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Solving singular integral equations by using orthogonal polynomials

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Abstract In this paper, a special technique is studied by using the orthogonal Chebyshev polynomials to get approximate solutions for singular and hyper-singular integral equations of the first kind. A singular integral equation is converted to a system of algebraic equations based on using special properties of Chebyshev series. The error bounds are also stated for the regular part of approximate solution of singular integral equations. The efficiency of the method is illustrated through some examples.

Keywords. Singular integral equations, Chebyshev systems, Approximate quadratures.

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1. Introduction

Let us consider a singular integral equation of the form

$$\int_{-1}^{1} \frac{\psi(t)}{(t-x)^{\alpha}}\, dt + \int_{-1}^{1} K(t,x)\psi(t)dt = f(x), \quad -1 < x < 1, \quad \alpha \in \mathbf{N}, \tag{1.1}$$

where the functions $K(t,x)$ and $f(x)$ are given real-valued Hölder continues functions and $\psi(t)$ is the unknown function to be determined.

The first integral in (1.1) is known as the finite part integral, which was firstly introduced by J. Hadamard who generalized the Cauchy singular integral equations [11]. The singularities of Eq. (1.1) would be mentioned as Cauchy or Hadamard type for $\alpha = 1$ and $\alpha > 1$, respectively.

Equations of the form (1.1) arise in the study of obtaining solutions to mixed boundary value problems of mathematical physics and theoretical mechanics, particularly in the areas of elasticity, aerodynamics, and unsteady aerofoil theory [12].

The classical theoretical and numerical studies on the Eq. (1.1) in the case $\alpha = 1$ are given in [15] and [7] by Muskhelishivi and Goldberg, respectively. The recent studies on the Eq. (1.1) can be found in [5, 8, 9, 10, 14, 17, 18]. For example, Eshkuvatov and Long [5] presented new quadrature formulas to approximate the singular integrals of Cauchy type by utilizing linear spline interpolation, modified discrete vortex and product integration methods. But they only considered the case $K(t,x) = 0$ and $\alpha = 1$, i.e., the characteristic singular integral equation. Kashfi and Shahmorad [10] constructed an approximate solution for Eq. (1.1) in the case $\alpha = 1$ by using Chebyshev polynomials of the first and second kinds. Setia [17] developed a numerical method to solve various types of Cauchy singular integral equations by using Bernstein polynomials. Karczmarek et. al. [9] have approximated the solution of

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such Cauchy type singular integral equations by using Jacobi polynomials. Tsalamengas [18] derived quadrature rules for Cauchy and Hadamard type singular integrals of the form

$$I^{\pm}(K,\psi,t)=\int_{-1}^{1}\omega(t)\psi(t)K(t,x)dt, \qquad \omega(t)=(1-t^2)^{\pm 1/2},$$

from corresponding quadrature rules for logarithmic singular integrals.

For the case $\alpha=1$ a convergence analysis of Galerkin and collocation methods for Eq. (1.1) has been given by Miel [14]. Zhang et. al. [19] discussed a pointwise superconvergence of the composite trapezoidal rule for Hadamard finite part integrals.

A special case of Eq. (1.1) is the famous characteristic singular integral equation

$$\int_{-1}^{1}\frac{\psi(t)}{t-x}\,dt=f(x), \qquad -1<x<1. \tag{1.2}$$

For the Hölder continues $f(x)$, the Eq. (1.2) has analytical solution in four different forms given by

$$\psi(t):=\omega_r(t)\phi_r(t), \qquad r=1,2,3,4, \tag{1.3}$$

where

$$\phi_r(t)=a_0-\frac{1}{\pi^2}\int_{-1}^{1}\frac{f(x)}{\omega_r(x)(x-t)}\,dx, \qquad -1<t<1, \tag{1.4}$$

and

$$\omega_r(t)=\frac{\lambda_r(t)}{\sqrt{1-t^2}}\,, \qquad \lambda_r(t)=\begin{cases}1, & r=1,\\ 1-t^2, & r=2,\\ 1+t, & r=3,\\ 1-t, & r=4,\end{cases} \tag{1.5}$$

(See [5, 7, 12, 17]). In the case $r=1$, a_0 is an arbitrary constant, whereas in other cases we have $a_0=0$. For $\int_{-1}^{1}\psi(t)dt=Const.$ the uniqueness of solution is also guaranteed in the case $r=1$. A necessary and sufficient condition for existence of solution in the case $r=2$ is given by

$$\int_{-1}^{1}\frac{f(x)}{\omega_2(x)}\,dx=0\,.$$

An application of Eq. (1.2) were given in [1] by reducing a system of dual integral equations to Cauchy type singular integral equations.

Let $\alpha=1$. By using (1.3)-(1.4) and a simple manipulation, Eq. (1.1) is converted to the linear Fredholm integral equation

$$\phi_r(t)+\int_{-1}^{1}\omega_r(\tau)K_r(t,\tau)\phi_r(\tau)d\tau=F_r(t), \quad -1<t<1, \quad r=1,2,3,4, \tag{1.6}$$

where

$$\begin{cases} F_r(t)=-\frac{1}{\pi^2}\int_{-1}^{1}\frac{f(x)}{\omega_r(x)(x-t)}\,dx,\\ \\ K_r(t,\tau)=-\frac{1}{\pi^2}\int_{-1}^{1}\frac{K(\tau,x)}{\omega_r(x)(x-t)}\,dx.\end{cases}$$

Since $K(t,x)$ is Hölder continuous, then $K_r(t,\tau)$ is generally integrable and one may proceed to solve Eq. (1.6) numerically [7].

In the next section, we investigate our new methodology for approximating the solution of Eq. (1.1) for $\alpha=1$ in the four above-mentioned cases.

2. Approximate solution of Cauchy integral equation

The Chebyshev polynomial is like a fine jewel that reveals different characteristics under illumination from varying positions [16]. There are many applications of Chebyshev polynomials, specially in solving differential and integral equations. After presenting a brief introduction about these polynomials, we apply them to construct approximate solution for Eq. (1.1).

An infinite series of the form

$$\sum_{i=0}^{\infty} a_i \, P_{r,i}(t), \quad r = 1, 2, 3, 4,$$

is called a Chebyshev series. For $t = \cos(\theta)$, we have

$$P_{r,j}(t) = \begin{cases} T_j(t) = \cos(j\theta), & r = 1, \\ U_j(t) = \sin\left((j+1)\theta\right)/\sin(\theta), & r = 2, \\ V_j(t) = \cos\left((j+\frac{1}{2})\theta\right)/\cos(\frac{\theta}{2}), & r = 3, \\ W_j(t) = \sin\left((j+\frac{1}{2})\theta\right)/\sin(\frac{\theta}{2}), & r = 4, \end{cases} \tag{2.1}$$

which are known as the first, second, third and fourth kind Chebyshev polynomial, respectively.

The functions $P_{r,j}$ satisfy the orthogonality properties

$$\mu_{ij}^r := \frac{1}{\pi} < P_{r,i}, P_{r,j} >_r = \begin{cases} 0, & i \neq j, \\ 1, & i = j = 0\,, \quad r = 1, \\ 1/2, & i = j \neq 0, \quad r = 1, \\ 1/2, & i = j, \quad r = 2, \\ 1, & i = j, \quad r = 3\,,\,4, \end{cases} \tag{2.2}$$

with respect to the inner product

$$< f, g >_r = \int_{-1}^{1} \omega_r(t) f(t) g(t)\, dt\ ,$$

where $\omega_r(t)$, the weight function, is determined from (1.5).

Now, we recall the following theorem from [13].

Theorem 2.1. *As a Cauchy principle value integral, we have*

$$Q_{r,i}(x) := \frac{1}{\pi} \int_{-1}^{1} \frac{\omega_r(t)\, P_{r,i}(t)}{t - x}\, dt \ = \begin{cases} U_{i-1}(x), & r = 1, \\ -T_{i+1}(x), & r = 2, \\ W_i(x), & r = 3, \\ -V_i(x), & r = 4. \end{cases} \tag{2.3}$$

For every integrable function f on $I = [-1,\, 1]$, there is a Chebyshev expansion

$$f(x) := \sum_{j=0}^{\infty} c_j \, P_{r,j}(x), \quad r = 1, 2, 3, 4,$$

associated with the Chebyshev polynomial of kind r, where

$$c_j = \frac{1}{\pi\, \mu_{jj}^r} \int_{-1}^{1} \omega_r(t) f(x) P_{r,j}(x)\, dx, \quad j = 0, 1, \ldots\, .$$

Now, we are in a position to use the Chebyshev polynomials to construct an approximate solution to the Eq. (1.1) in the case $\alpha = 1$. To do this, we set

$$\phi_r(t) := \sum_{i=0}^{\infty} a_i\, P_{r,i}(t), \quad r = 1, 2, 3, 4, \tag{2.4}$$

in (1.3) and

$$K(t,x) \simeq \kappa(t,x) := \sum_{j=0}^{M} b_j(x)\, P_{r,j}(t), \quad r = 1, 2, 3, 4, \tag{2.5}$$

where the coefficients a_i,s are unknown, but $b_j(x)$,s are determined as

$$b_j(x) = \frac{1}{\pi\, \mu_{jj}^r} \int_{-1}^{1} \omega_r(t) \kappa(t,x) P_{r,j}(t)\, dt, \qquad j = 0, 1, \dots, M,$$

which may be calculated approximately by using a Gauss-Chebyshev quadrature rule, i.e.,

$$b_j(x) \simeq \frac{4}{\eta_r} \sum_{n=1}^{M+1} \lambda_r(\tau_{r,n}) \kappa(\tau_{r,n}, x) P_{r,j}(\tau_{r,n}), \quad j = 0, 1, \dots, M,$$

in which $\{\tau_{r,n}\}_{n=1}^{M+1}$ are the roots of $P_{r,M+1}(t)$, i.e.,

$$\tau_{r,n} = \begin{cases} \cos\left((2n-1)\pi/\eta_r\right), & r = 1, 3, \\ \cos\left(2n\pi/\eta_r\right), & r = 2, 4, \end{cases} \tag{2.6}$$

for

$$\eta_r = \begin{cases} 2M+2, & r = 1, \\ 2M+4, & r = 2, \\ 2M+3, & r = 3, 4. \end{cases}$$

Substituting from (1.3) and (2.4)-(2.5) into Eq. (1.1) for $\alpha = 1$ and using (2.2)-(2.3), gives the equations

$$\sum_{i=\sigma}^{\infty} a_i\, Q_{r,i}(x) + \sum_{i=0}^{M} \mu_{ii}^r\, a_i\, b_i(x) = F(x), \quad \sigma = \begin{cases} 1, & r = 1, \\ 0, & r = 2, 3, 4, \end{cases} \tag{2.7}$$

where $F(x) = f(x)/\pi$. In what follows, we show that each of these equations leads to a system of linear algebraic equations in the cases $r = 1, 2, 3, 4$.

Case 1. For $r = 1$, the relations (2.4)-(2.5) take the forms

$$\phi_1(t) := \textstyle\sum_{i=0}^{\infty} a_i\, T_i(t), \qquad \text{and} \qquad \kappa(t,x) := \textstyle\sum_{j=0}^{M}{}' \, b_j(x)\, T_j(t),$$

where $\sum{}'$ denotes that the first term in the summation is halved. Therefore, Eq. (2.7) can be rewritten as

$$\sum_{i=1}^{\infty} a_i U_{i-1}(x) + \frac{1}{2} \sum_{i=0}^{M} a_i b_i(x) = F(x). \tag{2.8}$$

Let the functions $F(x)$ and $b_i(x)$ be integrable with respect to the weight function $\omega_2(x)$ on $[-1\,,\ 1]$, then they can be expanded as

$$\begin{cases} b_i(x) \simeq \sum_{j=0}^{M} B_{ij}\, U_j(x), & i = 0, 1, \dots, M, \\ F(x) \simeq \sum_{j=0}^{M} c_j\, U_j(x), \end{cases} \tag{2.9}$$

where for $i, j = 0, 1, \dots, M$, the coefficients

$$\begin{cases} B_{ij} \simeq \frac{2}{\pi}\int_{-1}^{1}\sqrt{1-x^2}b_i(x)U_j(x)\,dx \simeq \frac{4}{\pi^2}\int_{-1}^{1}\int_{-1}^{1}\sqrt{\frac{1-x^2}{1-t^2}}K(t,x)T_i(t)\,U_j(x)\,dtdx, \\ c_j \simeq \frac{2}{\pi^2}\int_{-1}^{1}\sqrt{1-x^2}f(x)U_j(x)\,dx, \end{cases}$$

can be approximately computed by

$$\begin{cases} B_{ij} \simeq \frac{4}{(M+1)(M+2)}\sum_{m=1}^{M+1}\sum_{n=1}^{M+1}(1-\tau_{2,m}^2)K(\tau_{1,n},\tau_{2,m})T_i(\tau_{1,n})U_j(\tau_{2,m}), \\ c_j \simeq \frac{2}{\pi(M+2)}\sum_{m=1}^{M+1}(1-\tau_{2,m}^2)f(\tau_{2,m})U_j(\tau_{2,m}), \end{cases}$$

where $\tau_{r,n}$ is given by (2.6). Using (2.9) in (2.8) yields

$$\sum_{j=0}^{\infty}a_{j+1}U_j(x) + \frac{1}{2}\sum_{j=0}^{M}\sum_{i=0}^{M}B_{ij}a_iU_j(x) = \sum_{j=0}^{M}c_j\,U_j(x),$$

which is reduced to the linear system of algebraic equations

$$a_{j+1} + \frac{1}{2}\sum_{i=0}^{M}B_{ij}a_i = c_j, \qquad j = 0, 1, \dots, M, \tag{2.10}$$

along with $a_{j+1} = 0$ for $j > M$, by using the orthogonality property (2.2) for $r = 2$. By solving the under-determined system (2.10) for coefficients $\{a_j\}_{j=0}^{M+1}$ we obtain infinitely many approximations for $\phi_1(t)$ as $\varphi_1(t) = \sum_{i=0}^{M+1}a_iT_i(t)$, which results (using (1.3)) infinitely many approximate solutions for (1.1) as

$$\psi(t) = \omega_1(t)\phi_1(t) \simeq \omega_1(t)\varphi_1(t) = \frac{1}{\sqrt{1-t^2}}\sum_{i=0}^{M+1}a_iT_i(t). \tag{2.11}$$

Case 2. For $r = 2$, we set

$$\phi_2(t) := \sum_{i=0}^{\infty}a_i\,U_i(t) \qquad \text{and} \qquad \kappa(t,x) := \sum_{j=0}^{M}\,b_j(x)\,U_j(t),$$

then the relation (2.7) takes the form

$$-\sum_{i=0}^{M}a_iT_{i+1}(x) + \frac{1}{2}\sum_{i=0}^{M}a_ib_i(x) = F(x). \tag{2.12}$$

Using the expansions

$$\begin{cases} b_i(x) \simeq {\sum_{j=0}^{M}}'\,B_{ij}\,T_j(x)\,, \qquad i = 1, 2, \dots, M, \\ F(x) \simeq {\sum_{j=0}^{M}}'\,c_j\,T_j(x), \end{cases}$$

in Eq. (2.12), returns the over-determined linear algebraic system

$$\begin{cases} \frac{1}{2}\sum_{i=0}^{M}B_{ij}a_i = c_j, & j = 0, \\ -a_{j-1} + \frac{1}{2}\sum_{i=0}^{M}B_{ij}a_i = c_j, & j = 1, 2, \dots, M, \end{cases} \tag{2.13}$$

along with $a_{j-1} = 0$ for $j > M$, where

$$\begin{cases} B_{ij} \simeq \frac{4}{(M+1)(M+2)}\sum_{m=1}^{M+1}\sum_{n=1}^{M+1}(1-\tau_{2,n}^2)K(\tau_{2,n},\tau_{1,m})T_j(\tau_{1,m})U_i(\tau_{2,n}), \\ c_j \simeq \frac{2}{\pi(M+1)}\sum_{m=1}^{M+1}f(\tau_{1,m})T_j(\tau_{1,m}). \end{cases}$$

Then, the function $\psi(t)$ will be determined approximately from

$$\psi(t) = \omega_2(t)\phi_2(t) \simeq \omega_2(t)\varphi_2(t) = \sqrt{1-t^2}\sum_{i=0}^{M-1} a_i U_i(t). \tag{2.14}$$

Cases 3,4. For $r = 3, 4$, proceeding by the same way as we did in the cases $r = 1, 2$, we get the linear systems

$$(-1)^{r+1} a_j + \sum_{i=0}^{M} B_{ij} a_i = c_j, \qquad j = 0, 1, \ldots, M, \quad r = 3, 4 \tag{2.15}$$

along with $a_j = 0$ for $j > M$, which lead to the approximate solutions

$$\psi(t) = \omega_r(t)\phi_r(t) \simeq \omega_r(t)\varphi_r(t) = \begin{cases} \sqrt{\frac{1+t}{1-t}}\sum_{i=0}^{M} a_i V_i(t), & r = 3, \\ \sqrt{\frac{1-t}{1+t}}\sum_{i=0}^{M} a_i W_i(t), & r = 4. \end{cases} \tag{2.16}$$

3. Hadamard integral equation

Existence of the solution for the hyper-singular integral equations of the first kind investigated in the spacial case $\alpha = r = 2$, by Chen and Zhou [3].

To find an approximate solution of Eq. (1.1) for $\alpha > 1$ in the cases $r = 1, 2, 3, 4$, we first notice that

$$\int_{-1}^{1} \frac{\psi(t)}{(t-x)^{n+1}}\,dt = \frac{1}{n!}\frac{d^n}{dx^n}\int_{-1}^{1}\frac{\psi(t)}{t-x}\,dt, \qquad n \in \mathbf{N}.$$

Then, we substitute from (1.3) and (2.4)-(2.5) into Eq. (1.1) and use (2.2)-(2.3), to get

$$\frac{1}{(\alpha-1)!}\frac{d^{\alpha-1}}{dx^{\alpha-1}}\sum_{i=\sigma}^{\infty} a_i Q_{r,i}(x) + \sum_{i=0}^{M} \mu_{ii}^{r} a_i b_i(x) = F(x), \tag{3.1}$$

where

$$\sigma = \begin{cases} 1, & r = 1, \\ 0, & r = 2, 3, 4. \end{cases}$$

The next step is to convert Eq. (3.1) to the corresponding algebraic system. To do this, we firstly state the following lemma.

Lemma 3.1. *For $m \geq 1$, we have*

$$\frac{d}{dx}P_{r,m}(x) = \begin{cases} mU_{m-1}(x), & r = 1, \\ \sum_{k=0}^{[\frac{m-1}{2}]} 2(m-2k)U_{m-2k-1}(x), & r = 2, \\ \sum_{k=0}^{m-1} (-1)^k 2(m-k)U_{m-k-1}(x), & r = 3, \\ \sum_{k=0}^{m-1} 2(m-k)U_{m-k-1}(x), & r = 4, \end{cases} \tag{3.2}$$

where $P_{r,m}(x)$ is given by (2.1), and $U_j(x)$ is the Chebyshev polynomial of the second kind.

Proof. For $r = 1$, it directly follows from (2.1)

$$\frac{d}{dx}T_m(x) = \frac{d}{dx}\cos m\theta = \frac{\frac{d}{d\theta}\cos m\theta}{\frac{dx}{d\theta}} = m\frac{\sin m\theta}{\sin\theta} = mU_{m-1}(x).$$

For $r = 2, 3, 4$, the lemma is proved by induction. Here the details of the proof is stated for $r = 2$.
Let $m = 1$. Then

$$\frac{d}{dx}U_1(x) = \frac{d}{dx}2x = 2 = 2U_0(x) = \sum_{k=0}^{[\frac{m-1}{2}]} 2(m-2k)U_{m-2k-1}(x),$$

that is, (3.2) is true for $m = 1$. Let (3.2) is satisfied for $m = n$, i.e.,

$$\frac{d}{dx}U_n(x) = \sum_{k=0}^{[\frac{n-1}{2}]} 2(n-2k)U_{n-2k-1}(x).$$

For $m = n + 1$, we prove

$$\frac{d}{dx}U_{n+1}(x) = \sum_{k=0}^{[\frac{n}{2}]} 2(n-2k+1)U_{n-2k}(x).$$

To do this, we use the relation

$$U_{n+1}(x) - U_{n-1}(x) = 2T_{n+1}(x),$$

from [13] and get

$$\frac{d}{dx}U_{n+1}(x) = \frac{d}{dx}\left(2T_{n+1}(x) + U_{n-1}(x)\right).$$

Using (3.2) for $r = 1$ and the hypothesis of induction, results

$$\begin{aligned}
\frac{d}{dx}U_{n+1}(x) &= 2(n+1)U_n(x) + \sum_{k=0}^{[\frac{n-2}{2}]} 2(n-2k-1)U_{n-2k-2}(x) \\
&= 2(n+1)U_n(x) + \sum_{j=1}^{[\frac{n}{2}]} 2(n-2(j-1)-1)U_{n-2(j-1)-2}(x) \\
&= 2(n+1)U_n(x) + \sum_{j=1}^{[\frac{n}{2}]} 2(n-2j+1)U_{n-2j}(x) \\
&= \sum_{j=0}^{[\frac{n}{2}]} 2(n-2j+1)U_{n-2j}(x),
\end{aligned}$$

which proves (3.2) for $r = 2$ and $m = n + 1$.
For $r = 3, 4$, use the relation

$$V_{n+1}(x) + V_n(x) = W_{n+1}(x) - W_n(x) = 2T_{n+1}(x).$$

Remark 3.2. Since the operator $\frac{d^{\alpha-1}}{dx^{\alpha-1}}$ decreases degree of a polynomial $\alpha - 1$ times, we replace M by $M + \alpha - 1$ in (2.11), (2.14) and (2.16) to get appropriate approximate solution for Eq. (3.1).

Hence, for $\alpha = 2$ by using lemma 3.1 the system of algebraic equations corresponding to Eqs. (3.1) take the following forms for $j = 0, 1, \dots, M$,

$$\begin{cases} 2(j+1)\sum_{i=1}^{[\frac{M-j}{2}]+1} a_{2i+j} + \frac{1}{2}\sum_{i=0}^{M} B_{ij}a_i = c_j, & r = 1, \\ -(j+1)a_j + \frac{1}{2}\sum_{i=0}^{M} B_{ij}a_i = c_j, & r = 2, \\ 2(j+1)\sum_{i=j+1}^{M+1} (-1)^{(r-1)(i-j)}a_i + \sum_{i=0}^{M} B_{ij}a_i = c_j, & r = 3, 4, \end{cases} \tag{3.3}$$

where for $r = 1, 2, 3, 4$, we used (2.9).

4. Error bound

We recall the following definition and theorem from [10] describing an error bound for the approximate solution of Eq. (1.1) for $\alpha = 1$ in the special case $r = 1$.

Definition 4.1. Let $m \geq 0$, $0 < \nu \leq 1$. We say that a function $f(x)$, $x \in [-1, 1]$ belongs to the class $C^{m,\nu}$ if all the derivatives up to the order m inclusive exist and the m^{th} derivative belongs to the Hölder class $H(\nu)$:

$$|f^{(m)}(x) - f^{(m)}(y)| \leq k|x - y|^{\nu}, \quad \forall x, y \in [-1, 1],$$

where the constants k and ν are independent of choice of the points x, y.

Theorem 4.2. *Suppose that the functions $f(x)$ and $K(t, x)$ belong to the class $C^{m,\nu}$ (the second function of both variables) for some $m \geq 0$, $0 < \nu \leq 1$, and they are approximated by a finite Chebyshev series of order M. Moreover, for sufficiently large value of M, the homogeneous equation corresponding to (1.6) has only trivial solution. Then the system of linear equations (2.10) is nonsingular and*

$$\|\phi - \varphi\|_{\infty} \leq N\frac{\ln^2(M)}{M^{m+\nu}}, \tag{4.1}$$

where N is a constant independent of M.

By considering the following relation between the first and three other kinds of Chebyshev polynomials, the error bound (4.1) is confirmed in the cases $2 - 4$.

$$\omega_r(t)P_{r,j}(t) = \omega_1(t)\begin{cases} \frac{1}{2}\left[T_j(t) - T_{j+2}(t)\right], & r = 2, \\ T_j(t) + T_{j+1}(t), & r = 3, \\ T_j(t) - T_{j+1}(t), & r = 4. \end{cases}$$

For a bounded and convex domain $\Omega \subset R^n$, we have $C^{m+1}(\overline{\Omega}) \subset C^{m,1}(\overline{\Omega})$ [2]. Therefore, we deduce the following corollary.

Corollary 4.3. *Let $f(x)$ and $K(t, x)$ are polynomials of degree less than or equal to M (the second function of both variables). Then $f, K \in C^{m,1}$ $(m \to \infty)$ and so the right side of (4.1) tends to zero, i.e., $\phi = \varphi$.*

To have an error bound for the regular part of the approximate solution of Eq. (1.1) for $\alpha > 1$, we recall the following theorem from [6] which stated for $\alpha = 2$ and $r = 2$.

Theorem 4.4. *Let $f \in C^r([-1,1])$ and $K \in C^r([-1,1]\times[-1,1])$. Then the Galerkin approximations*

$$\varphi(t) = \sum_{i=0}^{M} a_i U_i(t),$$

for the solution of

$$\int_{-1}^{1} \sqrt{1-t^2}\frac{\phi(t)}{(t-x)^2}\,dt + \int_{-1}^{1} \sqrt{1-t^2}K(t,x)\phi(t)dt = f(x), \quad -1<x<1,$$

converge uniformly to ϕ, and there exists $n_0 \in \mathbf{N}$ such that

$$\|\phi - \varphi\|_\infty = O(M^{-r+2}),$$

for all $M \geq \max(n_0, r+1)$.

The Chebyshev polynomials of $1,3$ and 4 kinds have the following relations with the second kind:

$$\omega_r(t)P_{r,2j}(t) = \begin{cases} \omega_1(t)T_0(t) - 2\omega_2(t)\sum_{i=0}^{j-1} U_{2i}(t), & r=1, \\ \omega_1(t)(T_0(t)+T_1(t)) - 2\omega_2(t)\sum_{i=0}^{2j-1} U_i(t), & r=3, \\ \omega_1(t)(T_0(t)-T_1(t)) - 2\omega_2(t)\sum_{i=0}^{2j-1} (-1)^i U_i(t), & r=4, \end{cases}$$

and

$$\omega_r(t)P_{r,2j+1}(t) = \begin{cases} \omega_1(t)T_0(t) - 2\omega_2(t)\sum_{i=0}^{j-1} U_{2i+1}(t), & r=1, \\ \omega_1(t)(T_0(t)+T_1(t)) - 2\omega_2(t)\sum_{i=0}^{2j} U_i(t), & r=3, \\ \omega_1(t)(T_1(t)-T_0(t)) - 2\omega_2(t)\sum_{i=0}^{2j} (-1)^{i+1} U_i(t), & r=4. \end{cases}$$

By considering these relations and the facts that

$$\int_{-1}^{1} \frac{\omega_1(t)T_0(t)}{(t-x)^2}dt = \frac{d}{dx}\int_{-1}^{1} \frac{T_0(t)}{\sqrt{1-t^2}(t-x)}dt = \frac{d}{dx}\{0\} = 0,$$

and

$$\int_{-1}^{1} \frac{\omega_1(t)T_1(t)}{(t-x)^2}dt = \frac{d}{dx}\int_{-1}^{1} \frac{T_1(t)}{\sqrt{1-t^2}(t-x)}dt = \frac{d}{dx}U_0(x) = 0,$$

Theorem 4.4 holds for the other kinds of Chebyshev polynomials.

5. Examples

The following examples illustrate application of the method.

Example 1. Let

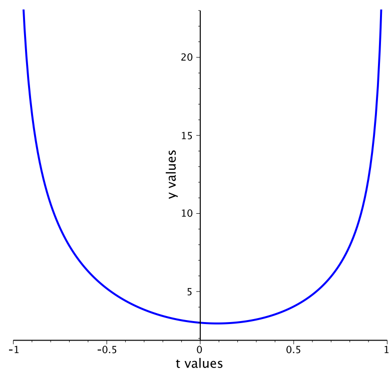
$$\int_{-1}^{1} \frac{\psi(t)}{t-x}\,dt + \int_{-1}^{1} (t-x)^2\psi(t)dt = \pi(5x^2+5x+2), \quad -1<x<1. \tag{5.1}$$

Find the unique solution in the case $r=1$, subject to $\int_{-1}^{1}\psi(t)dt = 5\pi$.

According to (1.1), we have

$$K(t,x) = (t-x)^2 \quad \text{and} \quad f(x) = \pi(5x^2+5x+2).$$

FIGURE 1. The plot of exact solution in Example 1.



We set

$$\psi(t) := \frac{1}{\sqrt{1-t^2}} \left\{ a_0\, T_0(t) + a_1\, T_1(t) + \ldots + a_M\, T_M(t) \right\}, \tag{5.2}$$

and expand $K(t,x)$ as

$$K(t,x) = b_0(x)T_0(t) + b_1(x)T_1(t) + b_2(x)T_2(t),$$

where

$$b_0(x) = x^2 + 1/2, \quad b_1(x) = -2x, \quad b_2(x) = 1/2\,.$$

Substituting these expansions in (5.1) and using (2.2)-(2.3) for $r = 1$, returns

$$\pi\Big(a_1U_0(x) + a_2U_1(x) + a_3U_2(x)\Big) + \frac{\pi}{2}\Big(2a_0b_0(x) + a_1b_1(x) + a_2b_2(x)\Big) = f(x). \tag{5.3}$$

Now, using the expansions

$$\begin{cases} F(x) := f(x)/\pi = (13/4)U_0(x) + (5/2)U_1(x) + (5/4)U_2(x), \\ \\ b_0(x) = (3/4)U_0(x) + (1/4)U_2(x), \quad b_1(x) = -U_1(x), \quad b_2(x) = (1/2)U_0(x), \end{cases}$$

in (5.3) and considering the linear independency of $\{U_j(x)\}$, yield

$$\begin{cases} (3/4)a_0 + a_1 + (1/4)a_2 = 13/4, \\ -(1/2)a_1 + a_2 = 5/2, \\ (1/4)a_0 + a_3 = 5/4, \end{cases} \tag{5.4}$$

along with $a_j = 0$ for $j > 3$. The system (5.4) is an under-determined linear system and its solution is given by

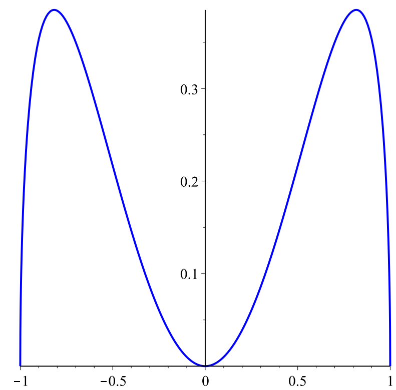
$$a_0 = 5 - 4C, \quad a_1 = -1 + (8/3)C, \quad a_2 = 2 + (4/3)C, \quad a_3 = C, \quad a_4 = \ldots = a_M = 0,$$

for some arbitrary arbitrary constant C. Therefore, Eq. (5.1) has an infinite set of solutions of the form (5.2). Using the uniqueness condition $\int_{-1}^{1} \psi(t)dt = 5\pi$ we get $C = 0$, then

$$\psi(t) = \frac{5T_0(t) - T_1(t) + 2T_2(t)}{\sqrt{1-t^2}} = \frac{4t^2 - t + 3}{\sqrt{1-t^2}}, \qquad -1 < t < 1,$$

(See Figure 1). This figure shows that the solution is unbounded around the endpoints $t = \pm 1$ (case $r = 1$).

Figure 2. Plot of the exact solution in Example 2.



Example 2. [3] Consider the hyper-singular integral equation,

$$\int_{-1}^{1} \frac{\psi(t)}{(t-x)^2}\,dt + \int_{-1}^{1} (t+x)\psi(t)dt = \frac{\pi}{8}(4+x-24x^2), \quad -1 < x < 1, \tag{5.5}$$

with the exact solution $\psi(t) = \sqrt{1-t^2}\, t^2$.

To solve this equation, we set

$$\psi(t) := \sqrt{1-t^2}\Big(a_0\, U_0(t) + a_1\, U_1(t) + \ldots + a_M\, U_M(t)\Big), \tag{5.6}$$

then the system (3.3) for $r = 2$, reduces to the linear system

$$\begin{cases} -a_0 + (1/4)a_1 = -1/4\ , \\ (1/4)a_0 - 2a_1 = 1/16\ , \\ -3a_2 = -3/4\ , \end{cases}$$

along with $a_j = 0$ for $j \geq 3$, which has the solution

$$a_0 = \frac{1}{4}, \quad a_1 = 0, \quad a_2 = \frac{1}{4}, \quad a_j = 0, \quad j = 3, 4, \ldots, M.$$

Hence, the solution (5.6) takes the form

$$\psi(t) = \sqrt{1-t^2}\, \frac{U_0(t) + U_2(t)}{4} = \sqrt{1-t^2}\, t^2, \qquad -1 < t < 1,$$

which is the exact solution and has bounded values at the endpoints $t = \pm 1$ (See Figure 2), while as in [3], the authors found the solution approximately.

Example 3. The general solution of the integral equation,

$$\int_{-1}^{1} \frac{\psi(t)}{(t-x)^2}\,dt + \int_{-1}^{1} \frac{x\sin(t)}{t}\psi(t)dt = 2\pi x, \quad -1 < x < 1,$$

with one degree of freedom is $\psi(t) = \frac{t^3+\nu t}{\sqrt{1-t^2}}$, where ν is any real number.

To solve this equation, we set

$$\psi(t) := \frac{1}{\sqrt{1-t^2}} \sum_{i=0}^{M} a_i\, T_i(t), \tag{5.7}$$

then solving the system (3.3) for $a_0, a_1, \ldots, a_M$, in the case $r = 1$, yield the solutions listed in Table 1, where C_1 and C_2 are arbitrary real constants.

TABLE 1. The values of multipliers of expansion (5.7).

M	a_0	a_1	a_2	a_3	$a_j, \quad j \geq 4$
4	$4(1-4C_2)$	C_1	0	C_2	0
5	$4.3636363636363636364(1-4C_2)$	C_1	0	C_2	0
7	$4.3488108720271800680(1-4C_2)$	C_1	0	C_2	0
10	$4.3491040489705664244(1-4C_2)$	C_1	0	C_2	0
20	$4.3491005148015315601(1-4C_2)$	C_1	0	C_2	0
50	$4.3491005148015315673(1-4C_2)$	C_1	0	C_2	0
100	$4.3491005148015315673(1-4C_2)$	C_1	0	C_2	0

Setting $C_2 = \frac{1}{4}$, yield

$$\psi(t) = \frac{1}{\sqrt{1-t^2}}\Big(\frac{1}{4}T_3(t) + C_1 T_1(t)\Big) = \frac{1}{\sqrt{1-t^2}}\Big(t^3 + (C_1 - \frac{3}{4})t\Big), \qquad -1 < t < 1,$$

which is the general solution (5.7).

Example 4. Consider the hyper-singular integral equation

$$\int_{-1}^{1} \frac{\psi(t)}{(t-x)^2}\, dt + \int_{-1}^{1} t\, e^x \psi(t) dt = -4\pi x + (\pi/4)e^x, \quad -1 < x < 1,$$

which has the exact solution $\psi(t) = \omega_2(t)\phi_2(t)$ with $\omega_2(t) = \sqrt{1-t^2}$ and $\phi_2(t) = 2t$.

To construct an approximate solution for this equation, let

$$\psi(t) \simeq \omega_2(t)\varphi_2(t), \quad \varphi_2(t) = \sum_{i=0}^{M} a_i\, U_i(t).$$

Then solving the system (3.3) in the case $r = 2$, yield the function $\varphi_2(t)$ for different values of M. For $M = 20$ the error function $E_M(t) = |\phi_2(t) - \varphi_2(t)|$ plotted in Figure 3.

Example 5. Let

$$\int_{-1}^{1} \frac{\psi(t)}{(t-x)^3}\, dt + \int_{-1}^{1} (2t-1)x\, \psi(t)\, dt = \pi, \quad -1 < x < 1. \tag{5.8}$$

The exact solution of this equation in the case $r = 3$, is given by

$$\psi(t) = \sqrt{\frac{1+t}{1-t}}\left(c_0 V_0(t) + c_1 V_1(t) + c_2 V_2(t) + c_3 V_3(x)\right).$$

where $c_0 = c_1 - 48c_3$ and $c_2 = \frac{1}{4} - c_3$.

To find the approximate solution in the case $r = 3$, according to (1.1), we have

$$\alpha = 3, \quad K(t,x) = (2t-1)x \quad \text{and} \quad f(x) = \pi.$$

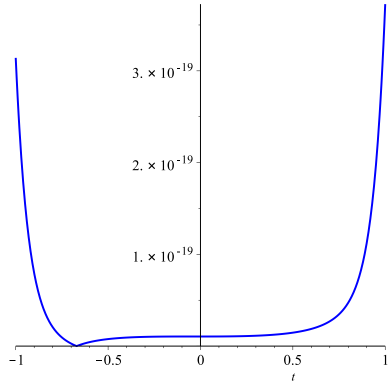
Then we set

$$\psi(t) := \sqrt{\frac{1+t}{1-t}}\left\{a_0\, V_0(t) + a_1\, V_1(t) + \ldots + a_M\, V_M(t)\right\},$$

and expand $K(t,x)$ as

$$K(t,x) = b_1(x)V_1(t), \qquad b_1(x) = x.$$

FIGURE 3. Plot of the error function E_M for Example 4.



Substituting these expansions in (5.8) and using (2.2)-(2.3) for $r = 3$ along with (3.1) (for $\alpha = 3$), returns

$$\frac{1}{2}\frac{d^2}{dx^2}\Big(a_0W_0(x) + a_1W_1(x) + \ldots + a_MW_M(x)\Big) + a_1b_1(x) = 1.$$

Applying (3.2) for $r = 3$ and then for $r = 2$, and using

$$b_1(x) = \frac{1}{2}U_1(x), \qquad F(x) := f(x)/\pi = U_0(x).$$

yield

$$a_0 = C_1 - 48C_2, \quad a_1 = C_1, \quad a_2 = \frac{1}{4} - C_2, \quad a_3 = C_2, \quad a_4 = \ldots = a_M = 0$$

for $M \geq 3$ and some arbitrary constants C_1 and C_2. Therefore, Eq.(5.8) has an infinite set of exact solutions of the form

$$\psi(t) = \sqrt{\frac{1+t}{1-t}}\left((C_1 - 48C_2)V_0(t) + C_1V_1(t) + (\frac{1}{4} - C_2)V_2(t) + C_2V_3(x)\right).$$

Example 6. Find the solution of integral equation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\psi(t)}{(t-x)^3}\,dt = 1 - 6x, \quad -1 < x < 1, \tag{5.9}$$

in the case $r = 4$.

Let $\psi(t) = \sqrt{\frac{1-t}{1+t}}\phi(t)$. Then by using (3.1), Eq. (5.9) can be rewritten as

$$\frac{1}{2\pi}\frac{d^2}{dx^2}\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\frac{\phi(t)}{t-x}\,dt = 1 - 6x, \quad -1 < x < 1,$$

which is reduced to

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\frac{\phi(t)}{t-x}\,dt = -2x^3 + x^2 + c_1x + c_2, \quad -1 < x < 1,$$

for some arbitrary constants c_1 and c_2. Putting $c_1 = 1$ and $c_2 = -\frac{1}{4}$, the right-hand side of the last equation will be $\frac{1}{4}W_3(x)$, therefore, using (2.3) for $r = 4$, we get

$$\phi(t) = \frac{1}{4}W_3(t) = 2t^3 + t^2 - t - \frac{1}{4}, \quad -1 < t < 1.$$

6. Conclusions

We generalized the idea of using Chebyshev polynomials for the numerical solution of singular integral equations of the form (1.1), and solved some singular and hyper-singular integral equations to illustrate the efficiency of the method.

The advantages of the presented method are

- using various Chebyshev series simultaneously to solve (1.1).
- obtaining the exact solution of problem (1.1), whenever the functions K and f are polynomials in comparison with other existing methods such as the method of [3].
- giving approximate solution with high accuracy, when the functions K and f are not polynomials (see the result of Example 4).

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