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Analysis of the stability and convergence of a finite difference approximation for stochastic partial differential equations

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Abstract In this paper, an implicit finite difference scheme is proposed for the numerical solution of stochastic partial differential equations (SPDEs) of Itô type. The consistency, stability, and convergence of the scheme are analyzed. Numerical experiments are included to show the efficiency of the scheme.

Keywords. Stochastic partial differential equations, Stochastic finite difference scheme, Stability, Consistency, Convergence.

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1. Introduction

Stochastic partial differential equations play a prominent role in a range of applications, including biology, chemistry, epidemiology, mechanics, microelectronics and, of course, finance. In general, obtaining analytical solutions for SPDEs is either difficult or impossible, therefore researchers are very interested in effective numerical methods for studying the behavior of these equations. In the literature, several methods have been proposed to solve the SPDEs from either numerically or analytically points of view. An analytical solution can be obtained in [3, 4, 8] for very few SPDEs. Allen [1] has constructed finite element and difference approximation of some SPDEs. Walsh [12] used the finite element methods for parabolic SPDEs and Roth [9] approximated the solution of some stochastic hyperbolic equations by finite difference methods. Kamrani and Hosseini [5] have studied explicit and implicit finite difference method for general SPDEs. Soheili *et al.* [10] presented two methods for solving SPDEs based on Saul'yev method and a high order finite difference scheme. Compact finite difference scheme for stochastic advection-diffusion equation has proposed by Soheili and Bishehniasar in [2].
This paper is organized as follows. In section 2, a review of the Crank-Nicolson

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method for deterministic advection-diffusion equations and stability of the scheme is analyzed. Afterward, we extend this finite difference scheme for an approximation of stochastic advection-diffusion equations. In section 3, consistency, stability and convergence of the proposed stochastic scheme have been discussed. Finally, in the last section, some numerical simulations to demonstrate the validity of the theoretical results are given.

2. Finite difference approximation for advection-diffusion equations

Consider the following stochastic advection-diffusion equation

$$u_t(x,t) + \nu u_x(x,t) = \gamma u_{xx}(x,t) + \sigma u(x,t)\dot{W}(t), \quad x \in [0,1], \quad t \in [0,1], \tag{2.1}$$

with initial condition $u(x,0) = u_0(x),\ 0 \le x \le 1$ and boundary conditions

$$u(0,t) = f_1(t), \quad u(1,t) = f_2(t), \qquad t \in [0,1],$$

where ν and γ are the positive parameters which are called the phase speed and the viscosity coefficient, respectively, and $W(t)$ is an one-dimensional Wiener process such that the white noise $\dot{W}(t)$ is a Gaussian distribution with zero mean [6]. Numerically, finite difference methods have vast applications for approximating the solution of SPDEs. These schemes discretize continuous space and time evenly into a distributed grid system, and the values of the state variables are evaluated at every node of the grid. By considering a uniform space Δx and time Δt grids in the time-space lattice, we can estimate the solution of the equation at the points of this lattice. The value of the approximate solution at the point $(k\Delta x, n\Delta t)$ will be denoted by u_k^n where n, k are integers.

2.1. **The Crank-Nicolson finite difference scheme for deterministic advection-diffusion equations.**

In this technique, the time and space derivatives in the partial differential equation (PDE) are approximated by finite difference replacements as the following

$$\begin{aligned} u_t(k\Delta x, n\Delta t) &\approx \frac{u_k^{n+1} - u_k^n}{\Delta t}, \\ u_x(k\Delta x, n\Delta t) &\approx \frac{u_{k+1}^n - u_{k-1}^n}{4\Delta x} + \frac{u_{k+1}^{n+1} - u_{k-1}^{n+1}}{4\Delta x}, \\ u_{xx}(k\Delta x, n\Delta t) &\approx \frac{1}{2}\left(\frac{u_{k+1}^n - 2u_k^n + u_{k-1}^n}{\Delta x^2} + \frac{u_{k+1}^{n+1} - 2u_k^{n+1} + u_{k-1}^{n+1}}{\Delta x^2} \right). \end{aligned} \tag{2.2}$$

This implicit finite difference scheme simplifies the solution SPDE (2.1) in the absence of the noise term takes the following form

$$\begin{aligned} &-\left(\frac{\nu\lambda}{4} + \frac{\gamma\rho}{2}\right) u_{k-1}^{n+1} + (1+\gamma\rho)u_k^{n+1} + \left(\frac{\nu\lambda}{4} - \frac{\gamma\rho}{2}\right) u_{k+1}^{n+1} \\ &\qquad = \left(\frac{\nu\lambda}{4} + \frac{\gamma\rho}{2}\right) u_{k-1}^n + (1-\gamma\rho)u_k^n + \left(\frac{\gamma\rho}{2} - \frac{\nu\lambda}{4}\right) u_{k+1}^n, \end{aligned} \tag{2.3}$$

where $\lambda = \frac{\Delta t}{\Delta x}$ and $\rho = \frac{\Delta t}{\Delta x^2}$. The Crank-Nicolson scheme can be shown to be unconditionally stable by use of the Von Neumann method of stability analysis [11] as follows. Assume that $\hat{u}^{n+1}$ is the Fourier-transformation of u^{n+1}, then the Fourier inversion formula gives us

$$u_m^{n+1} = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{\Delta x}}^{\frac{\pi}{\Delta x}} e^{im\Delta x\xi} \hat{u}^{n+1}(\xi) \mathrm{d}\xi, \tag{2.4}$$

where

$$\hat{u}^{n+1} = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-im\Delta x\xi} u_m^{n+1} \Delta x, \tag{2.5}$$

and ξ is a real variable. Substituting in equation (2.3), we get

$$\hat{u}^{n+1}(\xi) = g(\Delta x\xi, \Delta t, \Delta x)\hat{u}^n(\xi),$$

where $g(\Delta x\xi, \Delta t, \Delta x)$ is the amplification factor of the deterministic difference scheme. For fixed $\nu\lambda$ and $\gamma\rho$, the stability condition is $|g(\Delta x\xi, \Delta t, \Delta x)| \leq 1$. In this way, the amplification factor for the Crank-Nicolson method is

$$\begin{aligned} g(\Delta x\xi, \Delta t, \Delta x) &= \frac{(1-\gamma\rho) + \left(\frac{\nu\lambda}{4} + \frac{\gamma\rho}{2}\right) e^{-i\Delta x\xi} + \left(\frac{\gamma\rho}{2} - \frac{\nu\lambda}{4}\right) e^{i\Delta x\xi}}{(1+\gamma\rho) - \left(\frac{\nu\lambda}{4} + \frac{\gamma\rho}{2}\right) e^{-i\Delta x\xi} + \left(\frac{\nu\lambda}{4} - \frac{\gamma\rho}{2}\right) e^{i\Delta x\xi}} \\ &= \frac{1 - 2\gamma\rho \sin^2\left(\frac{\Delta x\xi}{2}\right) - i\frac{\nu\lambda}{2}\sin(\Delta x\xi)}{1 + 2\gamma\rho \sin^2\left(\frac{\Delta x\xi}{2}\right) + i\frac{\nu\lambda}{2}\sin(\Delta x\xi)}. \end{aligned}$$

Since $|g(\Delta x\xi, \Delta t, \Delta x)| \leq 1$, for all λ, ρ and Δx, the scheme is unconditionally stable.

2.2. **The Crank-Nicolson finite difference scheme for stochastic advection-diffusion equations.** We extend the proposed unconditional stable finite difference scheme to approximate the solutions of the stochastic advection-diffusion equations of the form (2.1) and investigate its performance in the stochastic case. Substituting partial derivatives (2.2) in Eq. (2.1), we obtain the stochastic Crank-Nicolson implicit finite difference scheme as follows:

$$\begin{aligned} &-\left(\frac{\nu\lambda}{4} + \frac{\gamma\rho}{2}\right) u_{k-1}^{n+1} + (1+\gamma\rho) u_k^{n+1} + \left(\frac{\nu\lambda}{4} - \frac{\gamma\rho}{2}\right) u_{k+1}^{n+1} \\ &= \left(\frac{\nu\lambda}{4} + \frac{\gamma\rho}{2}\right) u_{k-1}^{n} + (1-\gamma\rho) u_k^{n} + \left(\frac{\gamma\rho}{2} - \frac{\nu\lambda}{4}\right) u_{k+1}^{n} + \sigma u_k^n \Delta W_n, \end{aligned} \tag{2.6}$$

where $\lambda = \frac{\Delta t}{\Delta x}$, $\rho = \frac{\Delta t}{\Delta x^2}$, $\Delta W_n = W((n+1)\Delta t) - W(n\Delta t)$ and ΔW_n is a Gaussian distribution with mean 0 and variance Δt, *i.e.*, $\Delta W_n \sim N(0, \Delta t)$. In [7], the authors have considered the approximation of stochastic advection diffusion equation using a conditional stable stochastic difference scheme. It follows from [7] the stochastic difference for (2.1) is given by

$$u_k^{n+1} = \left(1+\nu\lambda-\frac{5}{2}\gamma\rho\right)u_k^n+\left(\frac{4}{3}\gamma\rho-\nu\lambda\right)u_{k+1}^n$$
$$+\gamma\rho\left(-\frac{1}{12}u_{k-2}^n+\frac{4}{3}u_{k-1}^n-\frac{1}{12}u_{k+2}^n\right)+\sigma u_k^n\Delta W_n, \tag{2.7}$$

where $\lambda=\frac{\Delta t}{\Delta x}$ and $\rho=\frac{\Delta t}{\Delta x^2}$. Throughout this paper we assume that for the stochastic finite difference scheme (2.6) the increments of Wiener process are independent of the state u_k^n. Essentially, it is important for the solution of stochastic difference scheme to converge to the solution of the SPDE. Consider an SPDE of the form $Lv=G$, where L denotes the differential operator and G is an inhomogeneity. Let u_k^n be a solution that is approximated by a stochastic finite difference scheme denoted by L_k^n, and applying the stochastic scheme to the SPDE, we have $L_k^n u_k^n = G_k^n$, where G_k^n is the approximation of the inhomogeneity. In order to get consistency, stability and convergence results, we will need a norm. Hence for a sequence $u=\{\dots,u_{-1},u_0,u_1,\dots\}$, the sup-norm is defined as $\|u\|_\infty=\sqrt{\sup_k |u_k|^2}$. We refer the reader to [9] for the following definitions of a stochastic difference scheme.

Definition 2.1. A stochastic difference scheme $L_k^n u_k^n = G_k^n$ is pointwise consistent with the SPDE $Lv=G$ at point (x,t), if we have for any continuously differentiable function $\Phi=\Phi(x,t)$, in mean square

$$\mathbb{E}\|\,(L\Phi-G)\,|_k^n-[L_k^n\Phi(k\Delta x,n\Delta t)-G_k^n]\,\|^2\to 0,$$

as $\Delta x\to 0$, $\Delta t\to 0$, and $(k\Delta x,(n+1)\Delta t)\to(x,t)$.

To investigate the stability of the stochastic difference scheme, we can apply the Von Neumann method for the stochastic difference scheme. From substituting Eq. (2.5) in the stochastic difference equation and the equality of the Fourier-transformation one obtains

$$\hat{u}^{n+1}(\xi)=\hat{u}^n(\xi)g(\Delta x\xi,\Delta t,\Delta x),$$

where $\hat{u}^n$ is the Fourier-transformation of u^n. So in this stability analysis a necessary and sufficient condition of stability is

$$\mathbb{E}|g(\Delta x\xi,\Delta t,\Delta x)|^2\le 1+K\Delta t.$$

Definition 2.2. A stochastic difference scheme $L_k^n u_k^n = G_k^n$ which approximate the SPDE $Lv=G$ is convergent in mean square at time t, if as $(n+1)\Delta t$ converges to t, then $\mathbb{E}\|u^{n+1}-v^{n+1}\|^2\to 0$, for $(n+1)\Delta t=t$, $\Delta x\to 0$ and $\Delta t\to 0$.

3. Consistency, stability and convergence of the stochastic Crank-Nicolson scheme

Theorem 3.1. *The stochastic difference scheme (2.6) is consistent in mean square with the stochastic partial differential equation (2.1).*

Proof. Let $\Phi(x,t)$ be a smooth function, then we have

$$L(\Phi)|_k^n = \Phi(k\Delta x,(n+1)\Delta t) - \Phi(k\Delta x, n\Delta t) + \nu\int_{n\Delta t}^{(n+1)\Delta t}\Phi_x(k\Delta x,s)\mathrm{d}s$$
$$-\gamma\int_{n\Delta t}^{(n+1)\Delta t}\Phi_{xx}(k\Delta x,s)\,\mathrm{d}s - \sigma\int_{n\Delta t}^{(n+1)\Delta t}\Phi(k\Delta x,s)\mathrm{d}W(s),$$

and from (2.3), we get

$$\begin{aligned}
L_k^n\Phi &= \Phi(k\Delta x,(n+1)\Delta t) - \Phi(k\Delta x,n\Delta t)\\
&+\nu\Delta t\Bigg[\frac{1}{4\Delta x}\Big(\Phi((k+1)\Delta x,n\Delta t) - \Phi((k-1)\Delta x,n\Delta t)\Big)\\
&+\frac{1}{4\Delta x}\Big(\Phi((k+1)\Delta x,(n+1)\Delta t) - \Phi((k-1)\Delta x,(n+1)\Delta t)\Big)\Bigg]\\
&-\gamma\Delta t\Bigg[\frac{1}{2\Delta x^2}\Big(\Phi((k+1)\Delta x,n\Delta t) - 2\Phi(k\Delta x,n\Delta t)\\
&+\Phi((k-1)\Delta x,n\Delta t)\Big) + \frac{1}{2\Delta x^2}\Big(\Phi((k+1)\Delta x,(n+1)\Delta t)\\
&-2\Phi(k\Delta x,(n+1)\Delta t) + \Phi((k-1)\Delta x,(n+1)\Delta t)\Big)\Bigg]\\
&-\sigma\Phi(k\Delta x,n\Delta t)(W((n+1)\Delta t) - W(n\Delta t)).
\end{aligned}$$

Therefore, if we use the square property of Itô integral, we will obtain

$$\begin{aligned}
&\mathbb{E}|L(\Phi)|_k^n - L_k^n\Phi|^2\\
&=\mathbb{E}\Bigg|\nu\int_{n\Delta t}^{(n+1)\Delta t}\Bigg(\Phi_x(k\Delta x,s) - \Bigg(\frac{1}{4\Delta x}\Big(\Phi((k+1)\Delta x,n\Delta t)\\
&-\Phi((k-1)\Delta x,n\Delta t))\Big) + \frac{1}{4\Delta x}\Big(\Phi((k+1)\Delta x,(n+1)\Delta t)\\
&-\Phi((k-1)\Delta x,(n+1)\Delta t)\Big)\Bigg)\Bigg)\mathrm{d}s - \gamma\int_{n\Delta t}^{(n+1)\Delta t}\Bigg(\Phi_{xx}(k\Delta x,s)\\
&-\Bigg(\frac{1}{2\Delta x^2}\Big(\Phi((k+1)\Delta x,n\Delta t) - 2\Phi(k\Delta x,n\Delta t) + \Phi((k-1)\Delta x,n\Delta t)\Big)\\
&+\frac{1}{2\Delta x^2}\Big(\Phi((k+1)\Delta x,(n+1)\Delta t) - 2\Phi(k\Delta x,(n+1)\Delta t)\\
&+\Phi((k-1)\Delta x,(n+1)\Delta t)\Big)\Bigg)\Bigg)\mathrm{d}s\\
&-\sigma\int_{n\Delta t}^{(n+1)\Delta t}\Big(\Phi(k\Delta x,s) - \Phi(k\Delta x,n\Delta t)\Big)\mathrm{d}W(s)\Bigg|^2
\end{aligned}$$

$$\leq 4\nu^2\mathbb{E}\bigg|\nu\int_{n\Delta t}^{(n+1)\Delta t}\Bigg(\Phi_x(k\Delta x,s)-\Bigg(\frac{1}{4\Delta x}\Big(\Phi((k+1)\Delta x,n\Delta t)$$
$$-\Phi((k-1)\Delta x,n\Delta t)\Big)+\frac{1}{4\Delta x}\Big(\Phi((k+1)\Delta x,(n+1)\Delta t)$$
$$-\Phi((k-1)\Delta x,(n+1)\Delta t)\Big)\Bigg)\Bigg)\mathrm{d}s\bigg|^2$$
$$+4\gamma^2\mathbb{E}\bigg|\int_{n\Delta t}^{(n+1)\Delta t}\Bigg(\Phi_{xx}(k\Delta x,s)$$
$$-\Bigg(\frac{1}{2\Delta x^2}\Big(\Phi((k+1)\Delta x,n\Delta t)-2\Phi(k\Delta x,n\Delta t)+\Phi((k-1)\Delta x,n\Delta t)\Big)$$
$$+\frac{1}{2\Delta x^2}\Big(\Phi((k+1)\Delta x,(n+1)\Delta t)-2\Phi(k\Delta x,(n+1)\Delta t)$$
$$+\Phi((k-1)\Delta x,(n+1)\Delta t)\Big)\Bigg)\Bigg)\mathrm{d}s\bigg|^2$$
$$+4\sigma^2\int_{n\Delta t}^{(n+1)\Delta t}\mathbb{E}|\Phi(k\Delta x,s)-\Phi(k\Delta x,n\Delta t)|^2\mathrm{d}s.$$

Since $\Phi(x,t)$ is a deterministic function, hence $\mathbb{E}|L(\Phi)|_k^n - L_k^n\Phi|^2 \to 0$, as $n,\, k \to \infty$. □

Theorem 3.2. *The stochastic Crank-Nicolson difference scheme (2.6) is unconditionally stable based on the Fourier-transformation analysis for the advection diffusion equation (2.1).*

Proof. Substituting (2.5) into Eq. (2.6) implies that

$$\left[-\left(\frac{\nu\lambda}{4}+\frac{\gamma\rho}{2}\right)e^{-i\Delta x\xi}+(1+\gamma\rho)+\left(\frac{\nu\lambda}{4}-\frac{\gamma\rho}{2}\right)e^{i\Delta x\xi}\right]\hat{u}^{n+1}(\xi)$$
$$=\left[\left(\frac{\nu\lambda}{4}+\frac{\gamma\rho}{2}\right)e^{-i\Delta x\xi}+(1-\gamma\rho)+\left(\frac{\gamma\rho}{2}-\frac{\nu\lambda}{4}\right)e^{i\Delta x\xi}+\sigma\Delta W_n\right]\hat{u}^n(\xi).$$

Then we get

$$\hat{u}^{n+1}(\xi)=\left\{\frac{\left(\frac{\nu\lambda}{4}+\frac{\gamma\rho}{2}\right)e^{-i\Delta x\xi}+(1-\gamma\rho)+\left(\frac{\gamma\rho}{2}-\frac{\nu\lambda}{4}\right)e^{i\Delta x\xi}}{-\left(\frac{\nu\lambda}{4}+\frac{\gamma\rho}{2}\right)e^{-i\Delta x\xi}+(1+\gamma\rho)+\left(\frac{\nu\lambda}{4}-\frac{\gamma\rho}{2}\right)e^{i\Delta x\xi}}\right.$$
$$\left.+\frac{\sigma\Delta W_n}{-\left(\frac{\nu\lambda}{4}+\frac{\gamma\rho}{2}\right)e^{-i\Delta x\xi}+(1+\gamma\rho)+\left(\frac{\nu\lambda}{4}-\frac{\gamma\rho}{2}\right)e^{i\Delta x\xi}}\right\}\hat{u}^n(\xi).$$

This means that the amplification factor of the stochastic Crank-Nicolson scheme is

$$g(\Delta x\xi, \Delta t, \Delta x) = \frac{\left(\frac{\nu\lambda}{4} + \frac{\gamma\rho}{2}\right) e^{-i\Delta x\xi} + (1-\gamma\rho) + \left(\frac{\gamma\rho}{2} - \frac{\nu\lambda}{4}\right) e^{i\Delta x\xi}}{-\left(\frac{\nu\lambda}{4} + \frac{\gamma\rho}{2}\right) e^{-i\Delta x\xi} + (1+\gamma\rho) + \left(\frac{\nu\lambda}{4} - \frac{\gamma\rho}{2}\right) e^{i\Delta x\xi}}$$
$$+ \frac{\sigma\Delta W_n}{-\left(\frac{\nu\lambda}{4} + \frac{\gamma\rho}{2}\right) e^{-i\Delta x\xi} + (1+\gamma\rho) + \left(\frac{\nu\lambda}{4} - \frac{\gamma\rho}{2}\right) e^{i\Delta x\xi}}.$$

Now by independence of the Wiener process and simple computations, we obtain

$$\mathbb{E}|g(\Delta x\xi, \Delta t, \Delta x)|^2 = \left[\frac{\left(\frac{\nu\lambda}{4} + \frac{\gamma\rho}{2}\right) e^{-i\Delta x\xi} + (1-\gamma\rho) + \left(\frac{\gamma\rho}{2} - \frac{\nu\lambda}{4}\right) e^{i\Delta x\xi}}{-\left(\frac{\nu\lambda}{4} + \frac{\gamma\rho}{2}\right) e^{-i\Delta x\xi} + (1+\gamma\rho) + \left(\frac{\nu\lambda}{4} - \frac{\gamma\rho}{2}\right) e^{i\Delta x\xi}}\right]^2$$
$$+ \left[\frac{\sigma}{-\left(\frac{\nu\lambda}{4} + \frac{\gamma\rho}{2}\right) e^{-i\Delta x\xi} + (1+\gamma\rho) + \left(\frac{\nu\lambda}{4} - \frac{\gamma\rho}{2}\right) e^{i\Delta x\xi}}\right]^2 \Delta t.$$

Hence for every γ, ρ, ν, λ and Δx we get

$$\left|\frac{\left(\frac{\nu\lambda}{4} + \frac{\gamma\rho}{2}\right) e^{-i\Delta x\xi} + (1-\gamma\rho) + \left(\frac{\gamma\rho}{2} - \frac{\nu\lambda}{4}\right) e^{i\Delta x\xi}}{-\left(\frac{\nu\lambda}{4} + \frac{\gamma\rho}{2}\right) e^{-i\Delta x\xi} + (1+\gamma\rho) + \left(\frac{\nu\lambda}{4} - \frac{\gamma\rho}{2}\right) e^{i\Delta x\xi}}\right|^2$$
$$= \left|\frac{1 - 2\gamma\rho\sin^2\left(\frac{\Delta x\xi}{2}\right) - i\frac{\nu\lambda}{2}\sin(\Delta x\xi)}{1 + 2\gamma\rho\sin^2\left(\frac{\Delta x\xi}{2}\right) + i\frac{\nu\lambda}{2}\sin(\Delta x\xi)}\right|^2 \le 1,$$

and

$$\left|\frac{\sigma}{-\left(\frac{\nu\lambda}{4} + \frac{\gamma\rho}{2}\right) e^{-i\Delta x\xi} + (1+\gamma\rho) + \left(\frac{\nu\lambda}{4} - \frac{\gamma\rho}{2}\right) e^{i\Delta x\xi}}\right|^2$$
$$= \left|\frac{\sigma}{1 + 2\gamma\rho\sin^2\left(\frac{\Delta x\xi}{2}\right) + i\frac{\nu\lambda}{2}\sin(\Delta x\xi)}\right|^2 \le K.$$

So

$$\mathbb{E}|g(\Delta x\xi, \Delta t, \Delta x)|^2 \le 1 + K\Delta t,$$

which proves the stability. □

Theorem 3.3. *Let $v \in H^1$, H^2, H^3, H^4. The stochastic Crank-Nicolson scheme (2.6) for Eq. (2.1) is convergent with respect to $\|\cdot\|_\infty$–norm when $\frac{\nu\lambda}{2} \le \gamma\rho \le 1$.*

Proof. We can rewrite the stochastic finite difference scheme (2.6) as

$$u_k^{n+1} = u_k^n - \nu\Delta t\left(\frac{u_{k+1}^n - u_{k-1}^n}{4\Delta x} + \frac{u_{k+1}^{n+1} - u_{k-1}^{n+1}}{4\Delta x}\right)$$
$$+ \gamma\Delta t\left(\frac{u_{k+1}^n - 2u_k^n + u_{k-1}^n}{2\Delta x^2} + \frac{u_{k+1}^{n+1} - 2u_k^{n+1} + u_{k-1}^{n+1}}{2\Delta x^2}\right)$$
$$+ \sigma u_k^n(W((n+1)\Delta t) - W(n\Delta t)).$$

We can represent the solution v_k^{n+1} by the Taylor's expansion $v_x(x,s)$ and $v_{xx}(x,s)$ with respect to the space variable as

$$
\begin{aligned}
v_k^{n+1} &= v_k^n - \nu \int_{n\Delta t}^{(n+1)\Delta t} v_x(x,s)|_{x=x_k} \mathrm{d}s + \gamma \int_{n\Delta t}^{(n+1)\Delta t} v_{xx}(x,s)|_{x=x_k} \mathrm{d}s \\
&+ \sigma \int_{n\Delta t}^{(n+1)\Delta t} v(x,s)|_{x=x_k} \mathrm{d}W(s) \\
&= v_k^n - \nu \int_{n\Delta t}^{(n+1)\Delta t} \Bigg(\frac{v_{k+1}^n - v_{k-1}^n}{4\Delta x} + \frac{v_{k+1}^{n+1} - v_{k-1}^{n+1}}{4\Delta x} \\
&- \frac{\Delta x^2}{4 \times 3!} \Big(v_{xxx}((k+\alpha_1)\Delta x, s) + v_{xxx}((k+\alpha_2)\Delta x, s) \\
&+ v_{xxx}((k+\alpha_3)\Delta x, s+\Delta t) \\
&+ v_{xxx}((k+\alpha_4)\Delta x, s+\Delta t) \Big) - \frac{\Delta t}{2} v_{xt}(k\Delta x, s+\eta\Delta t) \Bigg) \mathrm{d}s \\
&+ \gamma \int_{n\Delta t}^{(n+1)\Delta t} \Bigg(\frac{v_{k+1}^n - 2v_k^n + v_{k-1}^n}{2\Delta x^2} + \frac{v_{k+1}^{n+1} - 2v_k^{n+1} + v_{k-1}^{n+1}}{2\Delta x^2} \\
&- \frac{\Delta x^2}{2 \times 4!} \Big(v_{xxxx}((k+\beta_1)\Delta x, s) \\
&+ v_{xxxx}((k+\beta_2)\Delta x, s) + v_{xxxx}((k+\beta_3)\Delta x, s+\Delta t) \\
&+ v_{xxxx}((k+\beta_4)\Delta x, s+\Delta t) \Big) - \frac{\Delta t}{2} v_{xxt}(k\Delta x, s+\delta\Delta t) \Bigg) \mathrm{d}s \\
&+ \sigma \int_{n\Delta t}^{(n+1)\Delta t} v(x,s)|_{x=x_k} \mathrm{d}W(s),
\end{aligned}
$$

where $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \eta,\ \beta_1,\ \beta_2,\ \beta_3,\ \beta_4,\ \delta \in (0,1)$. Therefore we get

$$
\begin{aligned}
v_k^{n+1} &= v_k^n - \nu \int_{n\Delta t}^{(n+1)\Delta t} \Bigg(\frac{v_{k+1}^n - v_{k-1}^n}{4\Delta x} + \frac{v_{k+1}^{n+1} - v_{k-1}^{n+1}}{4\Delta x} \\
&- \frac{\Delta x^2}{4 \times 3!} \Big(v_{xxx}((k+\alpha_1)\Delta x, s) + v_{xxx}((k+\alpha_2)\Delta x, s) \\
&+ v_{xxx}((k+\alpha_3)\Delta x, s+\Delta t) \\
&+ v_{xxx}((k+\alpha_4)\Delta x, s+\Delta t) \Big) + \nu \frac{\Delta t}{2} v_{xx}(k\Delta x, s+\eta\Delta t)
\end{aligned}
$$

$$
\begin{aligned}
&- \gamma \frac{\Delta t}{2} v_{xxx}(k\Delta x, s+\eta\Delta t)\Big)\mathrm{d}s \\
&+ \gamma \int_{n\Delta t}^{(n+1)\Delta t} \Bigg(\frac{v_{k+1}^{n} - 2v_{k}^{n} + v_{k-1}^{n}}{2\Delta x^2} + \frac{v_{k+1}^{n+1} - 2v_{k}^{n+1} + v_{k-1}^{n+1}}{2\Delta x^2} \\
&- \frac{\Delta x^2}{2\times 4!}\Big(v_{xxxx}((k+\beta_1)\Delta x, s) + v_{xxxx}((k+\beta_2)\Delta x, s) \\
&+ v_{xxxx}((k+\beta_3)\Delta x, s+\Delta t) \\
&+ v_{xxxx}((k+\beta_4)\Delta x, s+\Delta t)\Big) + \nu \frac{\Delta t}{2} v_{xxx}(k\Delta x, s+\delta\Delta t) \\
&- \gamma \frac{\Delta t}{2} v_{xxxx}(k\Delta x, s+\delta\Delta t)\Bigg)\mathrm{d}s \\
&+ \nu\sigma \frac{\Delta t}{2} \int_{n\Delta t}^{(n+1)\Delta t} v_x(x,s)|_{x=x_k} \mathrm{d}W(s) \\
&- \gamma\sigma \frac{\Delta t}{2} \int_{n\Delta t}^{(n+1)\Delta t} v_{xx}(x,s)|_{x=x_k} \mathrm{d}W(s) \\
&+ \sigma \int_{n\Delta t}^{(n+1)\Delta t} v(x,s)|_{x=x_k} \mathrm{d}W(s).
\end{aligned}
$$

Let $z_k^n = v_k^n - u_k^n$, then

$$
\begin{aligned}
z_k^{n+1} = z_k^n - \nu \int_{n\Delta t}^{(n+1)\Delta t} \Bigg(&\frac{z_{k+1}^{n} - z_{k-1}^{n}}{4\Delta x} + \frac{z_{k+1}^{n+1} - z_{k-1}^{n+1}}{4\Delta x} \\
&- \frac{\Delta x^2}{4\times 3!}\Big(v_{xxx}((k+\alpha_1)\Delta x, s) + v_{xxx}((k+\alpha_2)\Delta x, s) \\
&+ v_{xxx}((k+\alpha_3)\Delta x, s+\Delta t) \\
&+ v_{xxx}((k+\alpha_4)\Delta x, s+\Delta t)\Big) + \nu \frac{\Delta t}{2} v_{xx}(k\Delta x, s+\eta\Delta t) \\
&- \gamma \frac{\Delta t}{2} v_{xxx}(k\Delta x, s+\eta\Delta t)\Bigg)\mathrm{d}s \\
&+ \gamma \int_{n\Delta t}^{(n+1)\Delta t} \Bigg(\frac{z_{k+1}^{n} - 2z_{k}^{n} + z_{k-1}^{n}}{2\Delta x^2} + \frac{z_{k+1}^{n+1} - 2z_{k}^{n+1} + z_{k-1}^{n+1}}{2\Delta x^2} \\
&- \frac{\Delta x^2}{2\times 4!}\Big(v_{xxxx}((k+\beta_1)\Delta x, s) + v_{xxxx}((k+\beta_2)\Delta x, s) \\
&+ v_{xxxx}((k+\beta_3)\Delta x, s+\Delta t)
\end{aligned}
$$

$$
\begin{aligned}
&+ v_{xxxx}((k+\beta_4)\Delta x, s+\Delta t)\Big) + \nu\frac{\Delta t}{2} v_{xxx}(k\Delta x, s+\delta\Delta t) \\
&- \gamma\frac{\Delta t}{2} v_{xxxx}(k\Delta x, s+\delta\Delta t)\Bigg)\mathrm{d}s \\
&+ \nu\sigma\frac{\Delta t}{2}\int_{n\Delta t}^{(n+1)\Delta t} v_x(x,s)|_{x=x_k}\mathrm{d}W(s) \\
&- \gamma\sigma\frac{\Delta t}{2}\int_{n\Delta t}^{(n+1)\Delta t} v_{xx}(x,s)|_{x=x_k}\mathrm{d}W(s) \\
&+ \sigma\int_{n\Delta t}^{(n+1)\Delta t} (v(x,s)|_{x=x_k} - u_k^n)\mathrm{d}W(s).
\end{aligned}
$$

Easily it follows that

$$
\begin{aligned}
&(1+\gamma\rho)z_k^{n+1} - \left(\frac{\nu\lambda}{4} + \frac{\gamma\rho}{2}\right) z_{k-1}^{n+1} + \left(\frac{\nu\lambda}{4} - \frac{\gamma\rho}{2}\right) z_{k+1}^{n+1} \\
&= (1-\gamma\rho)z_k^n + \left(\frac{\nu\lambda}{4} + \frac{\gamma\rho}{2}\right) z_{k-1}^{n} + \left(\frac{\gamma\rho}{2} - \frac{\nu\lambda}{4}\right) z_{k+1}^{n} \\
&- \nu\int_{n\Delta t}^{(n+1)\Delta t}\Bigg[-\frac{\Delta x^2}{4\times 3!}\Big(v_{xxx}((k+\alpha_1)\Delta x, s) + v_{xxx}((k+\alpha_2)\Delta x, s) \\
&+ v_{xxx}((k+\alpha_3)\Delta x, s+\Delta t) \\
&+ v_{xxx}((k+\alpha_4)\Delta x, s+\Delta t)\Big) + \nu\frac{\Delta t}{2} v_{xx}(k\Delta x, s+\eta\Delta t) \\
&- \gamma\frac{\Delta t}{2} v_{xxx}(k\Delta x, s+\eta\Delta t)\Bigg]\mathrm{d}s \\
&+ \gamma\int_{n\Delta t}^{(n+1)\Delta t}\Bigg[-\frac{\Delta x^2}{2\times 4!}\Big(v_{xxxx}((k+\beta_1)\Delta x, s) + v_{xxxx}((k+\beta_2)\Delta x, s) \\
&+ v_{xxxx}((k+\beta_3)\Delta x, s+\Delta t) \\
&+ v_{xxxx}((k+\beta_4)\Delta x, s+\Delta t)\Big) + \nu\frac{\Delta t}{2} v_{xxx}(k\Delta x, s+\delta\Delta t) \\
&- \gamma\frac{\Delta t}{2} v_{xxxx}(k\Delta x, s+\delta\Delta t)\Bigg]\mathrm{d}s \\
&+ \nu\sigma\frac{\Delta t}{2}\int_{n\Delta t}^{(n+1)\Delta t} v_x(x,s)|_{x=x_k}\mathrm{d}W(s) \\
&- \gamma\sigma\frac{\Delta t}{2}\int_{n\Delta t}^{(n+1)\Delta t} v_{xx}(x,s)|_{x=x_k}\mathrm{d}W(s) \\
&+ \sigma\int_{n\Delta t}^{(n+1)\Delta t} (v(x,s)|_{x=x_k} - u_k^n)\mathrm{d}W(s). \qquad (3.1)
\end{aligned}
$$

Applying $\mathbb{E}|\cdot|^2$ to (3.1) and by use of the inequality

$$\mathbb{E}|X+Y+Z+R+S|^2 \leq 4\mathbb{E}|X|^2 + 8\mathbb{E}|Y|^2 + 16\mathbb{E}|Z|^2 + 16\mathbb{E}|R|^2 + 2\mathbb{E}|S|^2,$$

we get

$$\begin{aligned}
&\mathbb{E}\left|(1+\gamma\rho)z_k^{n+1} - \left(\frac{\nu\lambda}{4} + \frac{\gamma\rho}{2}\right) z_{k-1}^{n+1} + \left(\frac{\nu\lambda}{4} - \frac{\gamma\rho}{2}\right) z_{k+1}^{n+1}\right|^2 \\
&\leq 4\mathbb{E}\left|(1-\gamma\rho)z_k^{n} + \left(\frac{\nu\lambda}{4} + \frac{\gamma\rho}{2}\right) z_{k-1}^{n} + \left(\frac{\gamma\rho}{2} - \frac{\nu\lambda}{4}\right) z_{k+1}^{n}\right|^2 \\
&\quad + 8\mathbb{E}\Bigg|-\nu\int_{n\Delta t}^{(n+1)\Delta t}\Bigg[-\frac{\Delta x^2}{4\times 3!}\Big(v_{xxx}((k+\alpha_1)\Delta x, s) \\
&\quad + v_{xxx}((k+\alpha_2)\Delta x, s) + v_{xxx}((k+\alpha_3)\Delta x, s+\Delta t) \\
&\quad + v_{xxx}((k+\alpha_4)\Delta x, s+\Delta t)\Big) + \nu\frac{\Delta t}{2} v_{xx}(k\Delta x, s+\eta\Delta t) \\
&\quad - \gamma\frac{\Delta t}{2} v_{xxx}(k\Delta x, s+\eta\Delta t)\Bigg]\mathrm{d}s \\
&\quad + \gamma\int_{n\Delta t}^{(n+1)\Delta t}\Bigg[-\frac{\Delta x^2}{2\times 4!}\Big(v_{xxxx}((k+\beta_1)\Delta x, s) \\
&\quad + v_{xxxx}((k+\beta_2)\Delta x, s) + v_{xxxx}((k+\beta_3)\Delta x, s+\Delta t) \\
&\quad + v_{xxxx}((k+\beta_4)\Delta x, s+\Delta t)\Big) + \nu\frac{\Delta t}{2} v_{xxx}(k\Delta x, s+\delta\Delta t) \\
&\quad - \gamma\frac{\Delta t}{2} v_{xxxx}(k\Delta x, s+\delta\Delta t)\Bigg]\mathrm{d}s\Bigg|^2 \\
&\quad + 4(\nu\sigma\Delta t)^2 \int_{n\Delta t}^{(n+1)\Delta t} \mathbb{E}|v_x(x,s)|_{x=x_k}|^2 \mathrm{d}s \\
&\quad + 4(\gamma\sigma\Delta t)^2 \int_{n\Delta t}^{(n+1)\Delta t} \mathbb{E}|v_{xx}(x,s)|_{x=x_k}|^2 \mathrm{d}s \\
&\quad + 2\sigma^2 \int_{n\Delta t}^{(n+1)\Delta t} \mathbb{E}|v(x,s)|_{x=x_k} - v_k^n + v_k^n - u_k^n|^2 \mathrm{d}s \\
&\leq 4\mathbb{E}\left|(1-\gamma\rho)z_k^{n} + \left(\frac{\nu\lambda}{4} + \frac{\gamma\rho}{2}\right) z_{k-1}^{n} + \left(\frac{\gamma\rho}{2} - \frac{\nu\lambda}{4}\right) z_{k+1}^{n}\right|^2
\end{aligned}$$

$$
\begin{aligned}
&+ 8\mathbb{E}\Bigg|-\nu \int_{n\Delta t}^{(n+1)\Delta t} \Bigg[-\frac{\Delta x^2}{4\times 3!}\Big(v_{xxx}((k+\alpha_1)\Delta x, s) \\
&+v_{xxx}((k+\alpha_2)\Delta x, s) + v_{xxx}((k+\alpha_3)\Delta x, s+\Delta t) \\
&+ v_{xxx}((k+\alpha_4)\Delta x, s+\Delta t)\Big) + \nu\frac{\Delta t}{2} v_{xx}(k\Delta x, s+\eta\Delta t) \\
&- \gamma\frac{\Delta t}{2} v_{xxx}(k\Delta x, s+\eta\Delta t)\Bigg]\mathrm{d}s \\
&+ \gamma \int_{n\Delta t}^{(n+1)\Delta t} \Bigg[-\frac{\Delta x^2}{2\times 4!}\Big(v_{xxxx}((k+\beta_1)\Delta x, s) \\
&+ v_{xxxx}((k+\beta_2)\Delta x, s) + v_{xxxx}((k+\beta_3)\Delta x, s+\Delta t) \\
&+ v_{xxxx}((k+\beta_4)\Delta x, s+\Delta t)\Big) + \nu\frac{\Delta t}{2} v_{xxx}(k\Delta x, s+\delta\Delta t) \\
&-\gamma\frac{\Delta t}{2} v_{xxxx}(k\Delta x, s+\delta\Delta t)\Bigg]\mathrm{d}s\Bigg|^2 \\
&+ 4(\nu\sigma\Delta t)^2 \int_{n\Delta t}^{(n+1)\Delta t} \mathbb{E}|v_x(x,s)|_{x=x_k}|^2\mathrm{d}s \\
&+ 4(\gamma\sigma\Delta t)^2 \int_{n\Delta t}^{(n+1)\Delta t} \mathbb{E}|v_{xx}(x,s)|_{x=x_k}|^2\mathrm{d}s \\
&+ 4\sigma^2 \int_{n\Delta t}^{(n+1)\Delta t} \mathbb{E}|v(x,s)|_{x=x_k} - v_k^n|^2\mathrm{d}s \\
&+ 4\sigma^2 \underbrace{\int_{n\Delta t}^{(n+1)\Delta t} \mathbb{E}|v_k^n - u_k^n|^2\mathrm{d}s}_{\mathbb{E}|z_k^n|^2\Delta t}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\mathbb{E}\left|(1+\gamma\rho)z_k^{n+1} - \left(\frac{\nu\lambda}{4}+\frac{\gamma\rho}{2}\right)z_{k-1}^{n+1} + \left(\frac{\nu\lambda}{4}-\frac{\gamma\rho}{2}\right)z_{k+1}^{n+1}\right|^2 \\
&\leq 4\left[\left(|1-\gamma\rho| + \left|\frac{\nu\lambda}{4}+\frac{\gamma\rho}{2}\right| + \left|\frac{\gamma\rho}{2}-\frac{\nu\lambda}{4}\right|\right)^2 + \sigma^2\Delta t\right]\sup_k \mathbb{E}|z_k^n|^2 \\
&+ 8\sup_k \mathbb{E}\Bigg|-\nu \int_{n\Delta t}^{(n+1)\Delta t} \Bigg[-\frac{\Delta x^2}{4\times 3!}\Big(v_{xxx}((k+\alpha_1)\Delta x, s) \\
&+v_{xxx}((k+\alpha_2)\Delta x, s) + v_{xxx}((k+\alpha_3)\Delta x, s+\Delta t)
\end{aligned}
$$

$$
\begin{aligned}
&+ v_{xxx}((k+\alpha_4)\Delta x, s+\Delta t)\Big) + \nu \frac{\Delta t}{2} v_{xx}(k\Delta x, s+\eta\Delta t) \\
&- \gamma \frac{\Delta t}{2} v_{xxx}(k\Delta x, s+\eta\Delta t)\Bigg]\mathrm{d}s \\
&+ \gamma \int_{n\Delta t}^{(n+1)\Delta t} \Bigg[-\frac{\Delta x^2}{2\times 4!}\Big(v_{xxxx}((k+\beta_1)\Delta x, s) \\
&+ v_{xxxx}((k+\beta_2)\Delta x, s) + v_{xxxx}((k+\beta_3)\Delta x, s+\Delta t) \\
&+ v_{xxxx}((k+\beta_4)\Delta x, s+\Delta t)\Big) + \nu \frac{\Delta t}{2} v_{xxx}(k\Delta x, s+\delta\Delta t) \\
&-\gamma \frac{\Delta t}{2} v_{xxxx}(k\Delta x, s+\delta\Delta t)\Bigg]\mathrm{d}s\Bigg|^2 \\
&+ 4(\nu\sigma\Delta t)^2 \sup_k \int_{n\Delta t}^{(n+1)\Delta t} \mathbb{E}|v_x(x,s)|_{x=x_k}|^2\mathrm{d}s \\
&+ 4(\gamma\sigma\Delta t)^2 \sup_k \int_{n\Delta t}^{(n+1)\Delta t} \mathbb{E}|v_{xx}(x,s)|_{x=x_k}|^2\mathrm{d}s \\
&+ 4\sigma^2 \sup_k \int_{n\Delta t}^{(n+1)\Delta t} \mathbb{E}|v(x,s)|_{x=x_k} - v_k^n|^2\mathrm{d}s.
\end{aligned}
$$

Let us introduce the notations $\psi_{1k} = v_{xxx}((k+\alpha_1)\Delta x, s) < \infty$, $\psi_{2k} = v_{xxx}((k+\alpha_2)\Delta x, s) < \infty$, $\psi_{3k} = v_{xxx}((k+\alpha_3)\Delta x, s+\Delta t) < \infty$, $\psi_{4k} = v_{xxx}((k+\alpha_4)\Delta x, s+\Delta t) < \infty$, $\psi_{5k} = v_{xx}(k\Delta x, s+\eta\Delta t) < \infty$, $\psi_{6k} = v_{xxx}(k\Delta x, s+\eta\Delta t) < \infty$, $\Theta_{1k} = v_{xxxx}((k+\beta_1)\Delta x, s) < \infty$, $\Theta_{2k} = v_{xxxx}((k+\beta_2)\Delta x, s) < \infty$, $\Theta_{3k} = v_{xxxx}((k+\beta_3)\Delta x, s+\Delta t) < \infty$, $\Theta_{4k} = v_{xxxx}((k+\beta_4)\Delta x, s+\Delta t) < \infty$, $\Theta_{5k} = v_{xxx}(k\Delta x, s+\delta\Delta t) < \infty$, $\Theta_{6k} = v_{xxxx}(k\Delta x, s+\delta\Delta t) < \infty$, $\psi'_{1k} = v_x(x,s) < \infty$ and $\psi''_{2k} = v_{xx}(x,s) < \infty$. Considering

$$
\begin{aligned}
\int_{n\Delta t}^{(n+1)\Delta t} \mathbb{E}|v(x,s)|_{x=x_k} - v_k^n|^2\mathrm{d}s &= \mathbb{E}\int_{n\Delta t}^{(n+1)\Delta t} |v(x,s)|_{x=x_k} - v_k^n|^2\mathrm{d}s \\
&\le \sup_{s\in[n\Delta t,(n+1)\Delta t]} |v(x,s)|_{x=x_k} - v(k\Delta x, n\Delta t)|^2\Delta t \le \psi'\Delta t,
\end{aligned}
$$

and using the hypothesis $\frac{\nu\lambda}{2} \le \gamma\rho \le 1$, we obtain

$$
\begin{aligned}
&\sup_k \mathbb{E}\left|(1+\gamma\rho)z_k^{n+1} - \left(\frac{\nu\lambda}{4} + \frac{\gamma\rho}{2}\right) z_{k-1}^{n+1} + \left(\frac{\nu\lambda}{4} - \frac{\gamma\rho}{2}\right) z_{k+1}^{n+1}\right|^2 \\
&\le 4\left(1+\sigma^2\Delta t\right) \sup_k \mathbb{E}|z_k^n|^2
\end{aligned}
$$

$$
\begin{aligned}
&+8\sup_k \mathbb{E}\Bigg|-\nu\int_{n\Delta t}^{(n+1)\Delta t}\Bigg[-\frac{\Delta x^2}{4\times 3!}\Big(\psi_{1k}+\psi_{2k}+\psi_{3k}+\psi_{4k}\Big)\\
&+\nu\frac{\Delta t}{2}\psi_{5k}-\gamma\frac{\Delta t}{2}\psi_{6k}\Bigg]\,\mathrm{d}s\\
&+\gamma\int_{n\Delta t}^{(n+1)\Delta t}\Bigg[-\frac{\Delta x^2}{2\times 4!}\Big(\Theta_{1k}+\Theta_{2k}+\Theta_{3k}+\Theta_{4k}\Big)\\
&+\nu\frac{\Delta t}{2}\Theta_{5k}-\gamma\frac{\Delta t}{2}\Theta_{6k}\Bigg]\,\mathrm{d}s\Bigg|^2\\
&+4\sup_k\int_{n\Delta t}^{(n+1)\Delta t}\left[(\nu\sigma\Delta t)^2\mathbb{E}|\psi'_{1k}|^2+(\gamma\sigma\Delta t)^2\mathbb{E}|\psi''_{2k}|^2\right]\mathrm{d}s\\
&+4\sigma^2\psi'\Delta t.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left(|1+\gamma\rho|-\left|\frac{\nu\lambda}{4}+\frac{\gamma\rho}{2}\right|+\left|\frac{\nu\lambda}{4}-\frac{\gamma\rho}{2}\right|\right)^2\sup_k\mathbb{E}|z_k^{n+1}|^2\\
&\le 4(1+\sigma^2\Delta t)\sup_k\mathbb{E}|z_k^n|^2+8\sup_k\mathbb{E}|\Psi_1|^2\Delta t\\
&+4\sup_k\mathbb{E}|\Psi_2|^2\Delta t+\Psi_4\Delta t,
\end{aligned}
$$

and consequently

$$
\begin{aligned}
\mathbb{E}\|z^{n+1}\|_\infty^2 &= \sup_k\mathbb{E}|z_k^{n+1}|^2 \le 4(1+\sigma^2\Delta t)\sup_k\mathbb{E}|z_k^n|^2+\Psi\Delta t\\
&\le 4(1+\sigma^2\Delta t)\mathbb{E}\|z^n\|_\infty^2+\Psi\Delta t\\
&\le \left(1+\sigma^2\frac{t}{n+1}\right)^{n+1}\sum_{j=1}^{n}(4\Psi\Delta t)^j+\Psi\Delta t\\
&\le e^{\sigma^2 t}\sum_{j=1}^{n}(4\Psi\Delta t)^j+\Psi\Delta t.
\end{aligned}
$$

When time step, *i.e.*, Δt, is tending to zero, we obtain

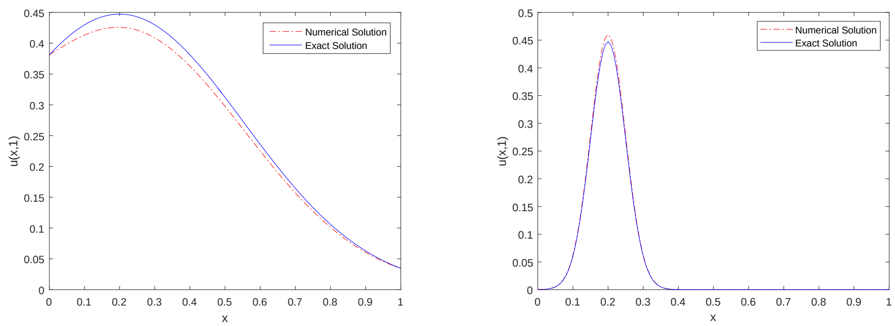
$$
\begin{aligned}
\mathbb{E}\|z^{n+1}\|_\infty^2 &\le (n-1)e^{\sigma^2 t}(4\Psi\Delta t)^2+4e^{\sigma^2 t}\Psi\Delta t+\Psi\Delta t\\
&\le te^{\sigma^2 t}(4\Psi)^2\Delta t+4e^{\sigma^2 t}\Psi\Delta t+\Psi\Delta t\\
&= (te^{\sigma^2 t}(4\Psi)^2+4e^{\sigma^2 t}\Psi+\Psi)\Delta t,
\end{aligned}
$$

or $\mathbb{E}\|z^{n+1}\|_\infty^2 \to 0$. □

4. Numerical aspects

Analytical studies always remain incomplete without numerical verification of the results. In this section, we present the numerical results of the stochastic difference scheme (2.6) on three test problems. Also, the convergence and stability of the stochastic difference scheme (2.6) are numerically investigated. Since $\Delta W_n \sim N(0, \Delta t)$,

FIGURE 1. Comparison between deterministic and stochastic numerical solution of (4.1) using the Crank-Nicolson scheme with $\gamma = 0.05$, $\sigma = 1.2$, $\Delta x = 0.01$, $\Delta t = 0.002$ and with $\gamma = 0.001$, $\sigma = 2$, $\Delta x = 0.008$, $\Delta t = 0.1$.



(a) $\gamma = 0.05$, $\sigma = 1.2$, $\Delta x = 0.01$, $\Delta t = 0.002$. (b) $\gamma = 0.001$, $\sigma = 2$, $\Delta x = 0.008$, $\Delta t = 0.1$.

hence to generate Wiener increments ΔW_n in MATLAB environment of random numbers the generator, randn(# traj,# step) is used, such that each call to randn(# traj,# step) creates a $\#traj \times \#step$ matrix of independent $N(0,1)$ samples.

Example 4.1. *Consider the stochastic diffusion equation of the form*

$$u_t(x,t) = \gamma u_{xx}(x,t) + \sigma u(x,t)\dot{W}(t), \quad x \in [0,1], \quad t \in [0,1], \tag{4.1}$$

with initial condition

$$u(x,0) = \exp\left(-\frac{(x-0.2)^2}{\gamma}\right),$$

and boundary conditions

$$u(0,t) = \frac{1}{\sqrt{4t+1}} \exp\left(-\frac{0.04}{\gamma(4t+1)}\right),$$
$$u(1,t) = \frac{1}{\sqrt{4t+1}} \exp\left(-\frac{0.64}{\gamma(4t+1)}\right).$$

In absence of the noise term, the exact solution is

$$u(x,t) = \frac{1}{\sqrt{4t+1}} \exp\left(-\frac{(x-0.2)^2}{\gamma(4t+1)}\right).$$

In this example, in order to qualify numerical results of the considered stochastic diffusion equation, we plot in Figure 1 the stochastic solution using stochastic Crank-Nicolson scheme (2.6) with $\gamma = 0.05$, $\sigma = 1.2$, $\Delta x = 0.01$, $\Delta t = 0.002$ and with $\gamma = 0.001$, $\sigma = 2$, $\Delta x = 0.008$, $\Delta t = 0.1$ as well the exact solution. In Table 2, some numerical results are presented for solving the stochastic diffusion equation (4.1) using the unconditional stable Crank-Nicolson scheme. Because of the significant property *of stability of this stochastic implicit method, we have not any restriction for considering the space and time step sizes and refinement of the computational domain does not impose any restriction on the stability scheme. So numerically implicit and unconditional stability of this stochastic method could be used to approximate the solution of stochastic diffusion equation. In Table 1, some numerical results are presented for solving the stochastic diffusion equation (4.1) using the conditional stable scheme* (2.7)*. The exact deterministic solution and numerical solution of the stochastic diffusion equation (4.1) using the stochastic Crank-Nicolson scheme are shown in Figure 3 and Figure 4 on a* 500×500 *grid during the time interval* $[0, 1]$ *for* $\gamma = 0.001$, $\sigma = 0.01$ *and for* $\gamma = 0.002$, $\sigma = 0.03$, *respectively. Also by fix* $\gamma = 0.001$, $\sigma = 2$ *and* $M = 125$ *the stochastic scheme is convergent with* $N \geq 16$. *We have shown this in Table 3. For* $\gamma = 0.001$, $\sigma = 2$, $\Delta x = 0.008$ *and* $\Delta t = 0.1$, *numerical solution* (2.7) *is shown in Figure 2. From the numerical results of this example, we get that the obtained results from the scheme* (2.6) *quite agreed with the exact one.*

TABLE 1. Test white noise SPDE (4.1) by the stochastic scheme (2.7).

N	$E(u(0.2,1))$	$E((u(0.2,1))^2$
5	-1.2703	1.6137
15	0.1389	0.0193
25	0.4101	0.1682
40	0.4498	0.2023
50	0.4961	0.2461
60	0.4639	0.2152

TABLE 2. Test white noise SPDE (4.1) by the stochastic Crank-Nicolson scheme.

γ	σ	Δt	Δx	$E(u(0.2,1))$	$E((u(0.2,1))^2$
0.005	1	0.005	0.01	0.4680	0.2190
0.05	1.2	0.02	0.01	0.4353	0.1895
0.001	2	0.04	0.008	0.4736	0.2243
0.01	1.5	0.1	0.01	0.4599	0.2115

Example 4.2. *We consider another test example to approximate the solution of stochastic diffusion equation driven by the white noise of the form*

$$u_t(x,t) = \gamma u_{xx}(x,t) + \sigma u(x,t)\dot{W}(t), \quad x \in [0,1], \quad t \in [0,1], \tag{4.2}$$

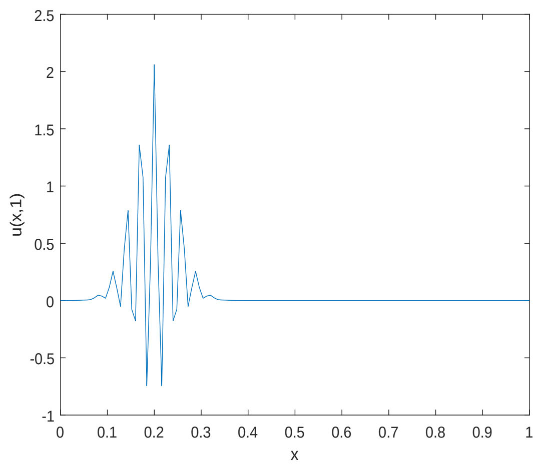
with initial condition

$$u(x,0) = \sin(\pi x), \qquad x \in [0,1],$$

and boundary conditions $u(0,t) = u(1,t) = 0$, *by use of the stochastic Crank–Nicolson scheme. The problem has an exact solution given by*

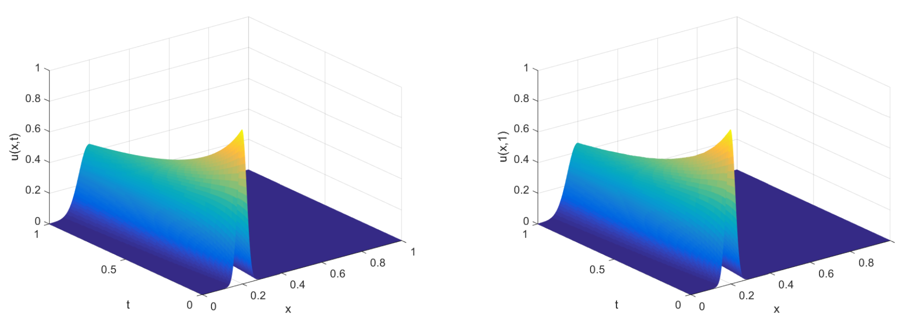
$$u(x,t) = e^{-\gamma\pi^2 t}\sin(\pi x).$$

FIGURE 2. Numerical solution of stochastic advection-diffusion equation by use of the scheme (2.7).



$\gamma = 0.001,\ \sigma = 2,\ \Delta x = 0.008,\ \Delta t = 0.1.$

FIGURE 3. The exact solution and numerical solution of (4.1) using the stochastic Crank-Nicolson method.

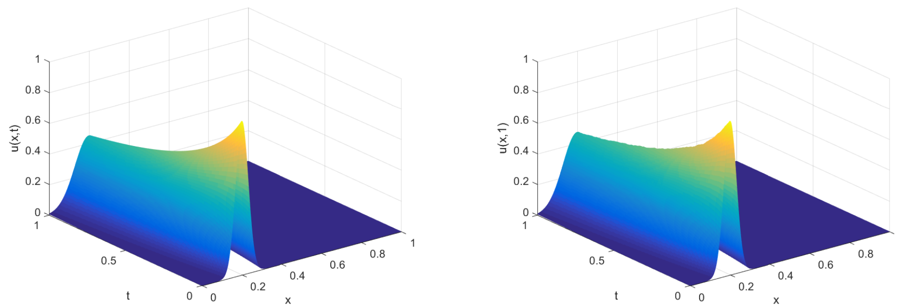


(a) The exact solution with $\gamma = 0.001$.

(b) The numerical solution with $\gamma = 0.001$ and $\sigma = 0.01$.

Figure 6 shows that the approximation of the stochastic advection diffusion equation using the stochastic difference scheme with $\gamma = 0.002$, $\sigma = 1.8$, $\Delta x = 0.01$, $\Delta t = 0.02$ *and with* $\gamma = 0.001$, $\sigma = 1.5$, $\Delta x = 0.01$, $\Delta t = 0.02$ *as well the exact solution. In order to examine the behavior of numerical solutions, we provide, in Table 6, averaged solution of (4.2) with some different values for diffusion and stochastic coefficients. In Table 4, some numerical results are presented for solving the stochastic diffusion equation (4.2) using the conditional stable scheme* (2.7). *Using above values, we*

FIGURE 4. The exact solution and numerical solution of (4.1) using the stochastic Crank-Nicolson method.



(a) The exact solution with $\gamma = 0.002$.

(b) The numerical solution with $\gamma = 0.002$ and $\sigma = 0.03$.

TABLE 3. Test white noise SPDE (4.1) by the stochastic Crank-Nicolson scheme.

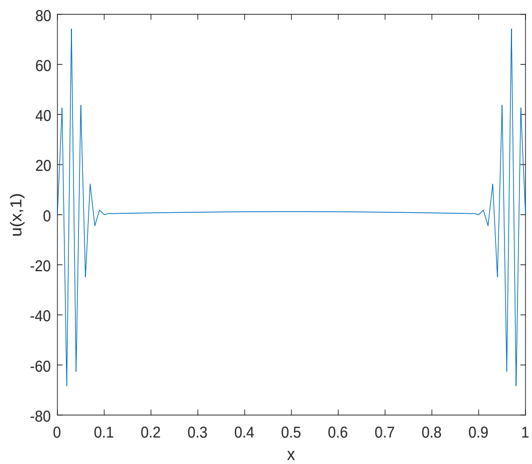
N	$E(u(0.2, 1))$	$E((u(0.2, 1))^2$
2	0.3970	0.1576
5	0.4474	0.2002
10	0.4598	0.2114
15	0.5111	0.2612
16	0.4629	0.2142
30	0.4114	0.1693
40	0.4184	0.1751

TABLE 4. Test of white noise SPDE (4.1) by the stochastic scheme (2.7).

N	$E(u(0.5, 1))$	$E((u(0.5, 1))^2$
5	1.2487	1.5591
10	1.2170	1.4810
15	1.1106	1.2333
30	1.0390	1.0796
40	1.0363	1.0740
45	1.0503	1.1031

see that stability conditions are hold for the scheme (2.7). *If we choose* $\gamma = 0.001$, $\sigma = 1.5$, $\Delta x = 0.01$ *and* $\Delta t > 0.04$, *i.e.,* $N < 25$, *we will see the scheme* (2.7) *is unstable. Figure* 5 *shows the approximation of the stochastic advection diffusion (4.2) using the stochastic difference scheme* (2.7) *with* $N = 10$. *In Table* 5, *some numerical results are demonstrated for solving the stochastic diffusion equation (4.2) using the conditional stable scheme* (2.7).

FIGURE 5. Numerical solution of stochastic advection-diffusion equation using the scheme (2.7).



$\gamma = 0.001,\ \sigma = 1.5,\ \Delta x = 0.01,\ \Delta t = 0.1.$

TABLE 5. Test white noise SPDE (4.1) by the stochastic scheme (2.7).

N	$E(u(0.5,1))$	$E((u(0.5,1))^2$
5	1.2487	1.5591
10	1.2170	1.4810
15	1.1106	1.2333
30	1.0390	1.0796
40	1.0363	1.0740
45	1.0503	1.1031

The exact deterministic solution and numerical solution of the stochastic diffusion equation (4.2) using the stochastic Crank–Nicolson scheme have been shown in Figures 8–9 on a 500×500 *grid during the time interval* $[0, 1]$ *for* $\gamma = 0.002$, $\sigma = 0.03$ *and for* $\gamma = 0.001$, $\sigma = 0.001$, *respectively. If* $\gamma = 0.001$, $\sigma = 1.5$ *and* $M = 100$, *the stochastic scheme is convergent if* $N \geq 10$. *This is obvious from Table 7. Figure 7 shows numerical solution of the scheme (2.7) for values* $\gamma = 0.001$, $\sigma = 1.5$, $\Delta x = 0.01$ *and* $\Delta t = 0.02$.

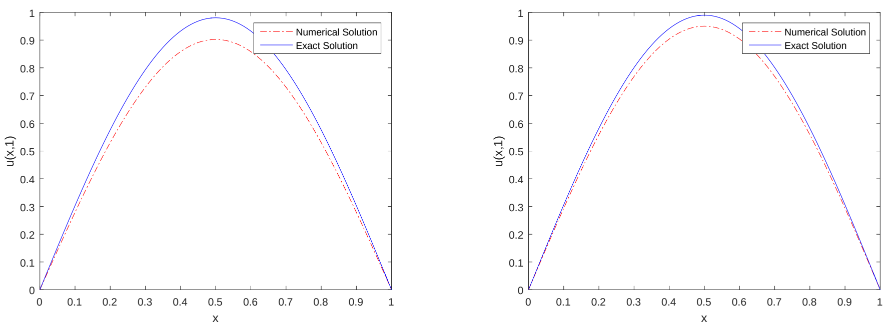
Example 4.3. *Consider the following SPDE*

$$u_t(x,t) + \nu u_x(x,t) = \gamma u_{xx}(x,t) + \sigma u(x,t)\dot{W}(t), \quad x \in [0,1], \quad t \in [0,1], \tag{4.3}$$

with the following initial condition

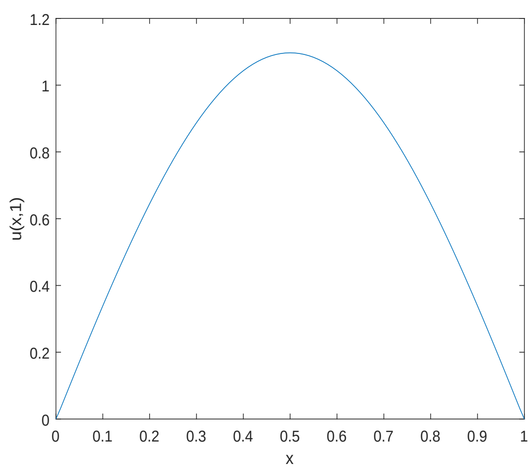
$$u(x,0) = \exp\left(-\frac{(x-0.5)^2}{\gamma}\right),$$

Figure 6. Comparison between deterministic and stochastic numerical solution of (4.2) using the Crank-Nicolson scheme with $\gamma = 0.002$, $\sigma = 1.8$, $\Delta x = 0.01$, $\Delta t = 0.02$ and with $\gamma = 0.001$, $\sigma = 1.5$, $\Delta x = 0.01$, $\Delta t = 0.02$.



(a) $\gamma = 0.002$, $\sigma = 1.8$, $\Delta x = 0.01$, $\Delta t = 0.02$. (b) $\gamma = 0.001$, $\sigma = 1.5$, $\Delta x = 0.01$, $\Delta t = 0.02$.

Figure 7. Numerical solution of stochastic advection-diffusion equation using the scheme (2.7).



$\gamma = 0.001$, $\sigma = 1.5$, $\Delta x = 0.01$, $\Delta t = 0.02$.

with the boundary conditions

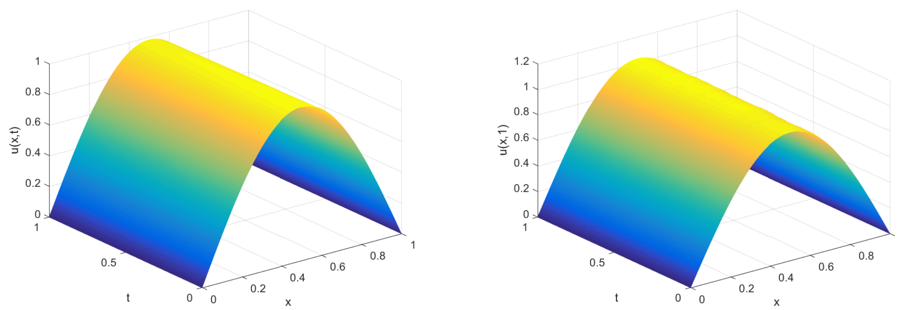
$$u(0,t) = \frac{1}{\sqrt{4t+1}} \exp\left(-\frac{(-0.5-\nu t)^2}{\gamma(4t+1)} \right),$$

$$u(1,t) = \frac{1}{\sqrt{4t+1}} \exp\left(-\frac{(0.5-\nu t)^2}{\gamma(4t+1)} \right).$$

TABLE 6. Test of white noise SPDE (4.2) by the stochastic Crank-Nicolson scheme.

γ	σ	Δt	Δx	$E(u(0.5,1))$	$E((u(0.5,1))^2$
0.01	0.5	0.005	0.005	0.9249	0.8554
0.01	1.5	0.02	0.01	0.8695	0.7561
0.001	1.2	0.1	0.025	1.0148	1.0299
0.005	0.8	0.01	0.02	0.9476	0.8979

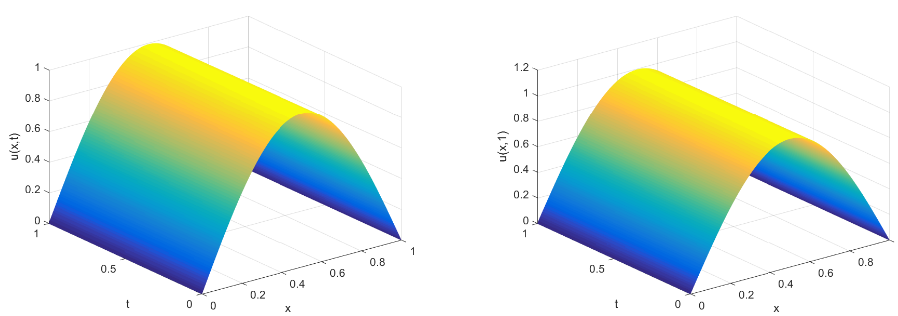
FIGURE 8. Exact solution and numerical solution of (4.2) using stochastic Crank-Nicolson scheme.



(a) The exact solution with $\gamma = 0.002$.

(b) The numerical solution with $\gamma = 0.002$ and $\sigma = 0.03$.

FIGURE 9. Exact solution and numerical solution of (4.2) using stochastic Crank-Nicolson scheme.



(a) The exact solution with $\gamma = 0.001$.

(b) The numerical solution with $\gamma = 0.001$ and $\sigma = 0.001$.

It is easy to verify that in the absence of the noise term, the exact solution is

$$u(x,t) = \frac{1}{\sqrt{4t+1}} \exp\left(-\frac{(x-0.5-\nu t)^2}{\gamma(4t+1)}\right).$$

Table 7. Test of white noise SPDE (4.2) by the stochastic Crank-Nicolson scheme.

N	$E(u(0.5,1))$	$E((u(0.5,1))^2$
2	0.9421	0.8876
5	0.9998	0.9997
10	1.0192	1.0387
15	1.0525	1.1077
20	1.1183	1.2507
40	1.0152	1.0307
100	1.1005	1.2111

Table 8. Test of white noise SPDE (4.3) by stochastic scheme (2.7).

N	$E(u(0.6,1))$	$E((u(0.6,1))^2$
80	$-2.6964E+20$	$7.2704E+40$
90	$-1.2982E+19$	$1.6853E+38$
100	$-8.4556E+16$	$7.1497E+33$
150	0.1114	0.0124
152	0.1162	0.0135
200	0.1068	0.0114

Table 9. Test white noise SPDE (4.3) by the stochastic Crank-Nicolson scheme.

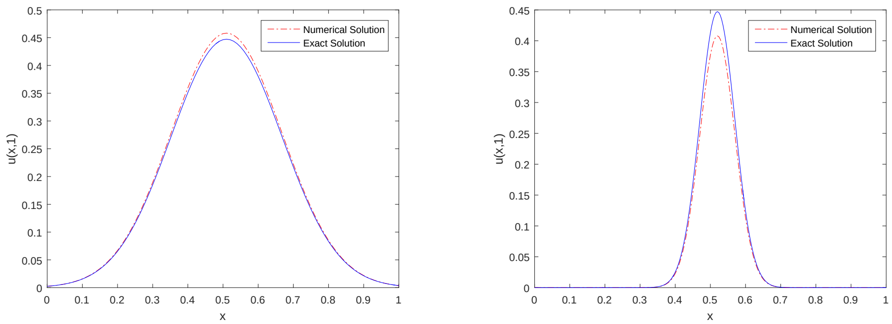
γ	ν	σ	Δt	Δx	$E(u(0.6,1))$	$E((u(0.6,1))^2$
0.005	0.1	1	0.005	0.01	0.4681	0.2191
0.05	0.05	1.2	0.01	0.01	0.4520	0.2043
0.05	0.01	2	0.1	0.02	0.4439	0.1970
0.01	0.03	1.5	0.05	0.008	0.4584	0.2101

Table 10. Test white noise SPDE (4.3) by the stochastic Crank-Nicolson scheme.

N	$E(u(0.6,1))$	$E((u(0.6,1))^2$
2	0.3882	0.1507
5	0.3236	0.1047
10	0.4339	0.1882
20	0.3343	0.1117
40	0.3900	0.1521
150	0.3910	0.1529
200	0.3940	0.1552

In Figure 10 the approximated solution of the stochastic advection-diffusion equation (4.3) by use of the stochastic Crank-Nicolson scheme is represented for $\gamma = 0.01$,

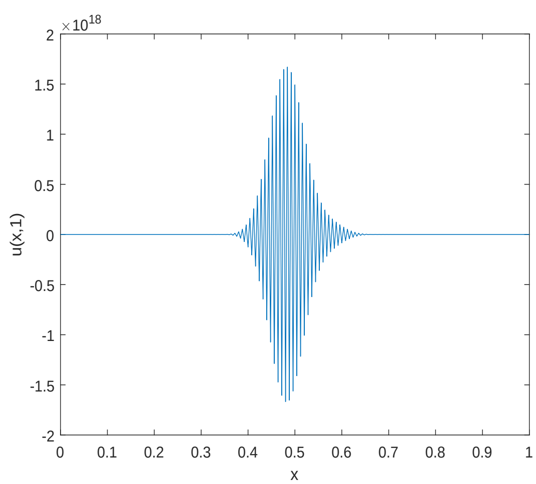
FIGURE 10. Comparison between deterministic and stochastic numerical solutions of (4.3) using the Crank-Nicolson scheme with $\gamma = 0.01$, $\nu = 0.01$, $\sigma = 2.5$, $\Delta x = 0.01$ and $\Delta t = 0.001$ and with $\gamma = 0.001$, $\nu = 0.02$, $\sigma = -2$, $\Delta x = 0.004$ and $\Delta t = 0.01$.



(a) $\gamma = 0.01$, $\nu = 0.01$, $\sigma = 2.5$, $\Delta x = 0.01$, $\Delta t = 0.001$.

(b) $\gamma = 0.001$, $\nu = 0.02$, $\sigma = -2$, $\Delta x = 0.004$, $\Delta t = 0.01$.

FIGURE 11. Numerical solution of stochastic advection diffusion equation using the scheme (2.7).

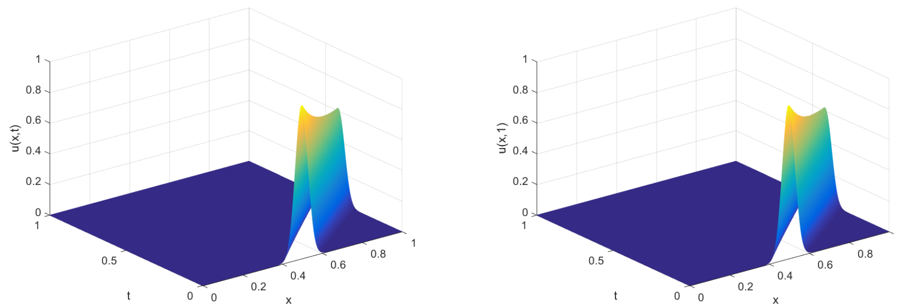


$\gamma = 0.001$, $\nu = 0.02$, $\sigma = -2$, $\Delta x = 0.004$, $\Delta t = 0.01$.

$\nu = 0.01$, $\sigma = 2.5$, $\Delta x = 0.01$, $\Delta t = 0.001$ *and for* $\gamma = 0.001$, $\nu = 0.02$, $\sigma = -2$, $\Delta x = 0.004$, $\Delta t = 0.01$.

In Table 8, some numerical results are presented for solving the stochastic diffusion equation (4.2) by use of the conditional stable scheme (2.7).

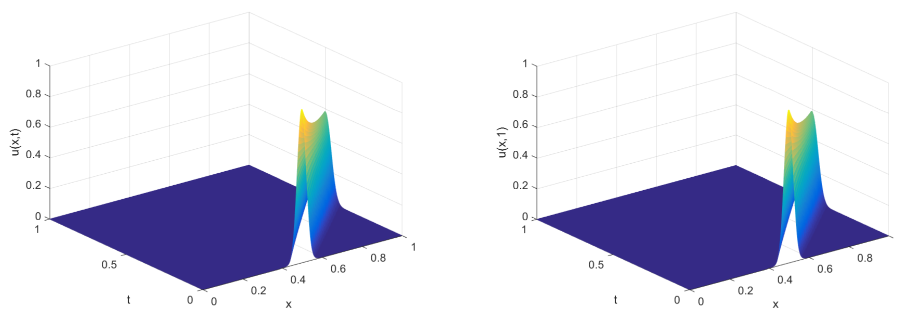
FIGURE 12. Exact solution and numerical solution of (4.3) using the stochastic Crank-Nicolson scheme.



(a) Exact solution with $\gamma = 0.002$ and $\nu = 1.2$.

(b) The numerical solution with $\gamma = 0.002$, $\nu = 1.2$ and $\sigma = 0.01$.

FIGURE 13. Exact solution and numerical solution of (4.3) using the stochastic Crank-Nicolson scheme.



(a) The exact solution with $\gamma = 0.001$ and $\nu = 1$.

(b) The numerical solution with $\gamma = 0.001$, $\nu = 1$ and $\sigma = 0.01$.

The computational results for approximating the solution of SPDE (4.3) are shown in Table 9 by consideration several values for time step and space size, ν, γ and σ. In Figures 12–13 we have shown the exact deterministic solution and the approximation of the stochastic advection diffusion equation using the stochastic Crank-Nicolson scheme on a 500×500 grid with $\gamma = 0.002$, $\nu = 1.2$, $\sigma = 0.01$ and $\gamma = 0.001$, $\nu = 1$, $\sigma = 0.01$ during the time interval $[0, 1]$. If we choose $\gamma = 0.01$, $\nu = 0.1$, $\sigma = -2$ and $M = 200$, we will conclude the convergence of the stochastic scheme for $N \geq 40$. It is obvious from Table 10. Figure 11 shows numerical solution of the scheme (2.7) for values $\gamma = 0.001$, $\nu = 0.02$, $\sigma = -2$, $\Delta x = 0.004$ and $\Delta t = 0.01$. The numerical results obtained by the scheme (2.6), compared to the scheme (2.7), show that the scheme (2.6) is significantly more effective and reliable than the scheme (2.7).

5. Conclusion

In this paper, a stochastic finite difference scheme has been applied for the solution of stochastic advection-diffusion equation. Also, we have provided analysis of consistency, stability, and convergence of the stochastic difference scheme. The scheme has applied to three problems have given in the paper, each with different boundary conditions and has given an initial condition. The numerical results have obtained by the stochastic difference scheme is compared with the exact solution and the scheme in [7], to verify the accuracy and efficiency of the stochastic difference scheme.

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