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Lie symmetry analysis for Kawahara-KdV equations

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Abstract We introduce a new solution for Kawahara-KdV equations. The Lie group analysis is used to carry out the integration of this equations. The similarity reductions and exact solutions are obtained based on the optimal system method.

Keywords. Lie symmetries, Symmetry analysis, Optimal system, Infinitesimal generators, Kawahara-KdV equations.

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1. Introduction

Nonlinear evolution equations (NLEEs) recently have been used to model many physical phenomena in various fields such as fluid mechanics, solid state physics, plasma physics, chemical physics, optical fiber and geochemistry. It is important to investigate the exact explicit solutions of NLEEs. Finding solutions of such an equation is an arduous task and only in certain special cases one can write down the solutions explicitly. However, a massive amount of work has been done in the past few decades and important progress has been made for obtaining exact solutions of NLEEs. In order to obtain the exact solutions, a number of methods have been proposed in the literature, some of the well known methods include the solitary wave ansatz method, the inverse scattering, Hirotas bilinear method, homogeneous balance method, Lie group analysis, etc. Among the above mentioned methods, the Lie group analysis method, which is also called the symmetry method, is one of the most effective methods to determine solutions of nonlinear partial differential equations [9].

The purpose of this paper is to use Lie group analysis to obtain some exact solutions of the Kawahara-KdV equation that is given by:

$$u_t + uu_x + u_{xxx} - \gamma u_{xxxxx} = 0, \tag{1}$$

and the modified Kawahara-KdV equation:

$$u_t + 3u^2 u_x + u_{xxx} - \mu u_{xxxxx} = 0, \tag{2}$$

and finally, the Rosenau-Kawahara equation:

$$u_t + u_{xxxxt} + u_x + uu_x + u_{xxx} - u_{xxxxx} = 0. \tag{3}$$

where $u := u(x,t)$ is a real function for all $x, t \in \mathbb{R}$ and $\gamma, \mu \geq 0$. These equations occur in the theory of magneto-acoustic waves in plasmas and propagation of nonlinear water-waves in the long-wave length region as in the case of KdV's equations. Moreover, the Kawahara equation is a model for small-amplitude gravity capillary waves on water of a finite depth when the Weber number is close to $\frac{1}{3}$ [12]. Lie group analysis is a very powerful tool for studying general properties of differential equations and for finding their solutions. A number of papers, including [4, 5, 10, 11, 13, 14, 15, 17], have been devoted the application of Lie group analysis to equations in financial mathematics.

2. Lie symmetry analysis

We shall now present Lie group method for Eqs. (1), (2), and (3). A Lie point symmetry of a partial differential equation (PDE) is an invertible transformation of the dependent and independent variables that leaves the equation unchanged. In general, determining all the symmetries of a partial differential equation is a formidable task [9]. Once one has determined the symmetry group of a system of differential equation, a number of applications become available. To start with, one can directly use the defining property of such a group and construct a new solutions to the system from known ones.

Let us consider a one-parameter Lie group of infinitesimal transformation for Kawahara-KdV equation:

$$\begin{aligned} x &\longrightarrow x + \epsilon\xi^1(x,t,u), \\ t &\longrightarrow t + \epsilon\xi^2(x,t,u), \\ u &\longrightarrow u + \epsilon\phi(x,t,u), \end{aligned} \tag{4}$$

with a small parameter $\epsilon \ll 1$, where the ξ^1 , ξ^2 and ϕ are the infinitesimals of the transformations for independent and dependent variable. The vector field associated with the above group of transformations can be written as

$$V = \xi^1(x,t,u)\partial_x + \xi^2(x,t,u)\partial_t + \phi(x,t,u)\partial_u. \tag{5}$$

Applying the fifth prolongation $pr^{(5)}V$ to Eq. (1), we will find the following determining equations:

$$
\begin{aligned}
&\xi^1_u = 0,\\
&\xi^2_u = 0,\\
&\phi_{uu} = 0,\\
&\xi^2_x = 0,\\
&-5\gamma\phi_{xu} + 10\gamma\xi^1_{xx} = 0,\\
&-\gamma\xi^2_t + 5\gamma\xi^1_x = 0,\\
&3\phi_{xu} - 3\xi^1_{xx} - 10\gamma\phi_{xxxu} + 5\gamma\xi^1_{xxxx} = 0,\\
&\phi_t + u\phi_x + \phi_{xxx} - \gamma\phi_{xxxxx} = 0,\\
&\xi^2_t - 3\xi^1_x - 10\gamma\phi_{xxu} + 10\gamma\xi^1_{xxx} = 0,\\
&\phi - \xi^1_t + u\xi^2_t - u\xi^1_x + 3\phi_{xxu} - \xi^1_{xxx} - 5\gamma\phi_{xxxxu} + \gamma\xi^1_{xxxxx} = 0.
\end{aligned} \tag{6}
$$

After solving the determining equations for the symmetry group of these equations, we have the following forms of the coefficient functions:

$$\xi^1 = c_1 + tc_2, \qquad \xi^2 = c_3, \qquad \phi = c_2,$$

where c_1, c_2 and c_3 are arbitrary constants. The result shows that the symmetry of Eq. (1) expressed by a finite three-dimensional point group containing translation in the independent variables and scaling transformations. The Lie algebra of infinitesimal symmetries of Eq. (1) is spanned by the following vectors:

$$V_1 = \frac{\partial}{\partial x}, \qquad V_2 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \qquad V_3 = \frac{\partial}{\partial t}.$$

This symmetry vector fields constitute a three-dimensional Lie algebra by

$$
\begin{aligned}
&[V_1, V_2] = [V_1, V_3] = [V_2, V_1] = [V_3, V_1] = 0,\\
&[V_3, V_2] = -V_1, \qquad [V_2, V_3] = V_1,
\end{aligned} \tag{7}
$$

the commutator table for V_i will be a 3×3 table, whose (i, j)-th entry expresses the Lie bracket $[V_i, V_j]$. The coefficient $C_{i,j,k}$ is the coefficient of V_i of the (i, j)-th entry of the table. The commutator Table 1, is resulted for those operators.

Table 1: *Commutator for the Lie algebra L of the Kawahara − KdV Equation.*

	V_1	V_2	V_3
V_1	0	0	0
V_2	0	0	V_1
V_3	0	$-V_1$	0

So,

$$C_{3,2,1} = -1 \qquad C_{2,3,1} = 1.$$

Remark 2.1. *V_1 is the casimir operator.*

Remark 2.2. *According to Table 1, it can be easily shown that the given algebra is solvable.*

Remark 2.3. *For modified Kawahara-KdV equation the Lie algebra is trivial.*

The symmetry group of Kawahara-KdV equation (3) will be generated by the vector field of the form

$$V = \xi^1(x,t,u)\frac{\partial}{\partial x} + \xi^2(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u},$$

where ξ^1, ξ^2 and ϕ depend on x, t and u. Applying the fifth prolongation $pr^{(5)}$ to (3) gives the following overdetermined system of linear partial differential equations:

$$
\begin{aligned}
&\xi^1_u = 0,\\
&\xi^2_u = 0,\\
&\phi_{uu} = 0,\\
&\xi^2_x = 0,\\
&4\phi_{xu} - 6\xi^1_{xx} = 0,\\
&6\phi_{xxu} - 4\xi^1_{xxx} = 0,\\
&-4\phi_{xxxu} - \xi^1_{xxxx} = 0,\\
&-4\xi^1_x - \phi_{xxxxu} = 0,\\
&\phi_{tu} - 5\phi_{xu} - 4\xi^1_{xt} + 10\xi^1_{xx} = 0,\\
&\phi_t + (1+u)\phi_x + \phi_{xxx} + \phi_{xxxxt} - \phi_{xxxxx} = 0,\\
&-\xi^1_t - \xi^2_t + 5\xi^1_x + \phi_{xxxxu} = 0,\\
&\xi^2_t - 3\xi^1_x + 4\phi_{xtu} - 10\phi_{xxu} - 6\xi^1_{xxt} + 10\xi^1_{xxx} - \phi_{xxxxu} = 0,\\
&3\phi_{xu} - 3\xi^1_{xx} + 6\phi_{xxtu} - 10\phi_{xxxu} - 4\xi^1_{xxxt} + 5\xi^1_{xxxx} = 0.
\end{aligned}
\tag{8}
$$

The infinitesimals are in an explicit form:

$$\xi^1 = k_2, \qquad \xi^2 = k_1, \qquad \phi = 0,$$

where k_1 and k_2 are arbitrary constants. The Lie algebra of infinitesimal symmetries of Eq. (3) is spanned by the following two vectors:

$$V_1 = \frac{\partial}{\partial t}, \qquad V_2 = \frac{\partial}{\partial x}.$$

Remark 2.4. *V_1 and V_2 are the casimir operator.*

Remark 2.5. *The given algebra is solvable.*

3. Classification of Group-Invariant Solutions

Recall first that, in general, each one parameter subgroup of the full symmetry group of a system will be correspond to a family of solutions, such solutions are called invariant solutions [16]. In this paper we are interesting only on the symmetry algebra $\mathfrak{g}$ of the Kawahara-KdV equation which is spanned by the vector fields V_1, V_2 and V_3, every one-dimensional subalgebra of $\mathfrak{g}$ is determined by a nonzero vector V of the form:

$$V = a_1V_1 + a_2V_2 + a_3V_3, \tag{9}$$

where a_i are arbitrary constants. We try to simplify many of the coefficients a_i as far as possible through application of adjoint to V. In order to obtaining the adjoint representation, we use the Lie series:

$$Ad(\exp(\varepsilon V_i))V_j = V_j - \varepsilon[V_i, V_j] + \frac{\varepsilon^2}{2}[V_i, [V_i, V_j]] - \cdots.$$

Furthermore, we can compute the adjoint representations of the vector fields. For the Kawahara-KdV equation, we have:

$$Ad(\exp(\varepsilon V_i))V_i = V_i, \qquad i = 1, 2, 3,$$

and

$$Ad(\exp(\varepsilon V_1))V_2 = V_2, \qquad Ad(\exp(\varepsilon V_1))V_3 = V_3,$$

$$Ad(\exp(\varepsilon V_2))V_1 = V_1, \qquad Ad(\exp(\varepsilon V_2))V_3 = V_3 - \varepsilon V_1,$$

$$Ad(\exp(\varepsilon V_3))V_2 = V_2 + \varepsilon V_1, \qquad Ad(\exp(\varepsilon V_3))V_1 = V_1.$$

Then from the mentioned computation and the commutation Table 1, we will obtain the following Table 2:

Table 2: *Adjoint representation of the Lie algebrag of the Kawahara−KdV Equation.*

Ad	V_1	V_2	V_3
V_1	V_1	V_2	V_3
V_2	V_1	V_2	$V_3 - \varepsilon V_1$
V_3	V_1	$V_2 + \varepsilon V_1$	V_3

Theorem 3.1. *The optimal system of one-dimensional subalgebras corresponds to (1) is expressed by*

(1) V_1,
(2) V_2,
(3) $\alpha V_2 + V_3$, *where* $\alpha \in \{-1, 0, 1\}$.

Proof. Let

$$V = a_1V_1 + a_2V_2 + a_3V_3, \tag{10}$$

be a nonzero vector field, we shall simplify V by using the adjoint representation given in Table 2. Let $a_3 \neq 0$, If we put $a_3 = 1$ and $V' = Ad(\exp(\varepsilon V_2))$ then we will have $V' = (a_1 - \varepsilon)V_1 + a_2V_2 + V_3$. Now if $\varepsilon = a_1$, the coefficient V_1 vanish, therefore $V' = a_2V_2 + V_3$.
The remaining one-dimensional subalgebras are spanned by vectors of the (10) from with $a_3 = 0$. By taking $\varepsilon = -a_1$, where a_1, a_2 are different from zero, we obtain $V' = Ad(\exp(-a_1V_3))V = V_2$. □

In what follows, we will have:

Case.1
Suppose first that $a_3 \neq 0$, we can assume $a_3 = 1$, so the vector filed V takes the form:

$$V = a_1V_1 + a_2V_2 + V_3,$$

refereing to the Table 2. if we act on such a V by $Ad(\exp(a_1V_2))$ then we can make the coefficient of V_1 equal to zero.
So the reduced vector field takes the form:

$$V = \alpha(t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}) + \frac{\partial}{\partial t}.$$

The invariants ς and χ can be found by integrating the corresponding characteristic system, which is:

$$\frac{dx}{\alpha t} = \frac{du}{\alpha} = \frac{dt}{1}, \tag{11}$$

where the solution are given by:

$$\chi = \alpha t^2 - 2x, \qquad and \qquad \varsigma = u - \alpha t.$$

The derivatives of u are given in terms of ς and χ as:

$$\begin{aligned} u_t &= 2\alpha t\varsigma' + \alpha; \\ u_x &= -2\varsigma'; \\ u_{xxx} &= -8\varsigma^{(3)}; \\ u_{xxxxx} &= -32\varsigma^{(5)}; \end{aligned} \tag{12}$$

Substituting the equations (12) into the Kawahara-KdV equation we obtain:

$$\alpha - 2\varsigma\varsigma' - 8\varsigma^{(3)} + 32\varsigma^{(5)} = 0. \tag{13}$$

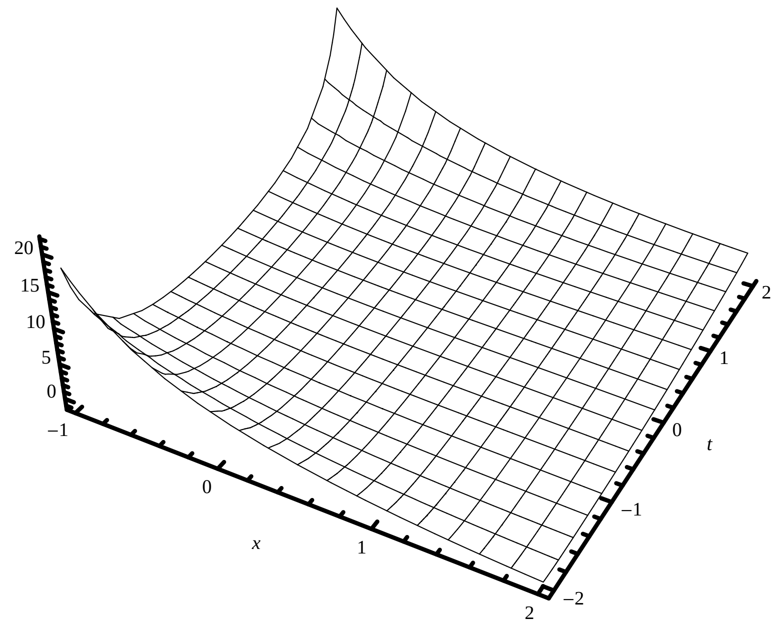

FIGURE 1. Plot of $u(t,x)$ obtained by initial conditions $\varsigma(0) = 1$, $\varsigma'(0) = 1$ and $\varsigma''(0) = 0$. and $\varsigma'''(0) = 0$, and $\varsigma^{(4)}(0) = 0$ for $\alpha = -1$.

Please see Fig.1. for more information.

Case.2

Suppose first that $a_3 = 0$ and $a_2 \neq 0$, we can assume that $a_2 = 1$, vector filed (9) takes the form:

$$V = a_1 V_1 + V_2.$$

Let $\varepsilon = -a_1$, we get

$$V' = Ad(\exp(-a_1 V_3))V = V_2,$$

in this case the corresponding vector field is:

$$V = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u},$$

so we have,

$$\frac{dx}{t} = \frac{du}{1}.$$

The solution are given by:

$$\chi = t, \qquad and \qquad \varsigma = u - \frac{x}{t}.$$

Finally the derivatives of u are given in terms of ς and χ, by the following relation:

$$u_t = \varsigma' - \frac{x}{t^2}; \tag{14}$$

$$u_x = \frac{1}{t}; \tag{15}$$

$$u_{xxx} = 0; \tag{16}$$

$$u_{xxxxx} = 0. \tag{17}$$

By substituting above equations into the Kawahara-KdV equation we obtain:

$$\chi\varsigma' + \varsigma = 0,$$

therefore

$$\varsigma = \frac{c}{t}.$$

Plot of $u(x,t)$ is shown in Fig 2.

Case.3

If $a_2 = 0, a_1 \neq 0$, we can assume that $a_1 = 1$, vector filed (9) takes the form:

$$V = V_1$$

therefore we have:

$$V = \frac{\partial}{\partial x}$$

and the invariants are:

$$\chi = t, \qquad and \qquad \varsigma = u.$$

The derivatives of u are given in terms of ς and χ as:

$$u_t = \varsigma'; \tag{18}$$

$$u_x = 0; \tag{19}$$

$$u_{xxx} = 0; \tag{20}$$

$$u_{xxxxx} = 0. \tag{21}$$

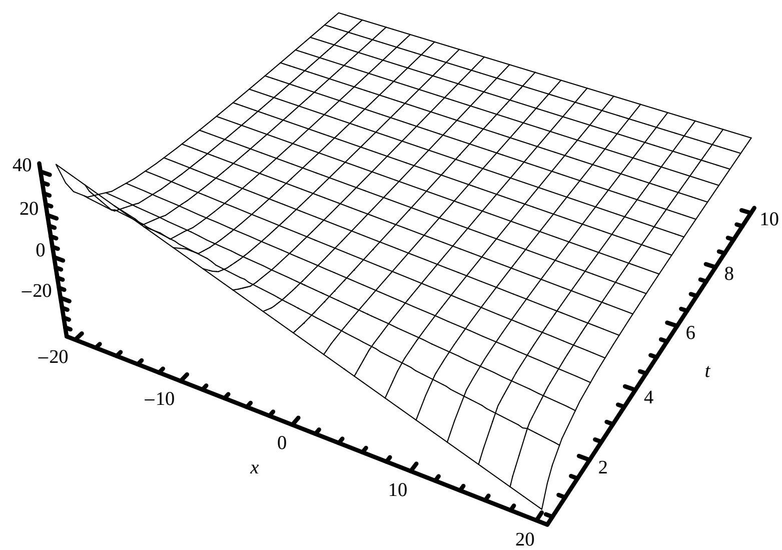

FIGURE 2. Plot of $u(t, x)$ obtained by initial conditions $\varsigma(0) = 1$ and $c = 1$.

By substituting above equations into the Kawahara-KdV equation we obtain:

$$\varsigma' = 0,$$

therefore,

$$\varsigma = c, \qquad and \qquad u = 0,$$

which is a trivial solution.

4. Conclusion

In this paper by using the criterion of invariance of the equation under the infinitesimal prolonged infinitesimal generators, we find the most Lie point symmetries group of the Kawahara-KdV equation. Looking the adjoint representation of the obtained symmetry group on its Lie algebra, we have find the preliminary classification of group-invariants solution. We have seen that the obtained reduced equation in such case can be transformed on known equation by using an appropriate change of the variables.

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