\mu_v(\mu_v + \eta_v)}\right] - (\delta_h + \mu_h)\frac{\Lambda_h}{\mu_h} > 0,\ \textit{for } R_0 > 1,\\
g\left(\frac{\Lambda_v}{\mu_1}\right) &= -(\delta_h + \mu_h)\left(\frac{\Lambda_h - d_1\Lambda_v}{\mu_h}\right) + \frac{1}{\mu_h}\left[\Lambda_h - D_4\frac{\Lambda_v}{\mu_1}\right],\\
&\quad \left[\beta_1 d_1 + \frac{\beta_2 \Lambda_h \mu_h \beta_v \eta_v d_1 \mu_1}{(\mu_v + \eta_v)\{\beta_v \mu_h d_1 \Lambda_v + \Lambda_h \mu_v \mu_1(1 - d_1)\}}\right] < 0,\\
g(A) &= -(\delta_h + \mu_h)\left(\frac{\Lambda_h - d_1\mu_1 A}{\mu_h}\right) + \frac{1}{\mu_h}\left[\Lambda_h - D_4 A\right],\\
&\quad \left[\beta_1 d_1 + \frac{\beta_2 \Lambda_h \mu_h \beta_v \eta_v d_1}{(\mu_v + \eta_v)\{\beta_v \mu_h d_1 E_h + \mu_v(\Lambda_h - d_1\mu_1 A)\}}\right] < 0,
\end{aligned}
$$

where,

$$
A = \frac{\Lambda_h}{(\gamma_h + \mu_h + \mu_1)},\ D_4 = (\mu_1 d_1 + \mu_h + \mu_h d_1 + \mu_h d_2)
$$

Here we observe that for $A < I_h < \frac{\Lambda_h}{\mu_1}$, $g(I_h)$ is always negative, i.e. there is no change of sign in $g(I_h)$. So there is no root of $g(I_h)$ in the interval $A < I_h < \frac{\Lambda_h}{\mu_1}$.

FIGURE 2. Showing the existence of one root of $g(I_h) = 0$ and polt $g(I_h)$ with I_h



Hence we can conclude that there is at least one root of $g(I_h) = 0$ in the interval $0 < I_h < A$

$$g(I_h') = -\frac{D_4}{\mu_h}\left[\beta_1 d_1 + \frac{\beta_2\mu_h\Lambda_v\beta_v\eta_v d_1}{(\mu_v+\eta_v)\{\beta_v\mu_h d_1 I_h + \mu_v(\Lambda_h - d_1\mu_1 I_h)\}}\right]$$

$$- \left[\frac{\beta_2\Lambda_h\mu_h\beta_v\eta_v d_1\{\Lambda_h - D_4 I_h\}}{\mu_h(\mu_v+\eta_v)\{\beta_v\mu_h d_1 I_h + \mu_v(\Lambda_h - d_1\mu_1 I_h)\}^2}\right]$$

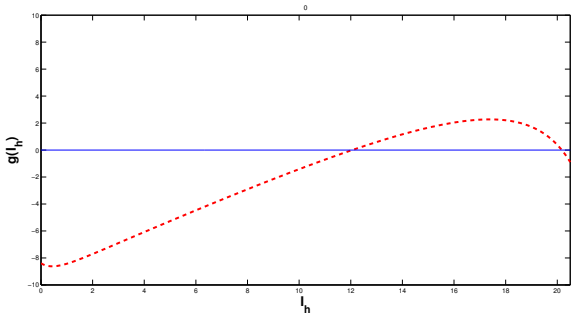
$$+ \frac{(\delta_h+\mu_h)\mu_1 d_1}{\mu_h} < 0$$

The above expression is negative under the condition $\Lambda_h > \mu_1 I_h$ and $\Lambda_h > (\mu_1 d_1 + \mu_h + \mu_h d_1 + \mu_h d_2) I_h$ then we can say that there exists unique positive root I_h^* (say) of $g(I_h') = 0$ in the interval $0 < I_h < A$. Also, it is clear that if $g(I_h') < 0$ at A then it must be negative for all I_h in the interval $0 < I_h < A$ Hence under this condition, we get the positive equilibrium point $E_1 = (N_h^*, E_h^*, I_h^*, R_h^*, N_v^*, E_v^*, I_v^*)$ and the fact is ploted in Figure 2 . But if $g(I_h')$ is not negative throughout the interval $0 < I_h < A$, then there is a possibility of more than one root of the given equation $g(I_h) = 0$. In general vector-borne disease model exhibits backward bifurcation which corresponds to endemic equilibrium points for $R_0 < 1$. For our model too, we get two positive roots of $g(I_h) = 0$ for some suitable set of parameters and the fact is ploted in Figure 3 . Hence we get two endemic equilibrium of the system (2.2).

4. EXISTENCE OF BIFURCATION

Here we analyze the existence of Backward bifurcation for the system (2.1). The phenomenon of backward bifurcation suggests that the stable disease-free equilibrium co-exists with a stable endemic equilibrium for $R_0 < 1$. This phenomenon has been observed in some epidemiological model [10], particularly Dengue, Malaria and Zika disease transmission models [7, 16, 23]. The backward bifurcation phenomenon has significances implications for public health practice, as it is related directly to whether or not the disease can be effectively controlled even when associated reproduction

FIGURE 3. Showing the existence of two roots of $g(I_h) = 0$ and plot of $g(I_h)$ with I_h



number $R_0 < 1$. The backward bifurcation property for human disease suggests that the standard incidence function is more suitable for modeling than mass action incident functions [7].

Let us consider the following change of variables $N_h = x_1, E_h = x_2, I_{sh} = x_3$, $I_{ah} = x_4, R_h = x_5, N_v = x_6, I_v = x_7$.

Also further by using vector notation $X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)^T$, our system (2.2) can be formulated as shown below

$$\frac{dX}{dt} = F(x), where\ F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7)^T$$

$$\begin{aligned}
x_1' &= \Lambda_h - \mu_h x_1 - \mu_1 x_3, \\
x_2' &= \beta_1(x_1 - x_2 - x_3 - x_4)\frac{x_3}{x_1} + \beta_2(x_1 - x_2 - x_3 - x_4)\frac{x_7}{x_1} - (\delta_h + \mu_h)x_2, \\
x_3' &= \delta_h x_2 + (\gamma_h + \mu_h + \mu_1)x_3, \\
x_4' &= \gamma_h x_3 - \mu_h x_4, \\
x_5' &= \Lambda_v - \mu_v x_5 \\
x_6' &= \beta_v(x_5 - x_6 - x_7)\frac{x_3}{x_1} - (\mu_v + \eta_v)x_6, \\
x_7' &= \eta_v x_6 - \mu_v x_7,
\end{aligned} \tag{4.1}$$

Consider the case $R_0 = 1$. Suppose, further, that $\beta_1 = \beta_1^*$ is chosen as a bifurcation parameter. Solving for $\beta_1 = \beta_1^*$ from $R_0 = 1$ gives

$$\beta_1^* = \frac{(\delta_h + \mu_h)(\gamma_h + \mu_h + \mu_1)}{\delta_h}\left(1 - \frac{\mu_h\beta_2\beta_v\eta_v\Lambda_v}{(\delta_h + \mu_h)(\gamma_h + \mu_h + \mu_1)(\mu_v + \eta_v)\mu_v^2\Lambda_h}\right)$$

The Jacobian of the above system (4.1) at disease-free equilibrium point E_0 with $\beta_1 = \beta_1^*$ is given by

$$J(\beta_1) = \begin{pmatrix} -\mu_h & 0 & -\mu_1 & 0 & 0 & 0 & 0 \\ 0 & -D_1 & 0 & 0 & 0 & 0 & \beta_2 \\ 0 & \delta_h & -D_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_h & -\mu_h & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mu_v & 0 & 0 \\ 0 & 0 & \beta_v \dfrac{x_5}{x_1} & 0 & 0 & -D_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta_v & -\mu_v \end{pmatrix}$$

Where, $x_1 = \dfrac{\Lambda_h}{\mu_h}, x_5 = \dfrac{\Lambda_v}{\mu_v}, D_1 = \delta_h + \mu_h; D_2 = \gamma_h + \mu_h + \mu_1; D_3 = \eta_v + \mu_v$.
According to Castillo-Chavez and Song [4], we use the center manifold theory and analyze it, which is shown below

Theorem 4.1. *(**Castillo-Chavez and Song** [4]). Consider the follwing general system of ordinary differential equations with a parameter ϕ,*

$$\frac{dx}{dt} = f(x, \phi), f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}, \text{ and } f \in \mathbb{C}^2(\mathbb{R}^n \times \mathbb{R})$$

without loss of generality, it is assumed that 0 is the equilibrium point of the system $(i.e. f(0, \phi)) \equiv 0$ for all ϕ and
1. $A = D_x f(0,0) = \left(\dfrac{\partial f_i}{\partial x_j}(0,0)\right)$ is the linearization matrix of the system around the equlibrium 0 with f evaluted at 0;
2. Zero is the simple eigenvalue of A and other eigenvalues of A has negatives real parts;
3. Matrix A has a right eigenvector w and a left eigenvector v corresponding to the zero eigenvalue.
Let f_k be the kth component of f and

$$a_1 = \sum_{k,i,j=1}^{n} v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(0,0)$$

$$b_1 = \sum_{k,i=1}^{n} v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \phi}(0,0)$$

then the local dynamics of the system around the equilibrium point 0 is totally determined by the signs a_1 and b_1.
(i). $a_1 > 0$, $b_1 > 0$. When $\phi < 0$ with $|\phi| \ll 1$, 0 is locally asymptotically stable, and there exists a positive unstable equilibrium; when $0 < \phi \ll 0$, 0 is unstable and there exists a negative and locally asymptotically stable equilibrium;
(ii). $a_1 < 0$, $b_1 < 0$. When $\phi < 0$ with $|\phi| \ll 1$, 0 is unstable; when $0 < \phi \ll 0$, 0 is locally asymptotically stable, and there exists a positive unstable equilibrium;
(iii). $a_1 > 0$, $b_1 < 0$. When $\phi < 0$ with $|\phi| \ll 1$ is unstable, and there exists a locally asymptotically stable negative equilibrium; when $0 < \phi \ll 0$, 0 is stable, and a positive unstable equilibrium appears;

(iv). $a_1 < 0$, $b_1 > 0$. When ϕ changes from negative to positive, 0 changes its stability from stable to unstable. Correspondingly a negative unstable equilibrium becomes positive and locally asymptotically stable.

4.1. **Eigenvalues of $J_{\beta_1^*}$.** It can easily seen that the Jacobian with $\beta_1 = \beta_1^*$ of the linearized system has a simple zero eigenvalue and the other eigenvalues have negative real parts. Hence, the center manifold theorem can be used to analyze the dynamics of the syster (4.1) near $\beta_1 = \beta_1^*$.
For the case when $R_0 = 1$, using the technique described in [7, 10], it can shown taht the matrix $J_{\beta_1^*}$ has a right eigenvector (corresponding to the zero eigenvalue) given by $w = [w_1, w_2, w_3, w_4, w_5, w_6, w_7]^T$, where

$$w_1 = -\frac{\mu_1}{\mu_h}, w_2 = 0, w_3 = 1, w_4 = \frac{\gamma_h}{\mu_h}, w_5 = 0, w_6 = 0, w_7 = 0.$$

Similarly, the matrix $J_{\beta_1^*}$ has a right eigenvector (corresponding to the zero eigenvalue) given by $w = [v_1, v_2, v_3, v_4, v_5, v_6, v_7]^T$, where

$$v_1 = 0, v_2 = 1, v_3 = \frac{\delta_h + \mu_h}{\delta_h}, v_4 = 0, v_5 = 0,$$

$$v_6 = \frac{\beta_2 \eta_v}{\mu_v(\mu_v + \eta_v)}, v_7 = \frac{\beta_2}{\mu_v}.$$

4.2. **Computation of a_1.** For the system (4.1), the associated non-zero partial derivatives at DFE (E_0) are given by

$$\begin{aligned}
\frac{\partial^2 f_2}{\partial x_3 \partial x_3} &= -\frac{2\beta_1}{x_1} = \frac{\partial^2 f_2}{\partial x_3 \partial x_3}, \\
\frac{\partial^2 f_2}{\partial x_4 \partial x_3} &= -\frac{\beta_1}{x_1} = \frac{\partial^2 f_2}{\partial x_3 \partial x_4}, \\
\frac{\partial^2 f_6}{\partial x_1 \partial x_3} &= -\frac{\beta_v x_5}{x_1^2} = \frac{\partial^2 f_2}{\partial x_3 \partial x_1}.
\end{aligned}$$

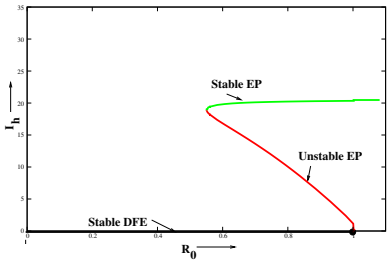
It follows from the above expressions that

$$\begin{aligned}
a_1 &= \sum_{k,i,j=1}^{n} v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(0,0) \\
&= -v_2 \left[2w_3 w_3 \left(\frac{2\beta_1}{x_1} \right) + 2w_4 w_3 \left(\frac{2\beta_1}{x_1} \right) \right] - 2v_6 w_1 w_3 \left(\frac{\beta_v x_5}{x_1^2} \right) \\
&= -\frac{4\beta_1(\mu_h + \gamma_h)}{\Lambda_h} + \frac{\beta_2 \beta_v \mu_1 \mu_h \eta_v \Lambda_h}{\mu_v^2(\mu_v + \eta_v)\Lambda_h^2}
\end{aligned}$$

4.3. **Computation of b_1.** For the system (4.1), the associated non-zero partial derivatives at DFE (E_0) are given by

$$\frac{\partial^2 f_2}{\partial x_3 \partial \beta_1} = 1$$

FIGURE 4. Plot diagram is infective populations with reproduction number showing the backward bifurcation by considering bifurcation parameter β_1.



It follows from the above expressions that

$$\begin{aligned} b_1 &= \sum_{k,i=1}^{n} v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \phi}(0,0) \\ &= v_2 w_3 = 1 > 0 \end{aligned}$$

Here, it is clear that the coefficient b_1 is positive and according to the Theorem (4.1),it will determine the phenomenon of backward bifurcation in our model. If the sign of the cofficient a_1 is positive, its implies that the model will undergo backward bifurcation aroud the disease-free equilibrim for $\beta_1 = \beta_1^*$ and the fact is demonestrated in Figure 4. This suggest that the disease-free is not globaly stable.

5. STABILITY ANALYSIS

5.1. Local Stability of Disease-Free Equilibrium(DFE).

Theorem 5.1. *If $R_0 < 1$, the disease-free equilibrium E_0 is locally asymptotically stable otherwise it is unstable.*

The Jacobian matrix of the system (2.2) at disease-free equilibrium point $E_0 = (N_h^0, 0, 0, 0, N_v^0, 0, 0)$ is obtained as follows:

$$J_0 = \begin{pmatrix} -\mu_h & 0 & -\mu_1 & 0 & 0 & 0 & 0 \\ 0 & -D_1 & 0 & 0 & 0 & 0 & \beta_2 \\ 0 & \delta_h & -D_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_h & -\mu_h & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mu_v & 0 & 0 \\ 0 & 0 & \beta_v \dfrac{N_v^0}{N_h^0} & 0 & 0 & -D_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta_v & -\mu_v \end{pmatrix}$$

where,

$$D_1 = \delta_h + \mu_h; D_2 = \gamma_h + \mu_h + \mu_1; D_3 = \eta_v + \mu_v.$$

Clearly, three eigenvalues of the matrix J_0 are $-\mu_h$,$-\mu_h$ and $-\mu_v$ and the remaining four eigenvalues are the roots of the following characteristics equation:

$$\lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d = 0$$

where,

$$\begin{aligned}
a &= 2\mu_h + \delta_h + \gamma_h + 2\mu_v + \eta_v + \mu_1 \\
b &= \mu_v(\mu_v + \eta_v) + (\mu_h + \delta_h)(\gamma_h + \mu_h + \mu_1) + (2\mu_v + \eta_v)(2\mu_h + \delta_h + \gamma_h + \mu_1) \\
c &= \mu_v(\mu_v + \eta_v) + (\mu_h + \delta_h)(\gamma_h + \mu_h + \mu_1) \\
d &= \mu_v(\mu_v + \eta_v)(\mu_h + \delta_h)(\gamma_h + \mu_h + \mu_1) - \beta_2\beta_v\delta_h\eta_v\frac{N_v^0}{N_h^0} \\
&= \mu_v(\mu_v + \eta_v)(\mu_h + \delta_h)(\gamma_h + \mu_h + \mu_1)(1 - R_2)
\end{aligned}$$

All conditions of Routh Hurtwiz criteria are satisfied as $a > 0, b > 0, c > 0, d > 0$ and $abc > a^2d + c^2$,whenever $R_2 < 1$. Hence all four eigenvalus of the characteristics equation are negative. Therefore the disease-free equilibrium E_0 is locally asymptotically stable if $R_0 < 1$

5.2. **Global Stability of Disease-Free Equilibrium(DFE).**

Theorem 5.2. *If $R_0 < 1$, then the disease-free equilibrium E_0 is globally asymptotically stable on Ω under some conditions.*

Proof. For the global stability of disease-free equilibrium, we follow the same method described in [12]. Consider the following Lyapunove function:

$$\begin{aligned}
L &= C_1\int_{S_h^0}^{S_h}\left(1 - \frac{S_h^0}{x}\right)dy + C_2E_h + C_3I_h + C_4\int_{S_v^0}^{S_v}\left(1 - \frac{S_v^0}{x}\right)dy \\
&\quad + C_5E_v + C_6I_v
\end{aligned}$$

The derivative of L along the solution of model (2.1) is

$$\begin{aligned}
\frac{dL}{dx} &= C_1\left(1 - \frac{S_h^0}{S_h}\right)\frac{dS_h}{dt} + C_2\frac{dE_h}{dt} + C_3\frac{dI_h}{dt} + C_4\left(1 - \frac{S_v^0}{S_v}\right)\frac{dS_v}{dt} \\
&\quad + C_5\frac{dE_v}{dt} + C_6\frac{dI_v}{dt}
\end{aligned}$$

Where, C_i , for $i = 1, 2, .., 6$ are positive constants to be chosen later

$$\begin{aligned}
\frac{dL}{dx} &= C_1\left(1 - \frac{S_h^0}{S_h}\right)\left[\Lambda_h - \beta_1\frac{S_hI_h}{N_h} - \beta_2\frac{S_hI_v}{N_h} - \mu_hS_h\right] \\
&+ C_2\left[\beta_1\frac{S_hI_v}{N_h} + \frac{S_hI_v}{N_h} - (\delta_h + \mu_h)E_h\right]
\end{aligned}$$

$$+ \quad C_3\left[\delta_h E_h(\mu_1+\mu_h+\gamma_h)I_h\right] + C_4\left(1-\frac{S_v^0}{S_v}\right)\left[\Lambda_v - \beta_v\frac{S_vI_h}{N_h} - \mu_v S_v\right]$$
$$+ \quad C_5\left[\beta_v\frac{S_vI_h}{N_h} - (\mu_v+\eta_v)E_v\right] + C_6(\eta_v E_v - \mu_v I_v)$$

using $S_h^0 = \frac{\Lambda_h}{\mu_h}$, $S_v^0 = \frac{\Lambda_v}{\mu_v}$ in the above equation and simplify, we get

$$\begin{aligned}\frac{dL}{dx} &= -C_1\mu_h\frac{(S_h^0-S_h)^2}{S_h} + (C_2-C_1)(\beta_1 I_h - \beta_2 I_v)\frac{S_h}{N_h}\\ &+ \left[C_3\delta_h - C_2(\delta_h+\mu_h)\right]E_h + \left[C_1\frac{\beta_1\Lambda_h}{\mu_h N_h} + C_4\frac{\beta_v\Lambda_v}{\mu_h N_v}\right]I_h\\ &- C_3(\mu_1+\mu_h+\gamma_h)I_h + (C_5-C_4)\beta_v\frac{S_vI_h}{N_h} - C_4\mu_v\frac{(S_v^0-S_v)^2}{S_v}\\ &+ \left[C_6\eta_v - C_5(\mu_v+\eta_v)\right]E_v + \left[C_1\frac{\beta_2\Lambda_h}{\mu_h N_h} - C_6\mu_v\right]I_v\end{aligned}$$

Let us choose the constants

$$c_1 = c_2 = \delta_h,\ c_4 = c_5 = \frac{\beta_1\Lambda_h\delta_h\eta_v}{N_h\mu_h\mu_v(\mu_v+\eta_v)},\ c_3 = \mu_h+\delta_h,\ c_6 = \frac{\beta_1\delta_h\Lambda_v}{N_h\mu_h\mu_v}$$

.

$$\begin{aligned}\frac{dL}{dx} &= -\delta_h\mu_h\frac{(S_h^0-S_h)^2}{S_h} - \frac{\beta_1\Lambda_h\delta_h\eta_v}{N_h\mu_h(\mu_v+\eta_v)}\frac{(S_v^0-S_v)^2}{S_v}\\ &\quad -(\delta_h+\mu_h)(\mu_1+\mu_h+\gamma_h)(1-R_2^2)\end{aligned}$$

Thus $\frac{dL}{dx} < 0$, for $R_0 \le 0$ and zero if and only if $S_h^0 = S_h$, $S_v^0 = S_v$,$E_h = I_h = R_h = 0$ and $E_v = I_v = 0$. Therefore the largest compact invariant set in Ω is the singleton set at E_0. So, the model (2.1) is globally asymptotically stable.

5.3. **Local Stability of Endemic Equilibrium(EE).**

Theorem 5.3. *When $R_0 > 1$, then endemic equilibrium E_1 is locally asymptotically stable under some conditions, otherwise it is unstable.*

The Jacobian matrix of the system (2.2) at endemic equilibrium point $E_1 = (N_h^*, E_h^*, I_h^*, R_h^*, N_v^*, E_v^*, I_v^*)$ is obtained as follows:

$$J_1 = \begin{pmatrix} -\mu_h & 0 & -\mu_1 & 0 & 0 & 0 & 0\\ m_{21} & m_{22} & m_{23} & m_{24} & 0 & 0 & m_{27}\\ 0 & \delta_h & -(\gamma_h+\mu_h+\mu_1) & 0 & 0 & 0 & 0\\ 0 & 0 & \gamma_h & -\mu_h & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -\mu_v & 0 & 0\\ m_{61} & 0 & m_{63} & 0 & m_{65} & m_{66} & m_{67}\\ 0 & 0 & 0 & 0 & 0 & \eta_v & -\mu_v \end{pmatrix}$$

where

$$\begin{aligned}
m_{21} &= \frac{(E_h^*+I_h^*+R_h^*)(\beta_1 I_h^*+\beta_v I_v^*)}{(N_h^*)^2};\\
m_{22} &= \frac{-(\beta_1 I_h^*+\beta_v I_v^*)}{N_h^*}-(\delta_h+\mu_h);\\
m_{23} &= \frac{\beta_1(N_h^*-E_h^*-2I_h^*-R_h^*)-\beta_2 I_v^*}{N_h^*};\\
m_{24} &= \frac{-(\beta_1 I_h^*+\beta_v I_v^*)}{N_h^*};\\
m_{27} &= \frac{\beta_1(N_h^*-E_h^*-I_h^*-R_h^*)}{N_h^*};\\
m_{61} &= \frac{-\beta_v(N_v^*-E_v^*-I_v^*)}{(N_v^*)^2};\\
m_{63} &= \frac{-\beta_v(N_v^*-E_v^*-I_v^*)}{N_v^*};\\
m_{65} &= m_{66}=\frac{-\beta_v I_h^*}{N_h^*};\\
m_{67} &= \frac{\beta_v I_h^*}{N_h^*}
\end{aligned}$$

Clearly, one eigenvalue of the matrix J_1 is $-\mu_v$ and remaining eigenvalues are the roots of the following polynomial equation:

$$\lambda^6+d_1\lambda^5+d_2\lambda^4+d_3\lambda^3+d_4\lambda^2+d_5\lambda+d_6=0$$

where

$$\begin{aligned}
d_1 &= 2\mu_h+\mu_v+k_1-m_{22}+m_{66},\\
d_2 &= k_1(\mu_h+\mu_h^2+m_{66}+\mu_v-m_{22}\mu_h)+\mu_h+m_{66}+2\mu_v\mu_h-2m_{22}\mu_h\\
&\quad -m_{23}\delta_h-m_{22}m_{66}-m_{66}-\mu_v,\\
d_3 &= k_1(\mu_v m_{66}+m_{22}m_{66}\mu_h+m_{22}\mu_h\mu_v-\mu_h m_{66}-\mu_h\mu_v-m_{66}\mu_h^2-\mu_1\mu_h^2)\\
&\quad +\mu_v\mu_h m_{66}-m_{22}m_{66}\mu_v+m_{66}\mu_h\mu_v+m_{22}m_{66}\mu_h+m_{22}\mu_h\mu_v\\
&\quad +\delta_h m_{23}(m_{66}-2\mu_h-\mu_v)-\delta_h\gamma_h\mu_v m_{24}+\mu_1\delta_h m_{21},\\
d_4 &= \mu_v\mu_h m_{22}m_{66}+k_1\mu_v\mu_h m_{22}m_{66}+\mu_h m_{22}(\mu_h+k_1)(m_{66}+\mu_v)\\
&\quad -k_1\mu_v\mu_h^2 m_{66}-k_1\mu_h^2 m_{22}+\delta_h m_{23}(\mu_v m_{66}-\mu_h^2+2\mu_h m_{66}-2\mu_h\mu_v)\\
&\quad +\mu_h\delta_h\gamma_h m_{24}+\delta_h\gamma_h m_{24}m_{66}-\delta_h\gamma_h\mu_1 m_{24}+\delta_h\eta_v m_{63}m_{67},\\
&\quad -\mu_1\delta_h m_{21}m_{66}+\mu_1\mu_v\delta_h m_{21}m_{66}-k_1\mu_h\mu_v m_{66}\\
d_5 &= k_1\mu_h^2 m_{22}(\mu_v+m_{66})-k_1\mu_h\mu_v m_{66}(\mu_h-m_{22})+\mu_h\mu_v m_{22}m_{66}(\mu_h+k_1)\\
&\quad +\mu_1\delta_h\eta_v m_{61}m_{27}+\delta_h m_{23}(2\mu_h\mu_v m_{66}+\mu_h^2\mu_v m_{66}+\mu_h^2 m_{66}-\mu_h^2\mu_v)\\
&\quad +\delta_h\gamma_h\mu_v m_{24}m_{66}+\delta_h\gamma_h\mu_h m_{24}m_{66}-\mu_h\mu_v\delta_h\gamma_h m_{24}+\delta_h\gamma_h\eta_v m_{24}m_{67}
\end{aligned}$$

$$
\begin{aligned}
& +2\delta_h\mu_h\eta_v m_{63}m_{67} \ + \mu_1\delta_h m_{21}(m_{66}-\mu_v) - \mu_1\mu_v\delta_h m_{21}m_{66} - \mu_1\delta_h\eta_v, \\
d_6 \ = \ & k_1\mu_v\mu_h^2 m_{22}m_{66} + \mu_h\delta_h\gamma_h m_{24} - \mu_1\mu_h\delta_h m_{21})(\mu_v m_{66} + \eta_v m_{67}) \\
& +\delta_h\mu_v\mu_h^2 m_{23}m_{66} + \mu_1\delta_h\eta_v m_{21}m_{67} + \eta_v\mu_v\delta_h\gamma_h m_{24}m_{67} \\
& -\mu_1\eta_v\delta_h m_{27}m_{61}m_{66} + \mu_h\mu_v\delta_h\gamma_h m_{24}m_{67},
\end{aligned}
$$

where

$$k_1 = \gamma_h + \mu_h + \mu_1$$

Thus the Routh-Hurwitz criterion, the above equation will give negative roots or negative real parts if the following condition are satisfied:

$$d_5 > 0, \begin{vmatrix} d_5 & d_3 \\ 1 & d_4 \end{vmatrix} > 0, \begin{vmatrix} d_5 & d_3 & d_1 \\ 1 & d_4 & d_2 \\ 0 & d_5 & d_3 \end{vmatrix} > 0,$$

$$\begin{vmatrix} d_5 & d_3 & d_1 & 0 \\ 1 & d_4 & d_2 & d_0 \\ 0 & d_5 & d_3 & d_1 \\ 0 & 1 & d_4 & d_2 \end{vmatrix} > 0, \begin{vmatrix} d_5 & d_3 & d_1 & 0 & 0 \\ 1 & d_4 & d_2 & d_0 & 0 \\ 0 & d_5 & d_3 & d_1 & 0 \\ 0 & 1 & d_4 & d_2 & d_0 \\ 0 & 0 & d_5 & d_3 & d_1 \end{vmatrix} > 0$$

,

Hence the endemic equilibrium point E_1 of the system is locally asymptotically stable, when $R_0 > 1$.

5.4. **Global Stability of Endemic Equilibrium(EE).** Here we analysis the global stability of the model (2.1) at endemic equilibrium E_1, the endemic steady state the system at E_1 is given by,

$$
\begin{aligned}
\Lambda_h &= (\beta_1 I_h^* + \beta_2 I_v^*)\frac{S_h^*}{N_h} + \mu_h S_h^*, \\
(\delta_h + \mu_h)E_h^* &= (\beta_1 I_h^* + \beta_2 I_v^*)\frac{S_h^*}{N_h}, \\
\delta_h E_h^* &= (\mu_1 + \mu_h + \gamma_h)I_h^*, \\
\frac{(\delta_h + \mu_h)(\mu_1 + \mu_h + \gamma_h)}{\delta_h} I_h^* &= (\beta_1 I_h^* + \beta_2 I_v^*)\frac{S_h^*}{N_h}, \\
\Lambda_v &= \frac{\beta_v S_v^* I_h^*}{N_h} + \mu_v S_v^*, \\
(\mu_v + \eta_v)E_v^* &= \frac{\beta_v S_v^* I_h^*}{N_h}, \\
\eta_v E_v^* &= \mu_v I_v^*, \\
\frac{\beta_v S_v^* I_h^*}{N_h} &= \frac{(\mu_v + \eta_v)\mu_v I_v^*}{\eta_v},
\end{aligned}
$$

Theorem 5.4. *If $R_0 > 1$, then the endemic equilibrium E_1 is globally asymptotically stable under some conditions.*

Proof. For the global stability of endemic equilibrium, we follow the same method described in [11]. Here we consider the following Lyapunov function:

$$L = \int_{S_h^0}^{S_h} \left(1 - \frac{S_h^0}{x}\right) dx + \int_{E_h^0}^{E_h} \left(1 - \frac{E_h^0}{x}\right) dx + \left(\frac{\delta_h + \mu_h}{\delta_h}\right) \int_{I_h^0}^{I_h} \left(1 - \frac{I_h^0}{x}\right) dx$$
$$+ \int_{S_v^0}^{S_v} \left(1 - \frac{S_v^0}{x}\right) dx + \int_{E_v^0}^{E_v} \left(1 - \frac{E_v^0}{x}\right) dx + \left(\frac{\eta_v + \mu_v}{\eta_v}\right) \int_{I_v^0}^{I_v} \left(1 - \frac{I_v^0}{x}\right) dx$$

The derivative of L along the solution of model (1) is

$$\begin{aligned} \frac{dL}{dx} &= \left(1 - \frac{S_h^0}{S_h}\right) \frac{dS_h}{dt} + \left(1 - \frac{E_h^0}{E_h}\right) \frac{dE_h}{dt} + \left(\frac{\delta_h + \mu_h}{\delta_h}\right) \left(1 - \frac{I_h^0}{I_h}\right) \frac{dI_h}{dt} \\ &+ \left(1 - \frac{S_v^0}{S_v}\right) \frac{dS_v}{dt} + \left(1 - \frac{E_v^0}{E_v}\right) \frac{dE_v}{dt} + \left(\frac{\eta_v + \mu_v}{\eta_v}\right) \left(1 - \frac{I_v^0}{I_v}\right) \frac{dI_v}{dt} \end{aligned}$$

Now from the mathematical model we put the expressions for $\frac{dS_h}{dt}$, $\frac{dE_h}{dt}$, $\frac{dI_h}{dt}, \frac{dS_v}{dt}, \frac{dE_v}{dt}, \frac{dI_v}{dt}$ in the above equation, which gives

$$\begin{aligned} \left(1 - \frac{S_h^0}{S_h}\right) \frac{dS_h}{dt} &= \left(1 - \frac{S_h^0}{S_h}\right) \left[\Lambda_h - \beta_1 \frac{S_h I_h}{N_h} - \beta_2 \frac{S_h I_v}{N_h} - \mu_h S_h\right] \\ &= \left(1 - \frac{S_h^0}{S_h}\right) \left[\mu_h S_h^* - \mu_h S_h - (\beta_1 I_h + \beta_2 I_v) \frac{S_h}{N_h}\right] \\ &\quad + \left(1 - \frac{S_h^0}{S_h}\right) \left[(\beta_1 I_h^* + \beta_2 I_v^*) \frac{S_h^*}{N_h}\right] \\ &= \mu_h S_h^* \frac{(S_h^* - S_h)^2}{S_h} + \left(1 - \frac{S_h^0}{S_h}\right) \left[(\beta_1 I_h^* + \beta_2 I_v^*) \frac{S_h^*}{N_h}\right] \\ &\quad - (\beta_1 I_h + \beta_2 I_v) \frac{S_h}{N_h} + (\beta_1 I_h^* + \beta_2 I_v^*) \frac{S_h^*}{N_h} \end{aligned}$$

$$\begin{aligned} \left(1 - \frac{E_h^0}{E_h}\right) \frac{dE_h}{dt} &= \left(1 - \frac{E_h^0}{E_h}\right) \left[\beta_1 \frac{S_h I_h}{N_h} + \beta_2 \frac{S_h I_v}{N_h} - (\delta_h + \mu_h) E_h\right] \\ &= (\beta_1 I_h + \beta_2 I_v) \frac{S_h}{N_h} - (\beta_1 I_h + \beta_2 I_v) \frac{S_h E_h^*}{N_h E_h} \\ &\quad - (\delta_h + \mu_h) E_h + (\delta_h + \mu_h) E_h^* \\ &= (\beta_1 I_h + \beta_2 I_v) \frac{S_h}{N_h} - (\beta_1 I_h + \beta_2 I_v) \frac{S_h E_h^*}{N_h E_h} \\ &\quad - (\delta_h + \mu_h) E_h + (\beta_1 I_h^* + \beta_2 I_v^*) \frac{S_h^*}{N_h} \end{aligned}$$

$$\left(\frac{\delta_h + \mu_h}{\delta_h}\right) \left(1 - \frac{I_h^0}{I_h}\right) \frac{dI_h}{dt} = \left(\frac{\delta_h + \mu_h}{\delta_h}\right) \left(1 - \frac{I_h^0}{I_h}\right) \delta_h E_h$$

$$
\begin{aligned}
& -\left(\frac{\delta_h+\mu_h}{\delta_h}\right)\left(1-\frac{I_h^0}{I_h}\right)(\gamma_h+\mu_h+\mu_1)I_{sh} \\
= \;& (\delta_h+\mu_h)E_h-(\delta_h+\mu_h)E_h\frac{I_h^*}{I_h} \\
& -\frac{(\delta_h+\mu_h)(\gamma_h+\mu_h+\mu_1)}{\delta_h}(I_h+I_h^*) \\
= \;& (\delta_h+\mu_h)E_h-(\beta_1 I_h^*-\beta_2 I_v^*)\frac{S_h^*E_hI_h^*}{N_hE_h^*I_h} \\
& -(\beta_1 I_h^*+\beta_2 I_v^*)\frac{S_h^*I_h^*}{N_hI_h}+(\beta_1 I_h^*+\beta_2 I_v^*)\frac{S_h^*}{N_h}
\end{aligned}
$$

$$
\begin{aligned}
\left(1-\frac{S_v^0}{S_v}\right)\frac{dS_v}{dt} &= \left(1-\frac{S_v^0}{S_v}\right)\left[\Lambda_v-\beta_v\frac{S_vI_h}{N_h}-\mu_vS_v\right] \\
&= \left(1-\frac{S_v^0}{S_v}\right)\left[\frac{\beta_vS_v^*I_h^*}{N_h}+\mu_vS_v^*-\frac{\beta_vS_vI_h}{N_h}-\mu_vS_v\right] \\
&= \mu_vS_v^*\frac{(S_v^*-S_v)^2}{S_v}+\left(1-\frac{S_v^0}{S_v}\right)\frac{\beta_vS_v^*I_h^*}{N_h}-\frac{\beta_vS_vI_h}{N_h}+\frac{\beta_vS_v^*I_h^*}{N_h}
\end{aligned}
$$

$$
\begin{aligned}
\left(1-\frac{E_v^0}{E_v}\right)\frac{dE_v}{dt} &= \left(1-\frac{E_v^0}{E_v}\right)\left[\beta_v\frac{S_vI_h}{N_h}-(\mu_v+\eta_v)E_v\right] \\
&= \beta_v\frac{S_vI_h}{N_h}-\beta_v\frac{S_vI_hE_v^0}{N_hE_v}-(\mu_v+\eta_v)(E_v+E_v^*) \\
&= \beta_v\frac{S_vI_h}{N_h}-\beta_v\frac{S_vI_hE_v^0}{N_hE_v}-(\mu_v+\eta_v)E_v+\frac{\beta_vS_v^*I_h^*}{N_h}
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{\eta_v+\mu_v}{\eta_v}\right)\left(1-\frac{I_v^0}{I_v}\right)\frac{dI_v}{dt} &= \left(\frac{\eta_v+\mu_v}{\eta_v}\right)\left(1-\frac{I_v^0}{I_v}\right)(\eta_vE_v-\mu_vI_v) \\
&= (\eta_v+\mu_v)E_v-\left(\frac{\eta_v+\mu_v}{\eta_v}\right)(\mu_vI_v+\mu_vI_v^*) \\
&\quad -(\eta_v+\mu_v)E_v\frac{I_v^*}{I_v} \\
&= (\eta_v+\mu_v)E_v-\frac{\beta_vS_v^*I_h^*}{N_h}\frac{E_vI_v^*}{E_v^*I_v}-\frac{\beta_vS_v^*I_h^*}{N_h}\frac{I_v^*}{I_v} \\
&\quad +\frac{\beta_vS_v^*I_h^*}{N_h}
\end{aligned}
$$

It follows that

$$
\begin{aligned}
L &= \mu_hS_h^*\frac{(S_h^*-S_h)^2}{S_h}+\mu_vS_v^*\frac{(S_v^*-S_v)^2}{S_v} \\
&+ (\beta_1 I_h^*+\beta_2 I_v^*)\frac{S_h^*}{N_h}\left[3-\frac{S_h^*}{S_h}-\frac{I_h}{I_h^*}-\frac{E_hI_h^*}{E_h^*I_h}+\frac{\beta_1 I_h+\beta_2 I_v}{\beta_1 I_h^*+\beta_2 I_v^*}\left(1-\frac{S_hE_h^*}{S_h^*E_h}\right)\right] \\
&+ \frac{\beta_vS_v^*I_h^*}{N_h}\left[3-\frac{S_v^*}{S_v}-\frac{I_v}{I_v^*}-\frac{E_vI_v^*}{E_v^*I_v}+\frac{I_v}{I_v^*}\left(1-\frac{S_vE_v^*}{S_v^*E_v}\right)\right]
\end{aligned}
$$

Therefore, the arithmetic mean (A.M.) is greater than or equal to geometric mean (G.M.), we have,

$$\left[3-\frac{S_h^*}{S_h}-\frac{I_h}{I_h^*}-\frac{E_hI_h^*}{E_h^*I_h}+\frac{\beta_1I_h+\beta_2I_v}{\beta_1I_h^*+\beta_2I_v^*}\left(1-\frac{S_hE_h^*}{S_h^*E_h}\right)\right]\leq 0$$

$$\left[3-\frac{S_v^*}{S_v}-\frac{I_v}{I_v^*}-\frac{E_vI_v^*}{E_v^*I_v}+\frac{I_v}{I_v^*}\left(1-\frac{S_vE_v^*}{S_v^*E_v}\right)\right]\leq 0$$

Thus it is easy to observed that $\frac{dL}{dt}\leq 0$ and equality $\frac{dL}{dT}=0$ hold only for $\frac{dS_h}{dS_h^*}=\frac{dE_h}{dE_h^*}=\frac{dI_h}{dI_h^*}$=1 and $\frac{dS_v}{dS_v^*}=\frac{dE_v}{dE_v^*}=\frac{dI_v}{dI_v^*}=1$ for which $S_h=S_h^*,E_h=E_h^*,I_h=I_h^*,S_v=S_v^*,E_v=E_v^*,I_v=I_v^*$
From the LaSells invariance principal [14] the endemic equilibrium E_1 of the given system is globally asymptotically stable for $R_0>1$

6. Sensitivity Analysis

In this section, we present the impact of the change in values of the parameters on the functional value of the basic reproduction number R_0. The sensitivity index of R_0 that depends differentiably on any of its parameter P as described [18, 22]

$$Y_P^{R_0}=\frac{P}{R_0}\frac{\partial R_0}{\partial P}$$

Here the parameter $\beta_1,\beta_2,\beta_v,\delta_h,\eta_v$ are the leading parameters, which control the basic reproduction number R_0. The sensitivity of R_0 are given below:

$$\begin{aligned}
Y_{\beta_1}^{R_0} &= \frac{\beta_1}{R_0}\left[\frac{\delta_h}{2D_1D_2}+\frac{\beta_1\delta_h}{2D_1D_2\sqrt{\bar{X}}}\right],\\
Y_{\beta_2}^{R_0} &= \frac{\beta_2\beta_v\delta_h\eta_v\mu_h\Lambda_v}{2R_0\Lambda_h\mu_v^2D_1D_2D_3\sqrt{\bar{X}}},\\
Y_{\beta_v}^{R_0} &= \frac{\beta_2\beta_v\delta_h\eta_v\mu_h\Lambda_v}{2R_0\Lambda_h\mu_v^2D_1D_2D_3\sqrt{\bar{X}}},\\
Y_{\delta_h}^{R_0} &= \frac{\delta_h}{R_0}\left[\frac{\beta_1}{2D_1D_2}+\frac{\beta_1\delta_h}{2D_1D_2\sqrt{\bar{X}}}\right],\\
Y_{\eta_v}^{R_0} &= \frac{\beta_2\eta_v\delta_h\eta_v\mu_h\Lambda_v}{2R_0\Lambda_h\mu_v^2D_1D_2D_3\sqrt{\bar{X}}},\\
Y_{\beta_2}^{R_0} &= Y_{\beta_v}^{R_0},
\end{aligned}$$

where,

$$\sqrt{\bar{X}}=\left(\frac{\beta_1\delta_h}{2D_1D_2}\right)^2+\frac{\beta_2\beta_v\delta_h\eta_v\mu_h\Lambda_v}{\Lambda_h\mu_v^2D_1D_2D_3}$$

As the above partial derivatives are positive, so we conclude that the basic reproduction number R_0 increases based on increase the control parameters. It is observed that $Y_{\beta_2}^{R_0}=Y_{\beta_v}^{R_0}$, hence we can conclude that minore changes in $\beta_1,\beta_2,\beta_v,\delta_h,\eta_v$, we

FIGURE 5. Influence of β_1 and β_2 on R_0

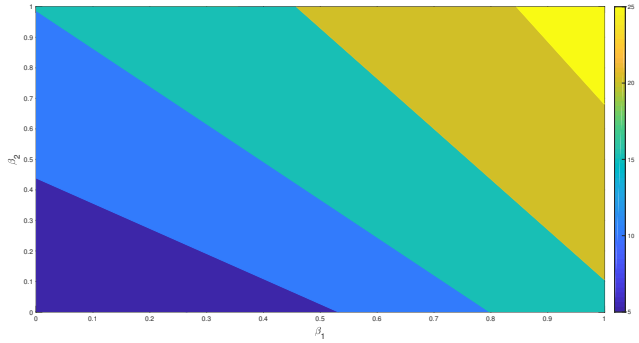
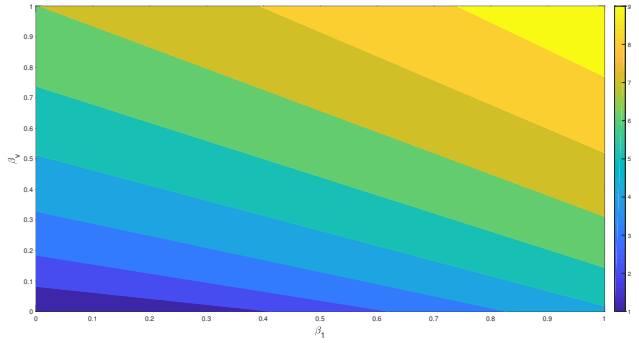


FIGURE 6. Influnce of β_1 *and* β_3 on R_0



will have same outcome on R_0. In Figures 5 ,6 and 7, we have demonestrated the effect of the parameters β_1, β_2,β_v on R_0

7. Numerical Simulation

For the Numerical simulation of the model, we consider all the parameters are in per day basis. First we consider the following set of parameters which corresponds to disease-free equilibrium.

$$\Lambda_h = 2; \Lambda_v = 40; \beta_1 = 0.05; \beta_2 = 0.05; \beta_v = 0.06;$$

$$\mu_h = 0.08; \mu_1 = 0.01; \gamma_h = 0.04; \delta_h = 0.01; \eta_v = 0.2; \mu_v = 0.1$$

For the above set of parameters we get $R_0 = 0.3162 < 1$ and the disease-free equilibrium point $E_0(35.53, 0, 0, 0, 398.33, 0, 0)$ is stable. This fact is demonstrated in Figure 8. Later, we change our parameter μ_h from 0.009 to 0.08 and γ_h from 0.009 to 0.04 and this leads to increase in R_0. Here $R_0 = 1.6165 > 1$, and the endemic equilibrium

FIGURE 7. Influence of β_2 and β_v on R_0

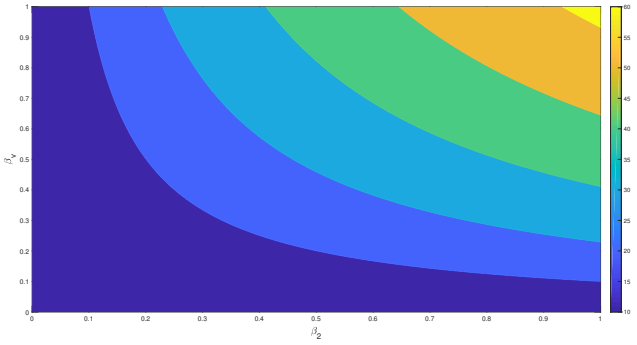
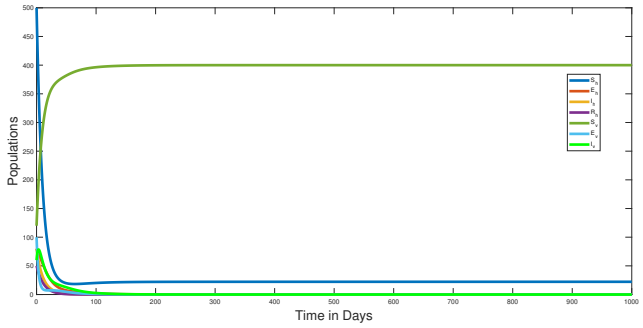


FIGURE 8. Variation of S_h E_h ,I_h,R_h S_v, E_v, I_v showing the stability of disease-free equilibrium point with $R_0 = 0.3162$.



$E_1(125.31, 51.35, 25.15, 27.31, 378.51, 12.31, 15.53)$ is stable. The stability of the equilibrium point E_1 is shown in Figure 9. The effect of different values the parameter (γ_h) which corresponds to infective human is demonstrated in Figure 10. It is clear that the parameter (γ_h) increase the infected population decreases. The effect of different values of the parameter (δ_h) which corresponds to exposed human is demonstrated in Figure 11.

8. Optimal Control Model

Here, we have extended our model (2.1) to optimal control problem by including three optimal control parameters ,namely, u_1 , u_2 and u_3. If u_1, u_2 and u_3 are equal to zero, then there is no effect being placed in these controls at time t and if they are equal to one then the maximum effect is applied. The control variable u_1 represents the reduction in the transmission between human to human. The control variable u_2 represents the use of insecticide-treated bed nets and the use of mosquito repulsive lotions and electronic devices, to reduce mosquito biting rate. The control variable u_3

FIGURE 9. Variation of S_h E_h ,I_h,R_h S_v, E_v, I_v showing the stability of endemic equilibrium point with $R_0 = 1.6165$.

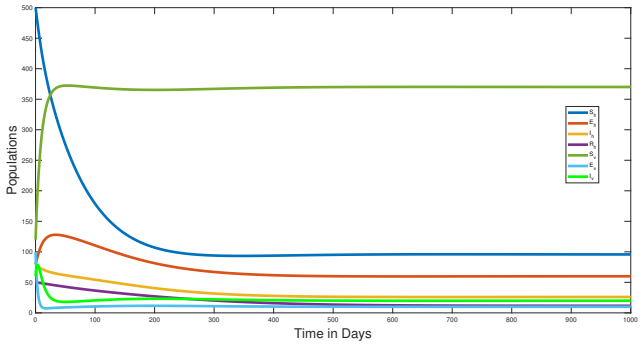


FIGURE 10. Variation of I_h with time showing different values of γ_h.

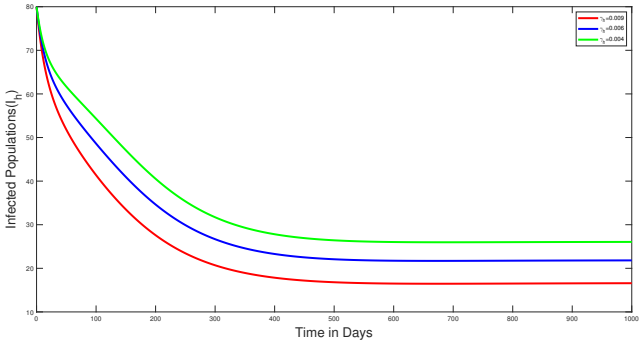
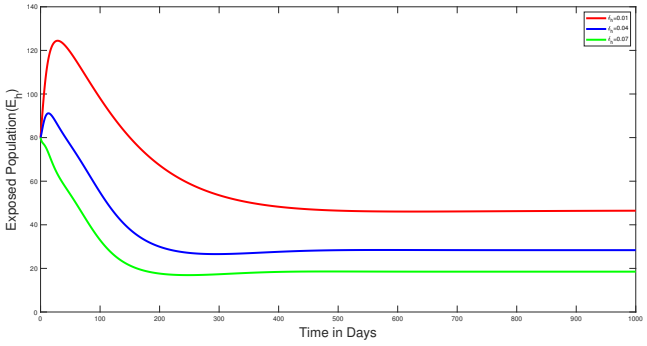


FIGURE 11. Variation of E_h with time showing different values of δ_h.



corresponds to the additional death rate of mosquitoes due to control efforts. Based on the above assumptions, the optimal control model as follows:

$$\begin{aligned}
\frac{dS_h}{dt} &= \Lambda_h - (1-u_1)\beta_1\frac{S_hI_h}{N_h} - (1-u_2)\beta_2\frac{S_hI_v}{N_h} - \mu_hS_h,\\
\frac{dE_h}{dt} &= (1-u_1)\beta_1\frac{S_hI_h}{N_h} + (1-u_2)\beta_2\frac{S_hI_v}{N_h} - (\delta_h+\mu_h)E_h,\\
\frac{dI_h}{dt} &= \delta_hE_h - (\gamma_h+\mu_h+\mu_1)I_h,\\
\frac{dR_h}{dt} &= \gamma_hI_h - \mu_hR_h,\\
\frac{dS_v}{dt} &= \Lambda_v - (1-u_2)\beta_v\frac{S_vI_h}{N_h} - (\mu_v+u_3)S_v,\\
\frac{dE_v}{dt} &= (1-u_2)\beta_v\frac{S_vI_h}{N_h} - (\mu_v+\eta_v+u_3)E_v,\\
\frac{dI_v}{dt} &= \eta_vE_v - (\mu_v+u_3)I_v,
\end{aligned} \tag{8.1}$$

8.1. **The Optimal Control Problem.** In this section, we analyze the behavior of the given model by using optimal control theory. The objective functional for fixed time t_f is given below:

$$J = \int_0^{t_f}\left[A_1(E_h+I_h) + A_2(S_v+E_v+I_v) + \frac{1}{2}A_3{u_1}^2 + \frac{1}{2}A_4u_2^2 + +\frac{1}{2}A_5u_3^2\right]dt$$

Here the parameter $A_1 \geq 0$, $A_2 \geq 0$, $A_3 \geq 0$, $A_4 \geq 0$, $A_5 \geq 0$ and they represent the weight constants.
Our objective is to find the control parameters ${u_1}^*$,${u_2}^*$, ${u_3}^*$ such that

$$J(u^*) = \min_{u\in\Omega}J(u_1,u_2,u_3), \tag{8.2}$$

where Ω is the control set and is defined as
$\Omega = \{u_1,u_2,u_3 : \text{measurable and } 0 \leq u_1 \leq 1\},\ 0 \leq u_2 \leq 1\},\ 0 \leq u_3 \leq 1\}$ and $t \in [0,t_f]$.
The Lagrangian of this problem is defined as :
$L(E_h,I_h,S_v,E_v,I_v,u_1,u_2,u_3) = A_1(E_h+I_h)+A_2(S_v+E_v+I_v)+\frac{1}{2}A_3{u_1}^2+\frac{1}{2}A_4u_2^2+\frac{1}{2}A_5u_3^2$
For our problem, we formed Hamiltonian $\mathcal{H}$:

$$\begin{aligned}
\mathcal{H} &= L(E_h,I_h,S_v,E_v,I_v,u_1,u_2,u_3) + \lambda_1\frac{dS_h}{dt} + \lambda_2\frac{dE_h}{dt} + \lambda_3\frac{dI_h}{dt} + \lambda_4\frac{dR_h}{dt}\\
&\quad +\lambda_5\frac{dS_v}{dt} + \lambda_6\frac{dE_v}{dt} + \lambda_7\frac{dI_v}{dt}
\end{aligned}$$

where λ_i,$(i = 1,2,....7)$ are the adjoint variables. Now the differential equation corresponding to adjoint variables can be written as

$$
\begin{aligned}
\frac{d\lambda_1}{dt} &= \mu_h\lambda_1 + (1-u_1)\beta_1\frac{I_h(N_h-S_h)}{N_h^2}(\lambda_1-\lambda_2)\\
&\quad +(1-u_2)\frac{\beta_2(N_h-S_h)I_v}{N_h^2}(\lambda_2-\lambda_1) + (1-u_2)\frac{k_vS_vI_h}{N_h^2}(\lambda_6-\lambda_5)\\
\frac{d\lambda_2}{dt} &= -A_1 + \mu_h\lambda_2 + (1-u_1)\beta_1\frac{S_hI_h}{N_h^2}(\lambda_2-\lambda_1) + (1-u_2)\frac{\beta_2S_hI_v}{N_h^2}(\lambda_2-\lambda_1)\\
&\quad +\delta_h(\lambda_2-\lambda_3) + (1-u_2)\frac{\beta_vS_vI_h}{N_h^2}(\lambda_6-\lambda_5)\\
\frac{d\lambda_3}{dt} &= -A_1 + (\mu_h+\mu_1)\lambda_3 + \gamma_h(\lambda_3-\lambda_4) + (1-u_1)\beta_1\frac{S_h(N_h-I_h)}{N_h^2}(\lambda_1-\lambda_2)\\
&\quad +(1-u_2)\beta_2\frac{S_hI_v}{N_h^2}(\lambda_2-\lambda_1) + (1-u_2)\frac{S_h(N_h-I_h)}{N_h^2}(\lambda_5-\lambda_6)\\
\frac{d\lambda_4}{dt} &= \mu_h\lambda_4 + (1-u_1)\frac{\beta_1S_hI_h}{N_h^2}(\lambda_2-\lambda_1) + (1-u_2)\frac{\beta_2S_hI_v}{N_h^2}(\lambda_1-\lambda_2)\\
&\quad +(1-u_2)\frac{\beta_vS_vI_h}{N_h^2}(\lambda_6-\lambda_5)\\
\frac{d\lambda_5}{dt} &= -A_2 + (\mu_v+u_3)\lambda_5 + (1-u_2)\frac{I_h}{N_h}(\lambda_5-\lambda_6)\\
\frac{d\lambda_6}{dt} &= -A_2 + (\mu_v+u_3)\lambda_6 - \eta_v(\lambda_6-\lambda_7)\\
\frac{d\lambda_7}{dt} &= -A_2 + (\mu_v+u_3)\lambda_7 + (1-u_2)\beta_2\frac{S_h}{N_h}(\lambda_1-\lambda_2) \qquad (8.3)
\end{aligned}
$$

Let $\widetilde{S_h}$, $\widetilde{E_h}$,$\widetilde{I_h}$,$\widetilde{R_h}$, $\widetilde{S_v}$, $\widetilde{E_v}$,$\widetilde{I_v}$ be the optimum values of S_h, E_h,I_h, R_h,S_v, E_v I_v respectively, and $\widetilde{\lambda_1}$, $\widetilde{\lambda_2}$, $\widetilde{\lambda_3}$,$\widetilde{\lambda_4}$,$\widetilde{\lambda_5}$,$\widetilde{\lambda_6}$,$\widetilde{\lambda_7}$ be the solution of the system (8.3)
By using [14, 15, 19], we state and prove the following theorem:

Theorem 8.1. *There exist optimal controls* $(u_1{}^*, u_2{}^*, u_3{}^*) \in \Omega$ *such that* $J(u_1{}^*, u_2{}^*, u_3{}^*) = \min J(u_1, u_2, u_3)$ *subject to system* (8.1).

Proof. To prove this theorem we use [15]. Here the state variables and the controls are positive. For this minimizing problem, the necessary convexity of the objective functional in (u_1, u_2, u_3) is satisfied. The control variable set $u_1, u_2, u_3 \in \Omega$ is also convex and closed by the definition. The integrand of the functional $A_1(E_h + I_h) + A_2(S_v + E_v + I_v) + \frac{1}{2}A_3{u_1}^2 + \frac{1}{2}A_4u_2^2 + \frac{1}{2}A_5u_3^2$ is convex on the control set Ω and the state variables are bounded.
Since there exist optimal controls for minimizing the functional subject to equations (8.1) and (8.3), we use Pontryagin's maximum principle to derive the necessary conditions to find the optimal solutions as follows:
If (x,u) is an optimal solution of an optimal control problem, then there exist a non-trivial vector function $\lambda = \lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$ satisfying the following equalities.

$$\frac{dx}{dt} = \frac{\partial H(t, x, u, \lambda)}{\partial \lambda}$$

$$0 = \frac{\partial H(t,x,u,\lambda)}{\partial u}$$

$$\frac{d\lambda}{dt} = -\frac{\partial H(t,x,u,\lambda)}{\partial x}$$

With the help of Pontryagin's maximum principle [15] and theorem (8.1), we prove the following theorem:

Theorem 8.2. *The optimal controls* $(u_1{}^*, u_2{}^*, u_3{}^*)$ *which minimizes* J *over the region* Ω *given by*

$$u_1{}^* = min\{1, max\left(0, \beta_1 \frac{S_h^* I_h^*}{A_3 N_h^*}(\lambda_1 - \lambda_2)\right)\}$$

$$u_2{}^* = min\{1, max\left(0, \frac{(\beta_2 S_h^* I_v^*)(\lambda_1 - \lambda_2) + (\beta_v S_v^* I_h^*)(\lambda_6 - \lambda_5)}{A_4 N_h^*}\right)\}$$

$$u_3{}^* = min\{1, max\left(\frac{S_v^*\lambda_5 + E_v^*\lambda_6 + I_v^*\lambda_7}{A_5}\right)\}$$

Proof. Using optimally condition :

$$\frac{\partial \mathcal{H}}{\partial u_1} = 0, \frac{\partial \mathcal{H}}{\partial u_2} = 0, \frac{\partial \mathcal{H}}{\partial u_3} = 0,$$

we get,

$$\frac{\partial \mathcal{H}}{\partial u_1} = u_1 A_3 + \beta_1 \frac{S_h^* I_h^*}{N_h^*}\lambda_2 - \beta_1 \frac{S_h^* I_h^*}{N_h^*}\lambda_1$$

This implies

$$u_1 = \beta_1 \frac{S_h^* I_h^*}{A_3 N_h^*}(\lambda_1 - \lambda_2) = \widetilde{u_1}$$

Proceeding similarly, we get

$$u_2 = \frac{(\beta_2 S_h^* I_v^*)(\lambda_1 - \lambda_2) + (\beta_v S_v^* I_h^*)(\lambda_6 - \lambda_5)}{A_4 N_h^*} = \widetilde{u_2}$$

$$u_3 = \frac{S_v^*\lambda_5 + E_v^*\lambda_6 + I_v^*\lambda_7}{A_5} = \widetilde{u_3}$$

Again upper and lower bounds for these control are 0 and 1 respectively. i.e. $u_1 = u_2 = u_3 = 0$ if $u_1 < 0$, $u_2 < 0$, $u_3 < 0$ and $u_1 = u_2 = u_3 = 1$ if $\widetilde{u_1} > 1$, $\widetilde{u_2} > 1$ and $\widetilde{u_2} > 1$ otherwise $u_1 = \widetilde{u_1}$, $u_2 = \widetilde{u_2}$ and $u_3 = \widetilde{u_3}$. Hence for these controls $u_1{}^*$,$u_2{}^*$ and $u_3{}^*$ we get optimum value of the function J.

FIGURE 12. The graph represents the susceptible humans with and without control.

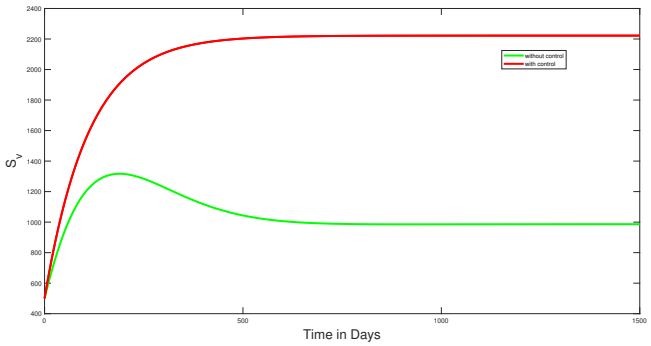


FIGURE 13. The graph represents exposed human with and without control

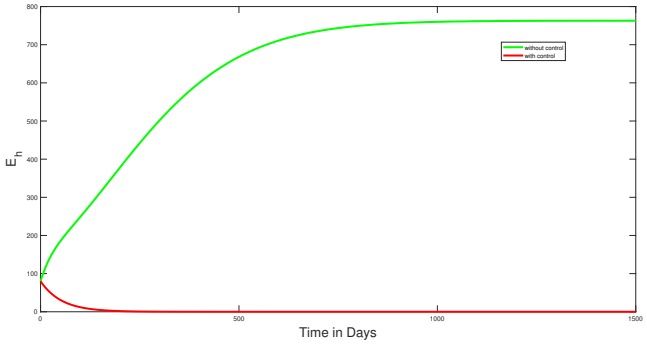


FIGURE 14. The graph represents susceptible mosquito with and without control

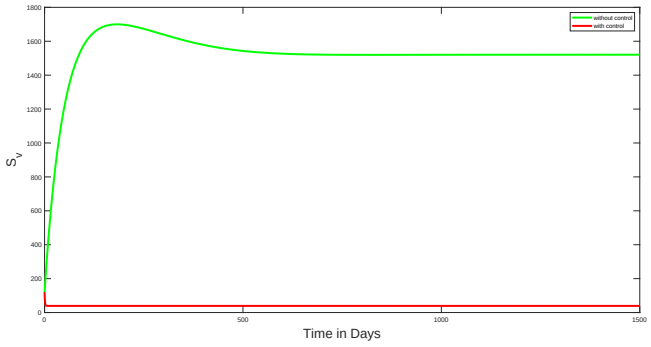
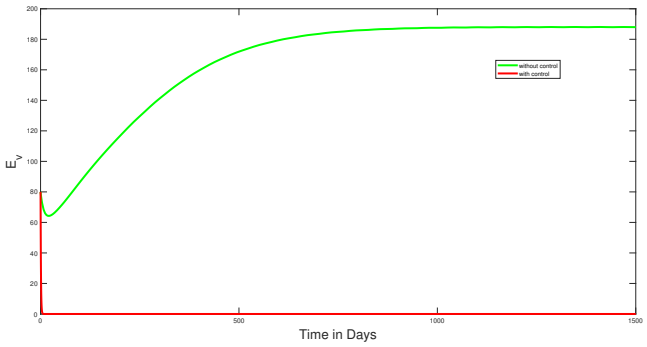


FIGURE 15. Variation of exposed mosquito with and without control



9. Numerical Simulation of Optimal Control

We simulate our optimal control model by keeping the parameters corresponding to stability of endemic equilibrium point E_1 of the model (2.1). With the help of MATLAB the optimal control model is simulated. We solve the optimality system by the iterative method with the help of forwarding and backward difference approximations [19]. Here in Figure 12, Figure 13, Figure14 and Figure 15, is plotted to observe the effects of optimal controls for susceptible humans S_h, infected humans I_h, susceptible mosquitos S_v and infected mosquitos I_v respectively are plotted to observe the effects of optimal controls against time with and without optimal control. It is easy to notice that optimal control is more effective in reducing the number of infectives is considered the period. The all three optimal control application is the best control strategy to minimize the number of infectives, which will reduce the spread of Zika virus.

10. Conclusion

In this paper, a mathematical model for the transmission dynamics of Zika virus is proposed and analyzed. For the dynamical behavior of the disease, we discussed the existence of equilibria and computed basic reproduction number (R_0) in detail. The disease-free equilibrium is locally and globally (with restrictions of parameters) asymptotically stable whenever the basic reproduction number $R_0 < 1$. Here we presented the existence of backward bifurcation which suggests that when $R_0 < 1$ is not completely sufficient for eradicating the disease from the specific region and this fact is demonstrated numerically. The backward bifurcation phenomenon has significances implications for public health practice, as it is related directly to whether or not the disease can be effectively controlled even when associated reproduction number $R_0 < 1$. Whenever the basic reproduction number $R_0 > 1$, then the endemic equilibrium is locally and globally asymptotically stable with restrictions of parameters. The sensitivity of different parameters of (R_0) is discussed and it is clear that (R_0) is very sensitive with respect to parametres involved with the model. The numerical

simulation is performed to support our mathematical results and and to compare our model with the existing model in [9]. Numerical simulation results indicate that the increase in the recovery rate of (γ_h) causes a decrease in the equilibrium level of the infective human.

We extended our model to the optimal control model and analyzed the optimal control strategy to eliminate the virus from the tropical region. All three optimal control parameters are the best control strategies to minimize the number of infectives, which will reduce the spread of the Zika virus. It is easy to notice that optimal control is more effective in reducing the number of infectives in a considered period. The control variable (u_1) represents the reduction in the transmission between human to human. The control variable (u_2) represents the use of insecticide-treated bed nets and the use of mosquito repulsive lotions and electronic devices, to reduce mosquito biting rate. The control variable (u_3) corresponds to the additional death rate of mosquitoes due to control efforts. The model is analyzed by using Pontryagin's Maximum Principle for better results. The numerical simulation is executed to observe the influence of optimal control. Finally, we can conclude that optimal control strategies give us better results to reduce the Zika virus infection.

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