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Solitary Wave solutions of the BK equation and ALWW system by using the first integral method

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Abstract Solitary wave solutions to the Broer-Kaup equations and approximate long water wave equations are considered challenging by using the first integral method.The exact solutions obtained during the present investigation are new. This method can be applied to nonintegrable equations as well as to integrable ones.

Keywords. First integral method; Broer-Kaup equations; Approximate long water wave equations.
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1. Introduction

Recently it is seen that the nonlinear phenomena are one of the most important subjects for study and they exist in all fields including either the scientific work or engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics, and so on. On the other hand, there is a lot of interest to find analytic and numerical solutions of these nonlinear equations by scientists.

In the present paper, we will seek new exact solutions of the following two nonlinear evolution equations: one is the Broer-Kaup(BK)equations [20, 21] in the form

$$u_t + uu_x + v_x = 0, \quad (1)$$
$$v_t + u_x + (uv)_x + u_{xxx} = 0,$$

which is used to model the bi-directional propagation of long waves in shallow water, and another is the approximate long water wave (ALWW) equations [21, 22, 23] in the form

$$u_t - uu_x - v_x + \alpha u_{xx} = 0, \quad (2)$$
$$v_t - (uv)_x - \alpha v_{xx} = 0,$$

where α is a real constant. The investigation of the exact solutions of nonlinear partial differential equations plays an important role in the study of

nonlinear physical phenomena. Nonlinear phenomena appear in a wide variety of scientific applications such as plasma physics, solid state physics, fluid dynamics. In order to better understand these nonlinear phenomena, many mathematicians and physical scientists make efforts to seek more exact solutions of them. Several powerful methods have been proposed to obtain exact solutions of nonlinear evolution equations, such as tanh-sech function method [1, 2, 3, 4], extended tanh function method [5, 6, 7, 8], hyperbolic function method [9], sine-cosine method [10, 11, 12], Jacobi elliptic function expansion method [13], F-expansion method [14], and the transformed rational function method [15].
The first integral method was first proposed by Feng [16] in solving Burgers-KdV equation which is based on the ring theory of commutative algebra. Recently, this useful method is widely used by many such as in [18, 19] and by the reference therein. The present paper investigates for applicability and effectiveness of the first integral method on nonlinear partial differential system.

2. Method Applied

For a given nonlinear partial differential equation

$$F(u, u_x, u_t, u_{xx}, u_{xt}, ...) = 0, \tag{3}$$

where $u = u(x,t)$ is the solution of nonlinear partial differential equation (3). We use the transformations,

$$u(x,t) = f(\xi), \tag{4}$$

where $\xi = kx + lt$. This enables us to use the following changes:

$$\frac{\partial}{\partial t}(.) = l\frac{\partial}{\partial \xi}(.),\ \frac{\partial}{\partial x}(.) = k\frac{\partial}{\partial \xi}(.),\ \frac{\partial^2}{\partial x^2}(.) = k^2\frac{\partial^2}{\partial \xi^2}(.),\ .... \tag{5}$$

using Eq.(5) to transfer the nonlinear partial differential equation (3) to nonlinear ordinary differential equation

$$G(f(\xi), \frac{\partial f(\xi)}{\partial \xi}, \frac{\partial^2 f(\xi)}{\partial \xi^2}, ...) = 0. \tag{6}$$

Next,we introduce a new independent variable

$$X(\xi) = f(\xi),\ Y = \frac{\partial f(\xi)}{\partial \xi}, \tag{7}$$

which leads a system of nonlinear ordinary differential equations

$$\frac{\partial X(\xi)}{\partial \xi} = Y(\xi), \tag{8}$$

$$\frac{\partial Y(\xi)}{\partial \xi} = F_1(X(\xi), Y(\xi)).$$

By the qualitative theory of ordinary differential equations [17] , if we can find the integrals to Eq.(8) under the same conditions, then the general solutions to Eq.(8) can be solved directly. However, in general, it is really difficult for us to realize this even for one first integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integrals, nor is there a logical way for telling us what these first integrals are. We will apply the Division Theorem to obtain one first integral to Eq.(8) which reduces Eq.(6) to a first order integrable ordinary differential equation. An exact solution to Eq.(3) is then obtained by solving this equation. Now, let us recall the Division Theorem:

Division Theorem. Suppose that $P(w,z)$ and $Q(w,z)$ are polynomials in $C[w,z]$; and $P(w,z)$ is irreducible in $C[w,z]$. If $Q(w,z)$ vanishes at all zero points of $P(w,z)$, then there exists a polynomial $G(w,z)$ in $C[w,z]$ such that

$$Q(w,z) = P(w,z)G(w,z).$$

3. Applications of the first integral method

In this section, we will construct new exact solutions of Broer-Kaup equations and approximate long water wave equations by using the method described in section 2.

3.1. Broer-Kaup equations

Considering the following transformation

$$u(x,t) = u(\xi),\ \ v(x,t) = v(\xi),\ \ \xi = kx + lt,$$

system (1) can be rewritten as

$$lu' + kuu' + kv' = 0, \tag{9}$$

$$lv' + ku' + k(uv)' + k^3u''' = 0. \tag{10}$$

Integrating (9) with respect to ξ, then we have

$$lu + \frac{k}{2}u^2 + kv = R_1, \tag{11}$$

where R_1 is integration constant. Rewrite this equation as follows

$$v(\xi) = \frac{R_1}{k} - \frac{l}{k}u(\xi) - \frac{1}{2}u^2(\xi). \tag{12}$$

Inserting Eq. (12) into Eq. (10) yields

$$k^3u''' - \frac{3k}{2}u^2u' - 3luu' + (R_1 + k - \frac{l^2}{k})u' = 0. \tag{13}$$

Integrating Eq. (13) once leads to

$$k^3u'' - \frac{k}{2}u^3 - \frac{3l}{2}u^2 + (R_1 + k - \frac{l^2}{k})u = R_2. \tag{14}$$

where R_2 is an integration constant. Rewrite this second-order ordinary differential equation as follows

$$u''(\xi) - \frac{1}{2k^2}u^3(\xi) - \frac{3l}{2k^3}u^2(\xi) + (\frac{R_1}{k^3} + \frac{1}{k^2} - \frac{l^2}{k^4})u(\xi) - \frac{R_2}{k^3} = 0. \tag{15}$$

Using (7) and (8), we can get

$$\dot{X}(\xi) = Y(\xi), \tag{16}$$

$$\dot{Y}(\xi) = \frac{1}{2k^2}X^3(\xi) + \frac{3l}{2k^3}X^2(\xi) + (\frac{l^2}{k^4} - \frac{R_1}{k^3} - \frac{1}{k^2})X(\xi) + \frac{R_2}{k^3}. \tag{17}$$

According to the first integral method, we suppose the $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (16)-(17), also

$$Q(X, Y) = \sum_{i=0}^{m} a_i(X)Y^i = 0$$

is an irreducible polynomial in the complex domain $C[X, Y]$ such that

$$Q(X(\xi), Y(\xi)) = \sum_{i=0}^{m} a_i(X(\xi))Y^i(\xi) = 0, \tag{18}$$

where $a_i(X)(i = 0, 1, ..., m)$, are polynomials of X and $a_m(X) \neq 0$. Eq. (18) is called the first integral to (16)-(17). Due to the Division Theorem, there exists a polynomial $g(X) + h(X)Y$, in the complex domain $C[X, Y]$ such that

$$\frac{dQ}{d\xi} = \frac{dQ}{dX}\frac{dX}{d\xi} + \frac{dQ}{dY}\frac{dY}{d\xi} = (g(X) + h(X)Y)\sum_{i=0}^{m} a_i(X)Y^i. \tag{19}$$

In this example, we take two different cases, assuming that $m = 1$ and $m = 2$ in (18).

Case A:

Suppose that $m = 1$, by comparing with the coefficients of $Y^i(i = 2, 1, 0)$ on both sides of (19), we have

$$\dot{a_1}(X) = h(X)a_1(X), \tag{20}$$

$$\dot{a_0}(X) = g(X)a_1(X) + h(X)a_0(X), \tag{21}$$

$$a_1(X)[\frac{1}{2k^2}X^3 + \frac{3l}{2k^3}X^2 + (\frac{l^2}{k^4} - \frac{R_1}{k^3} - \frac{1}{k^2})X + \frac{R_2}{k^3}] = g(X)a_0(X). \tag{22}$$

Since $a_i(X)$ $(i = 0, 1)$ are polynomials, then from (20) we deduce that $a_1(X)$ is constant and $h(X) = 0$. For simplicity, take $a_1(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X)) = 1$ only. Suppose that $g(X) = A_1X + B_0$, then we find $a_0(X)$,

$$a_0(X) = A_0 + B_0X + \frac{1}{2}A_1X^2, \tag{23}$$

where A_0 is arbitrary integration constant.
Substituting $a_0(X)$ and $g(X)$ into (22) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$A_1 = \frac{1}{k},\ B_0 = \frac{l}{k^2},\ R_1 = -k(A_0k + 1),\ R_2 = lkA_0, \tag{24}$$

$$A_1 = -\frac{1}{k},\ B_0 = -\frac{l}{k^2},\ R_1 = k(A_0k - 1),\ R_2 = -lkA_0, \tag{25}$$

where k, l and A_0 are arbitrary constants.
Using the conditions (24) in (18), we obtain

$$Y(\xi) = -A_0 - \frac{l}{k^2}X(\xi) - \frac{1}{2k}X^2(\xi). \tag{26}$$

Combining (26) with (16), we obtain the exact solution to equation (15) and then the exact solution to BK system (1) can be written as

$$u_1(x,t) = -\frac{l}{k} - \frac{\sqrt{2k^3A_0 - l^2}}{k}\ \tan(\frac{\sqrt{2k^3A_0 - l^2}}{2k^2}(kx + lt + \xi_0)), \tag{27}$$

$$v_1(x,t) = (-A_0k - 1 + \frac{l^2}{2k^2}) - \frac{2k^3A_0 - l^2}{2k^2}\ \tan^2(\frac{\sqrt{2k^3A_0 - l^2}}{2k^2}(kx + lt + \xi_0)),$$

where ξ_0 is an arbitrary constant.
Similarly, in the case of (25), from (18), we obtain

$$Y(\xi) = -A_0 + \frac{l}{k^2}X(\xi) + \frac{1}{2k}X^2(\xi), \tag{28}$$

and then the exact solution of BK system (1) can be written as

$$u_2(x,t) = -\frac{l}{k} - \frac{\sqrt{2k^3A_0 + l^2}}{k}\ \tanh(\frac{\sqrt{2k^3A_0 + l^2}}{2k^2}(kx + lt + \xi_0)), \tag{29}$$

$$v_2(x,t) = (A_0k - 1 + \frac{l^2}{2k^2}) - \frac{2k^3A_0 + l^2}{2k^2}\ \tanh^2(\frac{\sqrt{2k^3A_0 + l^2}}{2k^2}(kx + lt + \xi_0)),$$

where ξ_0 is an arbitrary constant.

Case B:

Suppose that $m = 2$, by equating the coefficients of $Y^i (i = 3, 2, 1, 0)$ on both sides of (19), we have

$$\begin{aligned} \dot{a_2}(X) &= h(X)a_2(X), & (30)\\ \dot{a_1}(X) &= g(X)a_2(X) + h(X)a_1(X), & (31)\\ \dot{a_0}(X) &= -2a_2(X)[\frac{1}{2k^2}X^3 + \frac{3l}{2k^3}X^2 + (\frac{l^2}{k^4} - \frac{R_1}{k^3} - \frac{1}{k^2})X + \frac{R_2}{k^3}] \\ &+ g(X)a_1(X) + h(X)a_0(X), & (32) \end{aligned}$$

$$a_1(X)[\frac{1}{2k^2}X^3 + \frac{3l}{2k^3}X^2 + (\frac{l^2}{k^4} - \frac{R_1}{k^3} - \frac{1}{k^2})X + \frac{R_2}{k^3}] = g(X)a_0(X). \tag{33}$$

Since $a_i(X)$ $(i = 0, 1, 2)$ are polynomials, then from (30) we deduce that $a_2(X)$ is constant and $h(X) = 0$. For simplicity, take $a_2(X) = 1$. Balancing the degrees of $g(X)$, $a_1(X)$ and $a_2(X)$, we conclude that $\deg(g(X)) = 1$ only. Suppose that $g(X) = A_1X + B_0$, then we find $a_1(X)$ and $a_0(X)$ as follows

$$a_1(X) = A_0 + B_0X + \frac{1}{2}A_1X^2, \tag{34}$$

$$\begin{aligned} a_0(X) &= d + \frac{1}{4}(-\frac{1}{k^2} + \frac{A_1^2}{2})X^4 + \frac{1}{3}(-\frac{3l}{k^3} + \frac{3}{2}B_0A_1)X^3 \\ &+ \frac{1}{2}(B_0^2 + A_1A_0 - \frac{2l^2}{k^4} + \frac{2}{k^2} + \frac{2R_1}{k^3})X^2 + (B_0A_0 - \frac{2R_2}{k^3})X. & (35) \end{aligned}$$

Substituting $a_0(X), a_1(X)$ and $g(X)$ in the last equation in (33) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it with aid Maple, we obtain

$$d = \frac{A_0^2}{4},\ k = \frac{2}{A_1},\ l = \frac{2B_0}{A_1^2},\ R_1 = -\frac{2(A_0 + A_1)}{A_1^2},\ R_2 = \frac{2B_0A_0}{A_1^3}, \tag{36}$$

$$d = \frac{A_0^2}{4},\ k = -\frac{2}{A_1},\ l = -\frac{2B_0}{A_1^2},\ R_1 = \frac{2(A_0 + A_1)}{A_1^2},\ R_2 = -\frac{2B_0A_0}{A_1^3}, \tag{37}$$

with A_0, B_0 and A_1 are arbitrary constants.
Using the conditions (36) and (37) into (18), we get

$$Y(\xi) = -\frac{A_0}{2} - \frac{B_0}{2}X(\xi) - \frac{A_1}{4}X^2(\xi). \tag{38}$$

Combining (38) with (16), we obtain the exact solution to equation (15) and then the exact solution to BK system (1) can be written as

$$u_3(x,t) = -\frac{B_0}{A_1} \mp \frac{\sqrt{2A_0A_1 - B_0^2}}{A_1} \tan[\frac{\sqrt{2A_0A_1 - B_0^2}}{4}(\frac{2}{A_1}x + \frac{2B_0}{A_1^2}t + \xi_0)], \tag{39}$$

$$v_3(x,t) = (\frac{B_0^2}{2A_1^2} - \frac{A_0 + A_1}{A_1}) \mp \frac{2A_0A_1 - B_0^2}{2A_1^2} \tan^2[\frac{\sqrt{2A_0A_1 - B_0^2}}{4}(\frac{2}{A_1}x + \frac{2B_0}{A_1^2}t + \xi_0)],$$

where ξ_0 is an arbitrary constant.

3.2. Approximate long water wave equations

Now, we will consider the approximate long water wave equations (2). Making the transformation $u(x,t) = u(\xi),\ v(x,t) = v(\xi)$ and $\xi = kx + lt$, we change the ALWW system (2) to the following ODEs

$$lu' - kuu' - kv' + \alpha k^2 u'' = 0, \tag{40}$$

$$lv' - k(uv)' - \alpha k^2 v'' = 0. \tag{41}$$

Integrating (40) with respect to ξ, then we have

$$lu - \frac{k}{2}u^2 - kv + \alpha k^2 u' = R_1, \tag{42}$$

where R_1 is integration constant. Rewrite this equation as follows

$$v(\xi) = \frac{l}{k}u - \frac{1}{2}u^2 + \alpha k u' - \frac{R_1}{k}. \tag{43}$$

Inserting Eq. (43) into Eq. (41) yields

$$(\frac{l^2}{k} + R_1)u' - 3luu' + \frac{3k}{2}u^2u' - \alpha^2k^3u''' = 0. \tag{44}$$

Integrating Eq. (44) once leads to

$$(\frac{l^2}{k} + R_1)u - \frac{3l}{2}u^2 + \frac{k}{2}u^3 - \alpha^2k^3u'' = R_2, \tag{45}$$

where R_2 is an integration constant. Rewrite this second-order ordinary differential equation as follows

$$u'' - \frac{1}{2\alpha^2k^2}u^3 + \frac{3l}{2\alpha^2k^3}u^2 - (\frac{l^2}{\alpha^2k^4} + \frac{R_1}{\alpha^2k^3})u + \frac{R_2}{\alpha^2k^3} = 0. \tag{46}$$

Using (7) and (8), we can get

$$\dot{X}(\xi) = Y(\xi), \tag{47}$$

$$\dot{Y}(\xi) = \frac{1}{2\alpha^2k^2}X^3(\xi) - \frac{3l}{2\alpha^2k^3}X^2(\xi) + (\frac{l^2}{\alpha^2k^4} + \frac{R_1}{\alpha^2k^3})X(\xi) - \frac{R_2}{\alpha^2k^3}. \tag{48}$$

According to the first integral method, we suppose the $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (47)-(48), also

$$Q(X,Y)=\sum_{i=0}^{m}a_i(X)Y^i=0$$

is an irreducible polynomial in the complex domain $C[X,Y]$ such that

$$Q(X(\xi),Y(\xi))=\sum_{i=0}^{m}a_i(X(\xi))Y^i(\xi)=0, \tag{49}$$

where $a_i(X)(i=0,1,...,m)$, are polynomials of X and $a_m(X)\neq 0$. Eq. (49) is called the first integral to (47)-(48). Due to the Division Theorem, there exists a polynomial $g(X)+h(X)Y$, in the complex domain $C[X,Y]$ such that

$$\frac{dQ}{d\xi}=\frac{dQ}{dX}\frac{dX}{d\xi}+\frac{dQ}{dY}\frac{dY}{d\xi}=(g(X)+h(X)Y)\sum_{i=0}^{m}a_i(X)Y^i. \tag{50}$$

In this example, we take two different cases, assuming that $m=1$ and $m=2$ in (49).

Case A:

Suppose that $m=1$, by comparing with the coefficients of $Y^i(i=2,1,0)$ on both sides of (50), we have

$$\dot{a_1}(X) = h(X)a_1(X), \tag{51}$$
$$\dot{a_0}(X) = g(X)a_1(X)+h(X)a_0(X), \tag{52}$$

$$a_1(X)[\frac{1}{2\alpha^2k^2}X^3(\xi)-\frac{3l}{2\alpha^2k^3}X^2(\xi)+(\frac{l^2}{\alpha^2k^4}+\frac{R_1}{\alpha^2k^3})X(\xi)-\frac{R_2}{\alpha^2k^3}]=g(X)a_0(X). \tag{53}$$

Since $a_i(X)$ $(i=0,1)$ are polynomials, then from (51) we deduce that $a_1(X)$ is constant and $h(X)=0$. For simplicity, take $a_1(X)=1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X))=1$ only. Suppose that $g(X)=A_1X+B_0$, then we find $a_0(X)$,

$$a_0(X)=A_0+B_0X+\frac{1}{2}A_1X^2, \tag{54}$$

where A_0 is arbitrary integration constant.

Substituting $a_0(X)$ and $g(X)$ into (53) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$A_1=\frac{1}{k\alpha},\ B_0=-\frac{l}{k^2\alpha},\ R_1=k^2\alpha A_0,\ R_2=l\alpha kA_0, \tag{55}$$

where k, l and A_0 are arbitrary constants.
Using the conditions (55) in (49), we obtain

$$Y(\xi) = -A_0 + \frac{l}{\alpha k^2} X(\xi) - \frac{1}{2\alpha k} X^2(\xi). \tag{56}$$

Combining (56) with (47), we obtain the exact solution to equation (46) and then the exact solution to ALWW system (2) can be written as

$$u_1(x,t) = \frac{l}{k} - \frac{\sqrt{2\alpha k^3 A_0 - l^2}}{k} \tan(\frac{\sqrt{2\alpha k^3 A_0 - l^2}}{2k^2\alpha}(kx + lt + \xi_0)), \tag{57}$$

$$v_1(x,t) = (\frac{l^2}{2k^2} - \frac{2\alpha k^3 A_0 - l^2}{2k^2} - \alpha k A_0) - \frac{2\alpha k^3 A_0 - l^2}{k^2} \tan^2(\frac{\sqrt{2\alpha k^3 A_0 - l^2}}{2\alpha k^2}(kx + lt + \xi_0)),$$

where ξ_0 is an arbitrary constant.

Case B:

Suppose that $m = 2$, by equating the coefficients of $Y^i (i = 3, 2, 1, 0)$ on both sides of (50), we have

$$\begin{aligned}
\dot{a_2}(X) &= h(X)a_2(X), && (58)\\
\dot{a_1}(X) &= g(X)a_2(X) + h(X)a_1(X), && (59)\\
\dot{a_0}(X) &= -2a_2(X)[\frac{1}{2\alpha^2 k^2}X^3(\xi) - \frac{3l}{2\alpha^2 k^3}X^2(\xi) + (\frac{l^2}{\alpha^2 k^4} + \frac{R_1}{\alpha^2 k^3})X(\xi) - \frac{R_2}{\alpha^2 k^3}]\\
&+ g(X)a_1(X) + h(X)a_0(X), && (60)
\end{aligned}$$

$$a_1(X)[\frac{1}{2\alpha^2 k^2}X^3(\xi) - \frac{3l}{2\alpha^2 k^3}X^2(\xi) + (\frac{l^2}{\alpha^2 k^4} + \frac{R_1}{\alpha^2 k^3})X(\xi) - \frac{R_2}{\alpha^2 k^3}] = g(X)a_0(X). \tag{61}$$

Since $a_i(X)$ $(i = 0, 1, 2)$ are polynomials, then from (58) we deduce that $a_2(X)$ is constant and $h(X) = 0$. For simplicity, take $a_2(X) = 1$. Balancing the degrees of $g(X)$, $a_1(X)$ and $a_2(X)$, we conclude that $\deg(g(X)) = 1$ only. Suppose that $g(X) = A_1 X + B_0$, then we find $a_1(X)$ and $a_0(X)$ as follows

$$a_1(X) = A_0 + B_0 X + \frac{1}{2}A_1 X^2, \tag{62}$$

$$\begin{aligned}
a_0(X) &= d + \frac{1}{4}(-\frac{1}{\alpha^2 k^2} + \frac{A_1^2}{2})X^4 + \frac{1}{3}(\frac{3l}{\alpha^2 k^3} + \frac{3}{2}B_0 A_1)X^3\\
&+ \frac{1}{2}(B_0^2 + A_1 A_0 - \frac{2l^2}{\alpha^2 k^4} - \frac{2R_1}{\alpha^2 k^3})X^2 + (B_0 A_0 + \frac{2R_2}{\alpha^2 k^3})X. && (63)
\end{aligned}$$

Substituting $a_0(X), a_1(X)$ and $g(X)$ in the last equation in (61) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it with aid Maple, we obtain

$$d = \frac{A_0^2}{4}, \ k = \frac{2}{\alpha A_1}, \ l = -\frac{2B_0}{\alpha A_1^2}, \ R_1 = \frac{2A_0}{\alpha A_1^2}, \ R_2 = -\frac{2B_0 A_0}{\alpha A_1^3}, \tag{64}$$

with A_0, B_0 and A_1 are arbitrary constants.
Using the conditions (64) into (49), we get

$$Y(\xi) = -\frac{A_0}{2} - \frac{B_0}{2}X(\xi) - \frac{A_1}{4}X^2(\xi). \tag{65}$$

Combining (65) with (47), we obtain the exact solution to equation (46) and then the exact solution to ALWW system (2) can be written as

$$u_2(x,t) = -\frac{B_0}{A_1} - \frac{\sqrt{2A_0A_1 - B_0^2}}{A_1} \tan[\frac{\sqrt{2A_0A_1 - B_0^2}}{4}(\frac{2}{\alpha A_1}x - \frac{2B_0}{\alpha A_1^2}t + \xi_0)], \tag{66}$$

$$\begin{aligned} v_2(x,t) =& (\frac{B_0^2}{2A_1^2} - \frac{A_0}{A_1} - \frac{2A_0A_1 - B_0^2}{2A_1^2}) \\ &- \frac{2A_0A_1 - B_0^2}{A_1^2} \tan^2[\frac{\sqrt{2A_0A_1 - B_0^2}}{4}(\frac{2}{\alpha A_1}x - \frac{2B_0}{\alpha A_1^2}t + \xi_0)], \end{aligned}$$

where ξ_0 is an arbitrary constant.

4. Conclusion:

In this present work we have presented a number of solitary wave solutions to the Broer-Kaup equations and approximate long water wave equations. The first integral method is a very powerful method for finding exact solutions of the nonlinear differential equations. From our results, we can see that the technique used in this paper is very effective and can be steadily applied to nonlinear problems.

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