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Existence of bound states for non-local fourth-order Kirchhoff systems

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Abstract This paper is concerned with existence of three solutions for non-local fourth-order Kirchhoff systems with Navier boundary conditions. Our technical approach is based on variational methods and the theory of the variable exponent Sobolev spaces.

Keywords. Navier condition, $(p(x), q(x))$-biharmonic systems, Variational method, Three critical points theorem, Variable exponent Sobolev spaces.

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1. Introduction

The purpose of this article is to establish the existence of multiple solutions of the following nonlocal elliptic system

$$\begin{cases} -M_1\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\right)\Delta\left(|\Delta u|^{p(x)-2}\Delta u\right) = \lambda F_u(x,u,v), & in\ \ \Omega, \\ -M_2\left(\int_\Omega \frac{1}{q(x)}|\Delta v|^{q(x)}dx\right)\Delta\left(|\Delta v|^{q(x)-2}\Delta u\right) = \lambda F_v(x,u,v), & in\ \ \Omega, \\ u = v = \Delta u = \Delta v = 0, & on\ \ \partial\Omega. \end{cases} \tag{1.1}$$

where $\Omega \subset (\mathbb{R}^N)$ $(N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $p(.),\ q(.) \in C^0(\overline{\Omega})$, $M_1,\ M_2$ are continuous functions, $\lambda > 0$ and $F \in C^0(\Omega \times \mathbb{R}^2)$.

We confine ourselves to the case, where $M_1 = M_2 := M$ for simplicity. Notice that the results we prove in what follows remain valid for $M_1 \neq M_2$ by adding some hypothesis on M_1 and M_2. We give some new criteria for guaranteeing that the problem (1.1) have at least three weak solutions by using a variational method and some critical point theorems due to Ricceri. Three critical points theorem of B.Ricceri has been widely used to solve differential equations, see for example [18, 20, 25].

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The fourth-order boundary value problem of nonlinearity furnishes a model to study traveling waves in suspension bridges, so it is important to physics. Recently, the existence of solutions to fourth-order boundary value problems have been studied in many papers. Molica Bisci and Repovš in [27] employing variational methods, studied the existence of multiple weak solutions for fourth- order elliptic equations. In [17, 24, 26], based on variational methods and critical point theory, the existence of multiple solutions for a class of elliptic Navier boundary problems.

Problem (1.1) is called a non-local problem because of the presence of the term M, which implies that the equation in (1.1) is no longer a pointwise identity.

Non-local operators can be seen as the infinitesimal generators of Lévy stable diffusion processes [2]. Moreover, they allow us to develop a generalization of quantum mechanics and also to describe the motion of a chain or an array of particles that are connected by elastic springs as well as unusual diffusion processes in turbulent fluid motions and material transports in fractured media(for more details see for example [1, 2, 20] and the references therein). Non-local differential equations are also called Kirchhoff-type equations, the study of Kirchhoff-type problems, which arise in various models of physical and biological systems, have received more attention in recent years. More precisely, Kirchhoff established a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2\right)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.2}$$

where $\rho,\ \rho_0,\ h,\ E\ and\ L$ are constants which represent some physical meanings. Equation (1.2) extends the classical D'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. see for example [15, 29, 38]. There are also many existence results on stationary Kirchhoff problems. For example, Autuori and Fiscella [3] obtained the existence of the asymptotic behavior of non-negative solutions for a class of stationary Kirchhoff problems driven by a fractional integro-differential operator. Baraket and Molica Bisci [6] proved the existence of multiple solutions for a perturbed Kirchhoff-type problem depending on two real parameters. Fiscella and Valdinocib [16] proved that the existence of non-negative solutions for a Kirchhoff type problem driven by a non-local integro-differential operator.

In [21] the authors established the existence of a weak solution for the following system:

$$\begin{cases} -\left[M_1\left(\displaystyle\int_\Omega|\Delta u|^p dx\right)\right]^{p-1}\Delta_p u=f(u,v)+\rho_1(x), & in\ \ \Omega,\\ -\left[M_2\left(\displaystyle\int_\Omega|\Delta v|^q dx\right)\right]^{p-1}\Delta_q v=f(x,v)+\rho_2(x), & in\ \ \Omega,\\ \dfrac{\partial u}{\partial\eta}=\dfrac{\partial v}{\partial\eta}=0, & on\ \ \partial\Omega,\end{cases} \tag{1.3}$$

where $M_1(t), M_2(t)\geq m_0>0$. We also recall that non-homogeneous $p(x)$-Kirchhoff operators have been used in the last decades to model various phenomena, see [8, 35] and the references therein. Indeed, recently, there has been an increasing interest in

studying systems involving somehow non-homogeneous $p(x)$-Laplace operators, motivated by the image restoration problem, by the modeling of electro-rheological fluids. The study of elliptic problems involving $p(x)$-biharmonic operators has interested in recent years, for the existence and multiplicity of solutions see [19, 22, 23, 28] for some recent work on this subject.
In [1] the authors established the existence and multiplicity of solutions for the following system:

$$\begin{cases} -M_1\Big(\displaystyle\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\Big)\Delta\Big(|\Delta u|^{p(x)-2}\Delta u\Big) = F_u(x,u,v), & in\ \ \Omega, \\ -M_2\Big(\displaystyle\int_\Omega \frac{1}{q(x)}|\Delta v|^{q(x)}dx\Big)\Delta\Big(|\Delta v|^{q(x)-2}\Delta u\Big) = F_v(x,u,v), & in\ \ \Omega, \\ u = v = \Delta u = \Delta v = 0, & on\ \ \partial\Omega. \end{cases} \tag{1.4}$$

Motivated by the above works, we are devoted to the existence of three solutions to problem (1.1). The article is organized as follows. We first present some necessary preliminary results on variable exponent Sobolev spaces. Next, we give the main results about the existence and multiplicity of weak solutions.

2. Preliminaries

For the reader's convenience, we recall some background facts concerning the variable exponent Lebesgue and Sobolev spaces and introduce some notation. For more details, we refer the reader to [12, 13, 14, 30] and the references therein. Consider the set

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}), h(x) > 1\ \forall x \in \overline{\Omega}\},$$

and for every $h \in C_+(\overline{\Omega})$, we define

$$\max\{2, \frac{N}{2}\} < h^- := \min\{h(x); x \in \overline{\Omega}\} \leq h^+ := \max\{h(x); x \in \overline{\Omega}\}.$$

For every $p \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(\cdot)}(\Omega) = \Big\{u : \Omega \to \mathbb{R}\ measurable\ and\ \int_\Omega |u(x)|^{p(x)}dx < \infty\Big\}.$$

This vector space is a Banach space if it is endowed with the Luxemburg norm, which is defined by

$$|u|_{p(\cdot)} = inf\Big\{\mu > 0; \int_\Omega |\frac{u(x)}{\mu}|^{p(x)} \leq 1\Big\}.$$

Proposition 2.1. [31, 39] $(L^{p(\cdot)}(\Omega), |u|_{p(\cdot)})$ *is separable, uniformly convex, reflexive and its dual space is* $L^{p'(\cdot)}(\Omega)$ *where* $p'(\cdot)$ *is the conjugate function of* $p(.)$, *i.e.*

$$\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1.$$

Moreover, for $u \in L^{p(\cdot)}(\Omega)$ *and* $v \in L^{p'}(\cdot)(\Omega)$

$$\Big|\int_\Omega u(x)v(x)dx\Big| \leq (\frac{1}{p^-}+\frac{1}{p'^-})|u|_{p(\cdot)}|v|_{p'(\cdot)} \leq 2|u|_{p(\cdot)}|v|_{p'(\cdot)}.$$

If L is a positive integer and $p \in C_+(\overline{\Omega})$, we define the variable exponent Sobolev space by

$$W^{L,p(\cdot)}(\Omega) = \Big\{u \in L^{p(\cdot)}; D^\alpha u \in L^{p(\cdot)}, |\alpha| \leq L\Big\},$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1}x_1 \dots \partial^{\alpha_N}x_N}$ for $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$ which is a multi-index and $|\alpha| = \sum_{i=1}^N \alpha_i$. The space $W^{L,p(\cdot)}(\Omega)$ equipped with the norm

$$\|u\|_{L,p(\cdot)} = \sum_{|\alpha|\leq L} |D^\alpha u|_{p(\cdot)},$$

becomes a separable, reflexive uniformly convex Banach space.
The space $W_0^{L,p(\cdot)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{L,p(\cdot)}(\Omega)$.

Proposition 2.2. [9] $W_0^{L,p(\cdot)}(\Omega)$ *is a separable, uniformly convex and reflexive Banach space.*

The function space $(W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega))$ is a separable and reflexive Banach space, where

$$\|u\|_{p(\cdot)} = \inf\Big\{\mu > 0 \ : \ \int_\Omega |\frac{\Delta u(x)}{\mu}|^{p(x)} \leq 1\Big\}. \tag{2.1}$$

Remark 2.3. According to [37], the norm $\|.\|_{2,p(\cdot)}$ is equivalent to the norm $|\Delta.|_{p(\cdot)}$ in the space $W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)$. Consequently, the norms $\|.\|_{2,p(\cdot)}$, $\|.\|_{p(\cdot)}$ and $|\Delta.|_{p(\cdot)}$ are equivalent.

In the following, we will use $\|.\|_{p(\cdot)}$ instead of $\|.\|_{2,p(\cdot)}$ on $W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)$. Similarly, we use $\|.\|_{q(\cdot)}$ instead of $\|.\|_{2,q(\cdot)}$ on $W^{2,q(\cdot)}(\Omega) \cap W_0^{1,q(\cdot)}(\Omega)$.
We denote by

$$X := \Big(W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)\Big) \times \Big(W^{2,q(\cdot)}(\Omega) \cap W_0^{1,q(\cdot)}(\Omega)\Big), \tag{2.2}$$

equipped with the norm

$$\|(u,v)\| = \|u\|_{p(\cdot)} + \|v\|_{q(\cdot)}. \tag{2.3}$$

Proposition 2.4. [31] *Let*

$$\rho(u) := \int_\Omega |u|^{p(x)}dx.$$

For $u, u_n \in L^{p(\cdot)(\Omega)}$, *we have,*

(1) $|u|_{p(\cdot)} < 1 (= 1; > 1) \Leftrightarrow \rho(u) < 1 (= 1; > 1)$,
(2) $|u|_{p(\cdot)} > 1 \Rightarrow |u|^{p^-} \leq \rho(u) \leq |u|^{p^+}$,

(3) $|u|_{p(\cdot)} < 1 \Rightarrow |u|^{p^+} \le \rho(u) \le \|u\|^{p^-}$,
(4) $|u_n|_{p(\cdot)} \to 0 \Leftrightarrow \rho(u_n) \to 0$,
(5) $|u_n|_{p(\cdot)} \to \infty \Leftrightarrow \rho(u_n) \to \infty$.

From Proposition 2.4 for $u \in L^{p(.)(\Omega)}$ the following inequalities hold:

$$\|u\|^{p^-} \le \int_\Omega |\Delta u|^{p(x)} dx \le \|u\|^{p^+}, \quad if\ \|u\| \ge 1, \tag{2.4}$$

$$\|u\|^{p^+} \le \int_\Omega |\Delta u|^{p(x)} dx \le \|u\|^{p^-}, \quad if\ \|u\| \le 1. \tag{2.5}$$

Proposition 2.5. [36] *If $\Omega \subset \mathbb{R}^N$ is a bounded domain, then the imbedding $W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ is compact whenever $\frac{N}{2} < p^-$.*

From Proposition 2.5, we know that when $p^-, q^- > \frac{N}{2}$, the embedding $X \hookrightarrow C^0(\overline{\Omega}) \times C^0(\overline{\Omega})$ is compact, and there exists a positive constant c such that

$$\|(u,v)\|_\infty = \|u\|_\infty + \|v\|_\infty \le c\|(u,v)\|, \ \forall (u,v) \in X. \tag{2.6}$$

Hereafter $M(t)$ is supposed to verify the following assumption:

($\mathbf{M}_1$) There exist $m_2 \ge m_1 > 0$ and $\alpha > 1$ such that

$$m_1 t^{\alpha-1} \le M(t) \le m_2 t^{\alpha-1}, \quad \forall t \in \mathbb{R}^+.$$

Put

$$\widehat{M}(t) = \int_0^t M(\tau) d\tau \quad (\forall t \in \mathbb{R}^+).$$

We have

$$\frac{m_1}{\alpha} t^\alpha \le \widehat{M}(t) \le \frac{m_2}{\alpha} t^\alpha. \tag{2.7}$$

Now, for every $(u,v) \in X$, we define the functionals Φ and Ψ

$$\Phi(u,v) = \widehat{M}\Big(\int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx\Big) + \widehat{M}\Big(\int_\Omega \frac{1}{q(x)} |\Delta v|^{q(x)} dx\Big),$$

$$\Psi(u,v) = \int_\Omega F(x,u,v) dx.$$

Standard arguments show that $I_1 = \Phi - \lambda\Psi$ is well defined on X and it is of class C^1 and for any $(z,w) \in X$,

$$\begin{aligned}\Phi'(u,v)(z,w) = {} & M\Big(\int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx\Big) \int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta z dx \\ & + M\Big(\int_\Omega \frac{1}{q(x)} |\Delta v|^{q(x)} dx\Big) \int_\Omega |\Delta v|^{q(x)-2} \Delta v \Delta w dx, \ \ \forall (u,v) \in X.\end{aligned}$$

$$\Psi'(u,v)(z,w) = -\int_\Omega F_u(x,u,v) z dx - \int_\Omega F_v(x,u,v) w dx. \ \forall (u,v) \in X.$$

$(u, v) \in X$ is called a (weak) solution of the problem (1.1) if

$$\Phi'(u, v)(z, w) - \lambda\Psi'(u, v)(z, w) = 0,$$

for every $(z, w) \in X$. We observe that a vector $(u, v) \in X$ is a solution of the problem (1.1) if and only if (u, v) is a critical point of the function I_1.

Lemma 2.6. [10] *Let* $I(u) = \int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx$.
We have the following assertions:

1. I' *is a bounded homeomorphism and strictly monotone operator.*
2. I' *is a mapping of type* (S_+)*, namely*

$$u_n \rightharpoonup u \quad \text{and} \quad \limsup_{n\to+\infty} I'(u_n)(u_n - u) \leq 0 \quad \text{implies} \quad u_n \to u.$$

Let X be a nonempty set and $\Phi, \Psi : X \to \mathbb{R}$ be two functions. For all $r, r_1, r_2 > \inf_X \Phi$, $r_2 > r_1$ and $r_3 > 0$,

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{(\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)) - \Psi(u)}{r - \Phi(u)},$$

$$\beta(r_1, r_2) := \inf_{u\in\Phi^{-1}(-\infty,r_1)} \sup_{v\in\Phi^{-1}[r_1,r_2)} \frac{\Psi(v) - \Psi(u)}{\Phi(v) - \Phi(u)},$$

$$\gamma(r_2, r_3) := \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)} \Psi(u)}{r_3},$$

$$\alpha(r_1, r_2, r_3) := \max\left\{\varphi(r_1), \varphi(r_2), \gamma(r_2, r_3)\right\}.$$

A central role in our arguments will be played by the three critical points theorem [4, Theorem 5.2]. For the reader's convenience we here recall as follows.

Theorem 2.7. *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gteaux differentiable functional whose Gâteaux derivative is compact, such that*

($\mathfrak{M}_1$) $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$,
($\mathfrak{M}_2$) *for every $u_1, u_2 \in X$ such that $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$, one has*

$$\inf_{s\in[0,1]} \Psi(su_1 + (1 - s)u_2) \geq 0.$$

Assume that there are three positive constants r_1, r_2, r_3 with $r_1 < r_2$, such that

($\mathfrak{M}_3$) $\alpha(r_1, r_2, r_3) < \beta(r_1, r_2)$.

Then, for each $\lambda \in]\frac{1}{\beta(r_1,r_2)}, \frac{1}{\alpha(r_1,r_2,r_3)}[$ the functional $\Phi - \lambda\Psi$ admits three critical points u_1, u_2, u_3 such that $u_1 \in \Phi^{-1}(]-\infty, r_1[)$, $u_2 \in \Phi^{-1}(]r_1, r_2[)$ and $u_3 \in \Phi^{-1}(]-\infty, r_2 + r_3[)$.

Definition 2.8. We say that $u \in X$ is a bound state of (1.1) if u is a critical point of I_1. A bound state $\tilde{u}$ is called ground state if its energy is minimal among all the bound states, namely

$$I_1(\tilde{u}) = \min \Big\{ I_1(u) : u \in X \setminus \{0\},\ I_1'(u) = 0 \Big\}.$$

3. Main results

In this section we establish our main result on the existence of at least three weak solutions for problem (1.1).

Fix $x^\star \in \Omega$ and choice a_1, a_2 with $0 < a_1 < a_2$, such that $B(x^\star, a_2) \subseteq \Omega$ denotes the open ball in Ω of radius a_2 and center $x^\star$. Put

$$\begin{aligned} \varsigma_p &:= \max \Big\{ [\frac{12(N+2)^2(a_1+a_2)}{(a_1-a_2)^3}]^{p^-}, [\frac{12(N+2)^2(a_1+a_2)}{(a_1-a_2)^3}]^{p^+} \Big\} \\ &\times \frac{2k^{p^-}\pi^{\frac{N}{2}}(a_2^N - a_1^N)}{\Gamma(1+\frac{N}{2})}, \end{aligned}$$

and

$$\begin{aligned} \varrho_p &:= \min \Big\{ [\frac{12(N+2)^2(a_1+a_2)}{(a_1-a_2)^3}]^{p^-}, [\frac{12(N+2)^2(a_1+a_2)}{(a_1-a_2)^3}]^{p^+} \Big\} \\ &\times \frac{2k^{p^-}\pi^{\frac{N}{2}}(a_2^N - a_1^N)}{\Gamma(1+\frac{N}{2})}, \end{aligned}$$

where $\Gamma(.)$ is Gamma function.

Lemma 3.1. *Assume that M satisfies (M_1) and there exist positive constants θ_i, ϑ_i and $\eta, \kappa \geq 1$ in which for $1 \leq i \leq 3$ such that*

$(\mathfrak{M}_4)$ $\theta_1 < \varrho_p^{\frac{1}{p^-}} \eta, \quad \eta < \min\{(\frac{p^+}{\varsigma_p} p^-)^{\frac{1}{p^+}} \theta_2^{\frac{p^-}{p^+}}, \theta_2\}$ *and* $\theta_2 < \theta_3$,

$(\mathfrak{M}_5)$ $\vartheta_1 < \varrho_q^{\frac{1}{q^-}} \kappa, \quad \kappa < \min\{(\frac{q^+}{\varsigma_q} q^-)^{\frac{1}{q^+}} \vartheta_2^{\frac{q^-}{q^+}}, \vartheta_2\}$ *and* $\vartheta_2 < \vartheta_3$.

Then there exist two positive constants r_1, r_2 and $(w_\eta, w_\kappa) \in X$ such that

$$r_1 \leq \Phi(w_\eta, w_\kappa) \leq r_2. \tag{3.1}$$

Proof. Let

$$w_\eta(x) := \begin{cases} 0, & x \in \overline{\Omega} \setminus B(x^0, a_2), \\ \frac{\eta(3(l^4-a_2^4)-4(a_1+a_2)(l^3-a_2^3)+6a_1a_2(l^2-a_2^2))}{(a_2-a_1)^3(a_1+a_2)}, & x \in B(x^0, a_2) \setminus B(x^0, a_1), \\ \eta, & x \in B(x^0, a_1), \end{cases}$$

where $l = dist(x, x^0) = \sqrt{\sum_{i=1}^N (x_i - x_i^0)^2}$. We have

$$\frac{\partial w_\eta(x)}{\partial x_i} = \begin{cases} 0, & x \in \overline{\Omega} \setminus B(x^0, a_2) \cup B(x^0, a_1), \\ \frac{12\eta(l^2(x_i-x_i^0)-l(a_1+a_2)(x_i-x_i^0)+a_1a_2(x_i-x_i^0))}{(a_2-a_1)^3(a_1+a_2)}, & x \in B(x^0, a_2) \setminus B(x^0, a_1), \end{cases}$$

$$\frac{\partial^2 w_\eta(x)}{\partial x_i^2} = \begin{cases} 0, & x \in \overline{\Omega} \setminus B(x^0, a_2) \cup B(x^0, a_1), \\ \frac{12\eta(a_1a_2+(2l-a_1-a_2)(x_i-x_i^0)^2/l-(a_1+a_2-l)l)}{(a_2-a_1)^3(a_1+a_2)}, & x \in B(x^0, a_2) \setminus B(x^0, a_1), \end{cases}$$

$$\sum_{i=1}^N \frac{\partial^2 w_\eta(x)}{\partial x_i^2} = \begin{cases} 0, & x \in \overline{\Omega} \setminus B(x^0, a_2) \cup B(x^0, a_1), \\ \frac{12\eta((N+2)l^2-(N+1)(a_1+a_2)l+Na_1a_2)}{(a_2-a_1)^3(a_1+a_2)}, & x \in B(x^0, a_2) \setminus B(x^0, a_1). \end{cases}$$

It is easy to verify that $(w_\eta, w_\kappa) \in X$ and in particular,

$$\frac{\varrho_p \eta^{p^-}}{p^+ c^{p^-}} \leq \int_{B(x^0,a_2)\setminus B(x^0,a_1)} \frac{1}{p(x)} |\Delta w_\eta(x)|^{p(x)} \leq \frac{\varsigma_p \eta^{p^+}}{p^- c^{p^-}},$$

and

$$\frac{\varrho_q \kappa^{q^-}}{q^+ c^{q^-}} \leq \int_{B(x^0,a_2)\setminus B(x^0,a_1)} \frac{1}{q(x)} |\Delta w_\kappa(x)|^{q(x)} \leq \frac{\varsigma_q \kappa^{q^+}}{q^- c^{q^-}}.$$

By (2.7), one has

$$\frac{m_1}{\alpha}(\frac{\varrho_p \eta^{p^-}}{p^+ c^{p^-}})^\alpha + \frac{m_1}{\alpha}(\frac{\varrho_q \kappa^{q^-}}{q^+ c^{q^-}})^\alpha \leq \Phi(w_\eta, w_\kappa) \leq \frac{m_2}{\alpha}(\frac{\varsigma_p \eta^{p^+}}{p^- c^{p^-}})^\alpha + \frac{m_2}{\alpha}(\frac{\varsigma_q \kappa^{q^+}}{q^- c^{q^-}})^\alpha.$$

Choose

$$r_1 = \frac{m_1}{\alpha}(\frac{1}{p^+}(\frac{\theta_1}{c})^{p^-})^\alpha + \frac{m_1}{\alpha}(\frac{1}{q^+}(\frac{\vartheta_1}{c})^{q^-})^\alpha, \tag{3.2}$$

and

$$r_2 = \frac{m_1}{\alpha}(\frac{1}{p^+}(\frac{\theta_2}{c})^{p^-})^\alpha + \frac{m_1}{\alpha}(\frac{1}{q^+}(\frac{\vartheta_2}{c})^{q^-})^\alpha. \tag{3.3}$$

By the assumptions $(\mathfrak{M}_4)$ and $(\mathfrak{M}_5)$, we obtain $r_1 < \Phi(w_\eta, w_\kappa) < r_2$. □

Lemma 3.2. *If conditions (M_1), $(\mathfrak{M}_4)$, $(\mathfrak{M}_5)$ and*

$(\mathfrak{M}_6)$ *$f(x,s,t) \geq 0$, for each $(x,s,t) \in \Omega \times [-\theta_3, \theta_3] \times [-\vartheta_3, \vartheta_3]$,*

$(\mathfrak{M}_7)$

$$\max\left\{ \frac{\int_\Omega F(x,\theta_1,\vartheta_1)}{\frac{m_1}{\alpha}(\frac{\theta_1^{p^-}}{p^+})^\alpha + \frac{m_1}{\alpha}(\frac{\vartheta_1^{q^-}}{q^+})^\alpha}, \frac{\int_\Omega F(x,\theta_2,\vartheta_2)}{\frac{m_1}{\alpha}(\frac{\theta_2^{p^-}}{p^+})^\alpha + \frac{m_1}{\alpha}(\frac{\vartheta_2^{q^-}}{q^+})^\alpha}, \frac{\int_\Omega F(x,\theta_3,\vartheta_3)}{\frac{m_1}{\alpha}(\frac{\theta_3^{p^-}-\theta_2^{p^-}}{p^+})^\alpha + \frac{m_1}{\alpha}(\frac{\vartheta_3^{q^-}-\vartheta_2^{q^-}}{q^+})^\alpha} \right\}$$
$$< \frac{\int_{B(x^0,a_1)} F(x,\eta,\kappa)dx - \int_\Omega F(x,\theta_1,\vartheta_1)dx}{\frac{m_2}{\alpha}(\frac{\varsigma_p \eta^{p^+}}{p^-})^\alpha + \frac{m_2}{\alpha}(\frac{\varsigma_q \kappa^{q^+}}{p^-})^\alpha},$$

are satisfied, then there exists one positive constant r_3 such that

$$\alpha(r_1, r_2, r_3) < \beta(r_1, r_2).$$

Proof. From $(\mathfrak{M}_6)$ and the definition of Ψ, we see that

$$\Psi(w_\eta, w_\kappa) = \int_\Omega F(x, w_\eta(x), w_\kappa(x))dx \geq \int_{B(x^0,a_1)} F(x,\eta,\kappa)dx. \tag{3.4}$$

Let

$$r_3 = \frac{m_1}{\alpha}(\frac{1}{p^+}(\frac{\theta_3^{p^-} - \theta_2^{p^-}}{c^{p^-}}))^\alpha + \frac{m_1}{\alpha}(\frac{1}{q^+}(\frac{\vartheta_3^{q^-} - \vartheta_2^{q^-}}{c^{q^-}}))^\alpha. \tag{3.5}$$

From the conditions $(\mathfrak{M}_4)$, $(\mathfrak{M}_5)$, we have $\theta_2 < \theta_3$ and $\vartheta_2 < \vartheta_3$, we achieve $r_3 > 0$. For all $(u, v) \in X$ with $\Phi(u, v) < r_1$, from (2.4) and (2.5),

$$\|u\|_{p(.)} \leq \max\left\{(\frac{\alpha(p^+)^\alpha}{m_1}(\frac{m_1}{\alpha}(\frac{1}{p^+}(\frac{\theta_1}{c})^{p^-})^\alpha))^{\frac{1}{\alpha p^+}}, (\frac{\alpha(p^+)^\alpha}{m_1}(\frac{m_1}{\alpha}(\frac{1}{p^+}(\frac{\theta_1}{c})^{p^-})^\alpha))^{\frac{1}{\alpha p^-}}\right\},$$

and

$$\|v\|_{q(.)} \leq \max\left\{(\frac{\alpha(q^+)^\alpha}{m_1}(\frac{m_1}{\alpha}(\frac{1}{q^+}(\frac{\vartheta_1}{c})^{q^-})^\alpha))^{\frac{1}{\alpha q^+}}, (\frac{\alpha(q^+)^\alpha}{m_1}(\frac{m_1}{\alpha}(\frac{1}{q^+}(\frac{\vartheta_1}{c})^{q^-})^\alpha))^{\frac{1}{\alpha q^-}}\right\}.$$

So, by Proposition 2.5, we have $\|u\|_\infty < \theta_1$ and $\|v\|_\infty < \vartheta_1$. From the definition of r_1, it follows that

$$\Phi^{-1}(-\infty, r_1) = \{(u, v) \in X; \Phi(u, v) < r_1\} \subseteq \{(u, v) \in X; |u| \leq \theta_1,\ |v| \leq \vartheta_1\}.$$

Thus, by using assumption $(\mathfrak{M}_6)$,

$$\begin{aligned} \sup_{(u,v)\in\Phi^{-1}(-\infty,r_1)} \int_\Omega F(x, u(x), v(x))dx &\leq \int_\Omega \sup_{\{|s|\theta_1, |t| \leq \vartheta_1\}} F(x, s, t)dx \\ &\leq \int_\Omega F(x, \theta_1, \vartheta_1)dx. \end{aligned} \tag{3.6}$$

Similarly,

$$\sup_{(u,v)\in\Phi^{-1}(-\infty,r_2)} \int_\Omega F(x, u(x), v(x))dx \leq \int_\Omega F(x, \theta_2, \vartheta_2)dx, \tag{3.7}$$

$$\sup_{(u,v)\in\Phi^{-1}(-\infty,r_2+r_3)} \int_\Omega F(x, u(x), v(x))dx \leq \int_\Omega F(x, \theta_3, \vartheta_3)dx. \tag{3.8}$$

Hence, since $(0, 0) \in \Phi^{-1}(-\infty, r_1)$ and $\Phi(0, 0) = \Psi(0, 0) = (0, 0)$, considering (3.2) and (3.6), one has

$$\begin{aligned} \varphi(r_1) &= \inf_{(u,v)\in\Phi^{-1}(-\infty,r_1)} \frac{\sup_{(u,v)\in\Phi^{-1}(-\infty,r_1)} \Psi(u, v) - \Psi(u, v)}{r_1 - \Phi(u, v)} \\ &\leq \frac{\sup_{(u,v)\in\Phi^{-1}(-\infty,r_1)} \Psi(u, v)}{r_1} \\ &= \frac{\sup_{(u,v)\in\Phi^{-1}(-\infty,r_1)} \int_\Omega F(x, u(x), v(x))dx}{r_1} \\ &\leq \frac{\int_\Omega F(x, \theta_1, \vartheta_1)dx}{\frac{m_1}{\alpha}(\frac{1}{p^+}(\frac{\theta_1}{c})^{p^-})^\alpha + \frac{m_1}{\alpha}(\frac{1}{q^+}(\frac{\vartheta_1}{c})^{q^-})^\alpha}. \end{aligned}$$

As above, we can obtain that

$$\begin{aligned}\varphi(r_2) &= \inf_{(u,v)\in\Phi^{-1}(-\infty,r_2)} \frac{\sup_{(u,v)\in\Phi^{-1}(-\infty,r_2)}\Psi(u,v)-\Psi(u,v)}{r_2-\Phi(u,v)}\\ &\le \frac{\sup_{(u,v)\in\Phi^{-1}(-\infty,r_2)}\Psi(u,v)}{r_2}\\ &= \frac{\sup_{(u,v)\in\Phi^{-1}(-\infty,r_2)}\int_\Omega F(x,u(x),v(x))dx}{r_2}\\ &\le \frac{\int_\Omega F(x,\theta_2,\vartheta_2)dx}{\frac{m_2}{\alpha}(\frac{1}{p^+}(\frac{\theta_2}{c})^{p^-})^\alpha+\frac{m_1}{\alpha}(\frac{1}{q^+}(\frac{\vartheta_2}{c})^{q^-})^\alpha},\end{aligned}$$

and

$$\begin{aligned}\gamma(r_2,r_3) &\le \frac{\sup_{(u,v)\in\Phi^{-1}(-\infty,r_2+r_3)}\Psi(u,v)}{r_3}\\ &= \frac{\sup_{(u,v)\in\Phi^{-1}(-\infty,r_2+r_3)}\int_\Omega F(x,u(x),v(x))dx}{r_3}\\ &\le \frac{\int_\Omega F(x,\theta_3,\vartheta_3)dx}{\frac{m_1}{\alpha}(\frac{1}{p^+}(\frac{\theta_3^{p^-}-\theta_2^{p^-}}{c^{p^-}}))^\alpha+\frac{m_1}{\alpha}(\frac{1}{q^+}(\frac{\vartheta_3^{q^-}-\vartheta_2^{q^-}}{c^{q^-}}))^\alpha}.\end{aligned}$$

Moreover, for each $(u,v)\in\Phi^{-1}(-\infty,r_1)$ one has

$$\begin{aligned}\beta(r_1,r_2) &\ge \frac{\int_{B(x^0,a_1)}F(x,\eta,\kappa)dx-\int_\Omega F(x,\theta_1,\vartheta_1)dx}{\Phi(w_\eta,w_\kappa)-\Phi(u,v)}\\ &\ge \frac{\int_{B(x^0,a_1)}F(x,\eta,\kappa)dx-\int_\Omega F(x,\theta_1,\vartheta_1)dx}{\frac{m_2}{\alpha}(\frac{S_p\eta^{p^+}}{p^-c^{p^-}})^\alpha+\frac{m_2}{\alpha}(\frac{S_q\kappa^{q^+}}{p^-c^{q^-}})^\alpha}.\end{aligned}$$

From $(\mathfrak{M}_7)$ we have $\alpha(r_1,r_2,r_3)<\beta(r_1,r_2)$. □

Theorem 3.3. *Assume (M_1), $(\mathfrak{M}_4)-(\mathfrak{M}_7)$ hold. Then for*

$$\lambda\in\Lambda:=\Bigg(\frac{\frac{m_2}{\alpha}(\frac{S_p\eta^{p^+}}{p^-c^{p^-}})^\alpha+\frac{m_2}{\alpha}(\frac{S_q\kappa^{q^+}}{p^-c^{q^-}})^\alpha}{\int_{B(x^0,a_1)}F(x,\eta,\kappa)dx-\int_\Omega F(x,\theta_1,\vartheta_1)dx},$$
$$\min\left\{\frac{\frac{m_1}{\alpha}(\frac{1}{p^+}(\frac{\theta_1}{c})^{p^-})^\alpha+\frac{m_1}{\alpha}(\frac{1}{q^+}(\frac{\vartheta_1}{c})^{q^-})^\alpha}{\int_\Omega F(x,\theta_1,\vartheta_1)},\frac{\frac{m_1}{\alpha}(\frac{1}{p^+}(\frac{\theta_2}{c})^{p^-})^\alpha+\frac{m_1}{\alpha}(\frac{1}{q^+}(\frac{\vartheta_2}{c})^{q^-})^\alpha}{\int_\Omega F(x,\theta_2,\vartheta_2)},\right.$$
$$\left.\frac{\frac{m_1}{\alpha}(\frac{1}{p^+}(\frac{\theta_3^{p^-}-\theta_2^{p^-}}{c^{p^-}}))^\alpha+\frac{m_1}{\alpha}(\frac{1}{q^+}(\frac{\vartheta_3^{q^-}-\vartheta_2^{q^-}}{c^{q^-}}))^\alpha}{\int_\Omega F(x,\theta_2,\vartheta_2)}\right\}\Bigg),$$

the problem (1.1) has at least three weak solutions (u_1,v_1), (u_2,v_2) and (u_3,v_3) such that $\max_{x\in\overline{\Omega}}|(u_1(x),v_1(x))|<\theta_1+\vartheta_1$, $\max_{x\in\overline{\Omega}}|(u_2(x),v_2(x))|<\theta_2+\vartheta_2$ and $\max_{x\in\overline{\Omega}}|(u_3(x),v_3(x))|<\theta_3+\vartheta_3$.

Proof. Our approach is to apply Theorem 2.7 for the problem (1.1). We consider the auxiliary problem

$$\begin{cases} -M_1\Big(\displaystyle\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\Big)\Delta\Big(|\Delta u|^{p(x)-2}\Delta u\Big) = \lambda \widehat{F}_u(x,u,v), & in\ \ \Omega,\\ -M_2\Big(\displaystyle\int_\Omega \frac{1}{q(x)}|\Delta v|^{q(x)}dx\Big)\Delta\Big(|\Delta v|^{q(x)-2}\Delta u\Big) = \lambda \widehat{F}_v(x,u,v), & in\ \ \Omega,\\ u=v=\Delta u=\Delta v=0, & on\ \ \partial\Omega, \end{cases} \tag{3.9}$$

where $\widehat{F}_u \in C^0(\Omega\times\mathbb{R}^2)$. Define

$$\widehat{F}_u(x,u,v) := \begin{cases} F_u(x,0,v), & \xi < -\theta_3,\\ F_u(x,\xi,v), & -\theta_3\le \xi\le\theta_3,\\ F_u(x,\theta_3,v), & \xi>\theta_3, \end{cases}$$

and

$$\widehat{F}_v(x,u,v) := \begin{cases} F_v(x,u,0), & \zeta < -\vartheta_3,\\ F_v(x,u,\zeta), & -\vartheta_3\le \zeta\le\vartheta_3,\\ F_v(x,\vartheta_3,v), & \zeta>\vartheta_3. \end{cases}$$

If (u,v) is a weak solution of (3.9) such that $-\theta_3\le u(x)\le\theta_3$ and $-\vartheta_3\le v(x)\le\vartheta_3$ for every $x\in\Omega$, then, clearly it turns to be also a weak solution of (1.1). Hence, it is sufficient to show that our conclusion holds for (1.1).

By the definitions of Φ, Ψ, we know that Ψ is a differentiable functional. As well as it is sequentially weakly upper semicontinuous. Also, Ψ' is compact.

For any $(u,v)\in X$,

$$\begin{aligned} \Phi(u,v) &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\Big) + \widehat{M}\Big(\int_\Omega \frac{1}{q(x)}|\Delta u|^{q(x)}dx\Big)\\ &\ge \frac{m_1}{\alpha}\Big(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\Big)^\alpha + \frac{m_1}{\alpha}\Big(\int_\Omega \frac{1}{q(x)}|\Delta u|^{q(x)}dx\Big)^\alpha\\ &\ge \frac{m_1}{\alpha(p^+)^\alpha}\|u\|_{p(.)}^{\alpha p^-} + \frac{m_1}{\alpha(q^+)^\alpha}\|u\|_{q(.)}^{\alpha q^-}, \end{aligned}$$

which implies Φ is coercive. Moreover, Φ is continuously differentiable on X and its derivative admits a continuous inverse X^* (see [36, Lemma 1]). Furthermore, Φ is sequentially weakly lower semicontinuous, and by Lemma 2.6 Φ' is strictly monotone operator, considering Proposition [5, 1.5.10], we conclude that Φ is a strictly convex functional.

Therefore, we can use Theorem 2.7 to obtain the result. By Lemma 3.2, the condition $(\mathfrak{M}_3)$ of Theorem 2.7 is clearly satisfied. Now, we verify that the assumption $(\mathfrak{M}_2)$ holds. Let $z_1=(u_1,v_1)$ and $z_2=(u_2,v_2)$ be two local minima for $\Phi-\lambda\Psi$. So z_1 and z_2 are critical points for $\Phi-\lambda\Psi$ and then, they are weak solutions of (1.1). Since we assumed f is non-negative, for fixed $\lambda>0$ we have $\lambda f(x,sz_1+(1-s)z_2)=\lambda f(x,su_1+(1-s)u_2,sv_1+(1-s)v_2)\ge 0$ for every $s\in[0,1]$. Therefore, Theorem

2.7 implies that for every

$$\lambda \in \Bigg(\frac{\frac{m_2}{\alpha}(\frac{\varsigma_p \eta^{p^+}}{p^- c^{p^-}})^\alpha + \frac{m_2}{\alpha}(\frac{\varsigma_q \kappa^{q^+}}{p^- c^{q^-}})^\alpha}{\int_{B(x^0,a_1)} F(x,\eta,\kappa)dx - \int_\Omega F(x,\theta_1,\vartheta_1)dx},$$
$$\min\left\{ \frac{\frac{m_1}{\alpha}(\frac{1}{p^+}(\frac{\theta_1}{c})^{p^-})^\alpha + \frac{m_1}{\alpha}(\frac{1}{q^+}(\frac{\vartheta_1}{c})^{q^-})^\alpha}{\int_\Omega F(x,\theta_1,\vartheta_1)}, \frac{\frac{m_1}{\alpha}(\frac{1}{p^+}(\frac{\theta_2}{c})^{p^-})^\alpha + \frac{m_1}{\alpha}(\frac{1}{q^+}(\frac{\vartheta_2}{c})^{q^-})^\alpha}{\int_\Omega F(x,\theta_2,\vartheta_2)}, \right.$$
$$\left. \frac{\frac{m_1}{\alpha}(\frac{1}{p^+}(\frac{\theta_3^{p^-}-\theta_2^{p^-}}{c^{p^-}}))^\alpha + \frac{m_1}{\alpha}(\frac{1}{q^+}(\frac{\vartheta_3^{q^-}-\vartheta_2^{q^-}}{c^{q^-}}))^\alpha}{\int_\Omega F(x,\theta_2,\vartheta_2)} \right\}\Bigg),$$

$\Phi - \lambda\Psi$ has three critical points (u_i, v_i), $i = 1, 2, 3$, in X such that $\Phi(u_1, v_1) < r_1$, $\Phi(u_2, v_2) < r_2$ and $\Phi(u_3, v_3) < r_2 + r_3$, that is, $\max_{x\in\overline{\Omega}} |(u_1(x), v_1(x))| < \theta_1 + \vartheta_1$, $\max_{x\in\overline{\Omega}} |(u_2(x), v_2(x))| < \theta_2 + \vartheta_2$ and $\max_{x\in\overline{\Omega}} |(u_3(x), v_3(x))| < \theta_3 + \vartheta_3$. □

Here, is a remarkable consequence of Theorem 3.3.

Theorem 3.4. *Assume that there exist positive constants θ_1, θ_2, ϑ_1, ϑ_4 and $\eta, \kappa > 1$ with $\theta_1 < \min\{\eta^{\frac{p^+}{p^-}}, \varrho_p^{\frac{1}{p^-}}\eta\}$, $\vartheta_1 < \min\{\kappa^{\frac{q^+}{q^-}}, \varrho_q^{\frac{1}{q^-}}\kappa\}$, $\eta < \min\{(\frac{p^+}{2\varsigma_p p^-})^{\frac{1}{p^+}}\theta_4^{\frac{p^-}{p^+}}, \theta_4\}$ and $\kappa < \min\{(\frac{q^+}{2\varsigma_q q^-})^{\frac{1}{q^+}}\vartheta_4^{\frac{q^-}{q^+}}, \vartheta_4\}$ such that*

($\mathfrak{M}_8$) *$f(x, s, t) \geq 0$, for each $(x, s, t) \in \Omega \times [-\theta_4, \theta_4] \times [-\vartheta_4, \vartheta_4]$,*
($\mathfrak{M}_9$)

$$\max\left\{ \frac{\int_\Omega F(x,\theta_1,\vartheta_1)dx}{\frac{m_1}{\alpha}(\frac{\theta_1^{p^-}}{p^+})^\alpha + \frac{m_1}{\alpha}(\frac{\vartheta_1^{q^-}}{q^+})^\alpha}, \frac{2\int_\Omega F(x,\theta_4,\vartheta_4)dx}{\frac{m_1}{\alpha}(\frac{\theta_4^{p^-}}{p^+})^\alpha + \frac{m_1}{\alpha}(\frac{\vartheta_4^{q^-}}{q^+})^\alpha} \right\}$$
$$< \frac{\int_{B(x^0,a_1)} F(x,\eta,\kappa)dx}{\frac{m_2}{\alpha}(\frac{(p^+\varsigma_p+p^-)\eta^{p^+}}{p^- p^+})^\alpha + \frac{m_2}{\alpha}(\frac{(q^+\varsigma_q+q^-)\kappa^{p^+}}{q^- q^+})^\alpha}.$$

Then there exists an open interval Λ' with the following property: for every

$$\lambda \in \Lambda' := \Bigg(\frac{\frac{m_2}{\alpha}(\frac{(p^+\varsigma_p+p^-)\eta^{p^+}}{p^- p^+ c^{p^-}})^\alpha + \frac{m_2}{\alpha}(\frac{(q^+\varsigma_q+q^-)\kappa^{p^+}}{q^- q^+ c^{q^-}})^\alpha}{\int_{B(x^0,a_1)} F(x,\eta,\kappa)dx},$$
$$\min\left\{ \frac{\frac{m_1}{\alpha}(\frac{1}{p^+}(\frac{\theta_1}{c})^{p^-})^\alpha + \frac{m_1}{\alpha}(\frac{1}{q^+}(\frac{\vartheta_1}{c})^{q^-})^\alpha}{\int_\Omega F(x,\theta_1,\vartheta_1)dx}, \frac{\frac{m_1}{\alpha}(\frac{1}{p^+}(\frac{\theta_4}{c})^{p^-})^\alpha + \frac{m_1}{\alpha}(\frac{1}{q^+}(\frac{\vartheta_4}{c})^{q^-})^\alpha}{2\int_\Omega F(x,\theta_4,\vartheta_4)dx} \right\}\Bigg),$$

problem (1.1) has at least three weak solutions (u_1, v_1), (u_2, v_2) and (u_3, v_3) such that $\max_{x\in\Omega} |(u_1(x), v_1(x))| < \theta_1 + \vartheta_1$, $\max_{x\in\Omega} |(u_2(x), v_2(x))| < \frac{1}{\sqrt[p^-]{2}}\theta_4 + \frac{1}{\sqrt[q^-]{2}}\vartheta_4$ and $\max_{x\in\Omega} |(u_3(x), v_3(x))| < \theta_4 + \vartheta_4$.

Proof. Set $\theta_2 = \frac{1}{\sqrt[p^-]{2}}\theta_4$, $\vartheta_2 = \frac{1}{\sqrt[q^-]{2}}\vartheta_4$, $\theta_3 = \theta_4$ and $\vartheta_3 = \vartheta_4$. So, from $(\mathfrak{M}_9)$ one has

$$\begin{aligned}
\frac{\int_\Omega F(x,\theta_2,\vartheta_2)dx}{\frac{m_1}{\alpha_1}(\frac{\theta_2^{p^-}}{p^+})^{\alpha_1}+\frac{m_1}{\alpha_1}(\frac{\vartheta_2^{q^-}}{q^+})^{\alpha_1}} &= \frac{2^{\alpha_1}\int_\Omega F(x,\frac{1}{\sqrt[q^-]{2}}\theta_4,\frac{1}{\sqrt[q^-]{2}}\vartheta_4)dx}{\frac{m_1}{\alpha_1}(\frac{\theta_4^{p^-}}{p^+})^{\alpha_1}+\frac{m_1}{\alpha_1}(\frac{\vartheta_4^{q^-}}{q^+})^{\alpha_1}}\\
&\le \frac{2^{\alpha_1}\int_\Omega F(x,\theta_4,\vartheta_4)dx}{\frac{m_1}{\alpha_1}(\frac{\theta_4^{p^-}}{p^+})^{\alpha_1}+\frac{m_1}{\alpha_1}(\frac{\vartheta_4^{q^-}}{q^+})^{\alpha_1}} \qquad (3.10)\\
&< \frac{\int_{B(x^0,a_1)} F(x,\eta,\kappa)}{\frac{m_2}{\beta_2}(\frac{(p^+\varsigma_p+p^-)\eta^{p^+}}{p^-p^+})^{\beta_2}+\frac{m_2}{\beta_2}(\frac{(q^+\varsigma_q+q^-)\kappa^{p^+}}{q^-q^+})^{\beta_2}},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\int_\Omega F(x,\theta_3,\vartheta_3)dx}{\frac{m_1}{\alpha_1}(\frac{\theta_3^{p^-}-\theta_2^{p^-}}{p^+})^{\alpha_1}+\frac{m_1}{\alpha_1}(\frac{\vartheta_3^{q^-}-\vartheta_2^{q^-}}{q^+})^{\alpha_1}} &= \frac{2^{\alpha_1}\int_\Omega F(x,\theta_3,\vartheta_3)dx}{\frac{m_1}{\alpha_1}(\frac{\theta_4^{p^-}}{p^+})^{\alpha_1}+\frac{m_1}{\alpha_1}(\frac{\vartheta_4^{q^-}}{q^+})^{\alpha_1}}\\
&< \frac{\int_{B(x^0,a_1)} F(x,\eta,\kappa)}{\frac{m_2}{\beta_2}(\frac{(p^+\varsigma_p+p^-)\eta^{p^+}}{p^-p^+})^{\beta_2}+\frac{m_2}{\beta_2}(\frac{(q^+\varsigma_q+q^-)\kappa^{p^+}}{q^-q^+})^{\beta_2}}.
\end{aligned} \qquad (3.11)$$

Moreover, since $\theta_1 < \eta^{\frac{p^+}{p^-}}$ and $\vartheta_1 < \kappa^{\frac{q^+}{q^-}}$, from $(\mathfrak{M}_9)$

$$\begin{aligned}
&\frac{\int_{B(x^0,a_1)} F(x,\eta,\kappa)dx - \int_\Omega F(x,\theta_1,\vartheta_1)dx}{\frac{m_2}{\beta_2}(\frac{\varsigma_p\eta^{p^+}}{p^-})^{\beta_1}+\frac{m_2}{\beta_2}(\frac{\varsigma_q\kappa^{q^+}}{q^-})^{\beta_2}}\\
&> \frac{\int_{B(x^0,a_1)} F(x,\eta,\kappa)dx}{\frac{m_2}{\beta_2}(\frac{\varsigma_p\eta^{p^+}}{p^-})^{\beta_2}+\frac{m_2}{\beta_2}(\frac{\varsigma_q\kappa^{q^+}}{q^-})^{\beta_2}} - \frac{\int_\Omega F(x,\theta_1,\vartheta_1)dx}{\frac{m_2}{\beta_2}(\frac{\varsigma_p\theta_1^{p^-}}{p^-})^{\beta_2}+\frac{m_2}{\beta_2}(\frac{\varsigma_q\vartheta_1^{q^-}}{q^-})^{\beta_2}}\\
&> \frac{\int_{B(x^0,a_1)} F(x,\eta,\kappa)dx}{\frac{m_2}{\beta_2}(\frac{\varsigma_p\eta^{p^+}}{p^-})^{\beta_2}+\frac{m_2}{\beta_2}(\frac{\varsigma_q\kappa^{q^+}}{q^-})^{\beta_2}} - \frac{\int_\Omega F(x,\theta_1,\vartheta_1)dx}{\frac{m_1}{\alpha_1}(\frac{\theta_1^{p^-}}{p^+})^{\alpha_1}+\frac{m_1}{\alpha_1}(\frac{\vartheta_1^{q^-}}{q^+})^{\alpha_1}}\\
&> \frac{\int_{B(x^0,a_1)} F(x,\eta,\kappa)dx}{\frac{m_2}{\beta_2}(\frac{\varsigma_p\eta^{p^+}}{p^-})^{\beta_2}+\frac{m_2}{\beta_2}(\frac{\varsigma_q\kappa^{q^+}}{q^-})^{\beta_2}} - \frac{\int_{B(x^0,a_1)} F(x,\eta,\kappa)}{\frac{m_2}{\beta_2}(\frac{(p^+\varsigma_p+p^-)\eta^{p^+}}{p^-p^+})^{\beta_2}+\frac{m_2}{\beta_2}(\frac{(q^+\varsigma_q+q^-)\kappa^{p^+}}{q^-q^+})^{\beta_2}}\\
&> \frac{\int_{B(x^0,a_1)} F(x,\eta,\kappa)}{\frac{m_2}{\beta_2}(\frac{(p^+\varsigma_p+p^-)\eta^{p^+}}{p^-p^+})^{\beta_2}+\frac{m_2}{\beta_2}(\frac{(q^+\varsigma_q+q^-)\kappa^{p^+}}{q^-q^+})^{\beta_2}}.
\end{aligned}$$

Thus, by $(\mathfrak{M}_9)$, (3.10) and (3.11) we have $(\mathfrak{M}_7)$ of Lemma 3.2, and it follows the conclusion. □

Theorem 3.5. *Let F_u, F_v be non-negative and nonzero functions such that*

$$\lim_{u\to 0^+}\frac{F_u(u,v)}{|u|^{p^--1}} = \lim_{u\to+\infty}\frac{F_u(u,v)}{|u|^{p^--1}} = 0, \qquad (3.12)$$

and

$$\lim_{v\to 0^+}\frac{F_v(u,v)}{|v|^{p^- -1}}=\lim_{v\to +\infty}\frac{F_v(u,v)}{|v|^{p^- -1}}=0. \tag{3.13}$$

Then, for every $\lambda>\mu$ where

$$\mu=\inf\left\{\frac{\frac{(p^+\varsigma_p+p^-)\eta^{p^+}}{p^-p^+c^{p^-}}+\frac{(q^+\varsigma_q+q^-)\kappa^{q^+}}{q^-q^+c^{q^-}}}{meas(B(x^0,a_1))F(\eta,\kappa)}\ :\ \eta,\kappa\geq 1, F(\eta,\kappa)>0\right\},$$

problem

$$\begin{cases} -M\Big(\displaystyle\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\Big)\Delta\Big(|\Delta u|^{p(x)-2}\Delta u\Big)=\lambda F_u(u,v), & in\ \ \Omega,\\ -M\Big(\displaystyle\int_\Omega \frac{1}{q(x)}|\Delta v|^{q(x)}dx\Big)\Delta\Big(|\Delta v|^{q(x)-2}\Delta u\Big)=\lambda F_v(u,v), & in\ \ \Omega,\\ u=v=\Delta u=\Delta v=0, & on\ \ \partial\Omega, \end{cases} \tag{3.14}$$

has at least two non-trivial weak solutions.

Proof. Fix $\lambda>\mu$ and let $\eta,\kappa\geq 1$ such that $F(\eta,\kappa)>0$ and

$$\lambda>\frac{\frac{(p^+\varsigma_p+p^-)\eta^{p^+}}{p^-p^+c^{p^-}}+\frac{(q^+\varsigma_q+q^-)\kappa^{q^+}}{q^-q^+c^{q^-}}}{meas(B(x^0,a_1))F(\eta,\kappa)}.$$

From (3.12) and (3.12) there is $\theta_1,\vartheta_1>0$ such that

$$\begin{aligned} &\theta_1<\min\{\eta^{\frac{p^+}{p^-}},\rho_p^{\frac{1}{p^-}}\eta\},\\ &\vartheta_1<\min\{\kappa^{\frac{q^+}{q^-}},\rho_q^{\frac{1}{q^-}}\kappa\},\\ &\frac{F(\theta_1,\vartheta_1)}{\frac{\theta_1^{p^-}}{p^+c^{p^-}}+\frac{\vartheta_1^{q^-}}{q^+c^{q^-}}}<\frac{1}{\lambda meas(\Omega)}, \end{aligned}$$

and $\theta_4,\vartheta_4>0$ such that

$$\begin{aligned} &\eta<\min\{(\frac{p^+}{2\varsigma_p p^-})^{\frac{1}{p^+}}\theta_4^{\frac{p^-}{p^+}},\theta_4\},\\ &\kappa<\min\{(\frac{q^+}{2\varsigma_q q^-})^{\frac{1}{q^+}}\vartheta_4^{\frac{q^-}{q^+}},\vartheta_4\},\\ &\frac{F(\theta_4,\vartheta_4)}{\frac{\theta_4^{p^-}}{p^+c^{p^-}}+\frac{\vartheta_4^{q^-}}{q^+c^{q^-}}}<\frac{1}{2\lambda meas(\Omega)}. \end{aligned}$$

Therefore, all assumptions of Theorem 3.4 are fulfilled and it ensures the conclusion. □

4. Conclusion

Non-local operators can be seen as the infinitesimal generators of Lévy stable diffusion processes. Moreover, they allow us to develop a generalization of quantum mechanics and also to describe the motion of a chain or an array of particles that are connected by elastic springs as well as unusual diffusion processes in turbulent fluid motions and material transports in fractured media. On the other hand, fourth-order boundary value problem of nonlinearity furnishes a model to study traveling waves in suspension bridges, so it is important to physics. In this manuscript, we deal with finding multiple weak solutions for nonlocal fourth-order Kirchhoff systems with Navier boundary conditions. The techniques used are based on variational method. Firstly, we presented the main results about the existence of at least three weak solutions for (1.1).

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