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Rational Chebyshev collocation approach in the solution of the axisymmetric stagnation flow on a circular cylinder

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Abstract In this paper, a spectral collocation approach based on the rational Chebyshev functions for solving the axisymmetric stagnation point flow on an infinite stationary circular cylinder is suggested. The Navier-Stokes equations which govern the flow, are changed to a boundary value problem with a semi-infinite domain and a third-order nonlinear ordinary differential equation by applying proper similarity transformations. The approach is named the rational Chebyshev collocation (RCC) method. This method reduces this nonlinear ordinary differential equation to an algebraic equations system. RCC method is a strong kind of the collocation technique to solve the problems of boundary value over a semi-infinite interval without truncating them to a finite domain. We also present the comparison of this work with others and show that the present method is more effective and precise.

Keywords. Axisymmetric flow, Stagnation point, Collocation method, Rational Chebyshev functions, Boundary value problem.

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1. Introduction

One of the main problems in fluid dynamics is the stagnation point flow. The fluid flow in the vicinity of a stagnation point is called stagnation point flow or stagnation flow and the area of stagnation would be where the mass and heat transfer rates and fluid pressure are highest.

The stagnation flow is studied for several decades due to the technical significance in a variety of industrial applications, like the cooling of electronic components and blades of gas turbine, the drying of films and papers, the metal and glass tempering during processing and painting of surfaces.

The two-dimensional stagnation point flow over a plate was initially introduced by Hiemenz [18]. He illustrated that the Navier-Stokes equations of this problem can be

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simplified to a third-order ordinary differential equation using similarity transformation. By Howarth [20], the solution of Hiemenz flow was later improved. Then, by Homann [19], three-dimensional axisymmetric stagnation point flow on a circular flat plate was investigated.

Wang [31] was among the first to investigate the axisymmetric stagnation flow on an infinite stationary circular cylinder. Many other problems in axisymmetric stagnation flow on cylinder were discussed by Gorla [9, 11, 10, 12, 13], Cunning et al. [6], Takhar et al. [30], Rahimi [23], Weidman et al. [32] and Saleh et al. [25].

In all of these researches, the Navier-Stokes equations which govern the flow have been reduced to a third-order ordinary differential equation of a boundary value problem with a semi-infinite domain by using similarity transformations. Due to the complexity or absence of analytical solutions, the simplified differential equations with two point boundary conditions, have been solved numerically with a fourth-order Runge-Kutta method along with a shooting method. Because of the asymptotic boundary condition, it would be needed some considerations to solve the differential boundary value problem (BVP) [24]. Lately, spectral methods have been successfully used to solve the boundary value problems defined on unbounded domains [21].

Spectral techniques are very applicable and effective methods to solve differential equations and are generally a kind of weighted residual methods. Spectral methods exhibit a particular group of approximation methods, that in a specific way, the residuals or errors are minimized and consequently generate the particular techniques like the collocation, Galerkin and Tau formulations [2]. In several studies, different types of spectral techniques have been considered to solve problems with particular boundary conditions or in bounded domains [3, 5, 8, 28, 29]. But, many problems exist in engineering and science defined in the unbounded intervals. Various spectral methods can be applied to solve problems in semi-infinite intervals and infinite domains. When the computational interval is unbounded, a variety of options are available that are classified into three main groups:

The first method would be the application of spectral methods based on the orthogonal polynomials over unbounded domains regarding a weight function, like the spectral methods of Hermite (for infinite domain) and Laguerre (for semi-infinite interval) [7, 14, 15, 27].

By choosing L large enough, the second method is truncating semi-infinite domain $[0, \infty)$ and infinite domain $(-\infty, \infty)$ to $[0, L]$ and $[-L, L]$ intervals respectively and applying the spectral methods to solve the problem. This strategy is called domain truncation [3].

The third method is applying the spectral collocation methods based on the rational orthogonal functions to solve the problems with unbounded intervals. Boyd [3, 4] defined a system of orthogonal polynomials for an infinite domain, called as the functions of rational Chebyshev, by mapping the Chebyshev orthogonal polynomials and used them to solve such problems. Also Guo et al. [16] introduced an orthogonal system of rational Legendre functions for solving differential equations on the half line.

Recently, some spectral methods based on the rational Chebyshev functions have been used to solve some types of boundary value problems in fluid dynamics on unbounded domains [1, 2, 21]. In all of these studies, the new basis functions have been generated by using a transformation that maps a semi-infinite interval $[0,\infty)$ into the finite domain $[-1,1]$. But the computational interval of the current problem is semi-infinite interval $[1,\infty]$ and this interval transforms into the finite domain $[-1,1]$ by use of proper mapping. Then the rational Chebyshev functions are formed by this mapping and a spectral collocation method based on these new basis functions is formulated and used for the analysis of the axisymmetric stagnation point flow on an infinite stationary cylinder and subsequently the nonlinear equation which govern this flow would be solved and also analyzed.

2. Problem Formulation

Let us consider the laminar, steady, incompressible flow of a viscous fluid in the neighborhood of an axisymmetric stagnation point flow on an infinite stationary cylinder. The flow model in cylindrical coordinates (r,θ,z) can be seen in Figure 1 with relevant velocity components (u,v,w). An external axisymmetric radial stagnation flow of strain rate $\overline{k}$ impinges on the cylinder with radius a and centered on $r=0$. The flow is axisymmetric about the z-axis and also symmetric to the $z=0$ plane. The stagnation line is at $z=0$ and $r=a$. The steady Navier-Stokes equations in cylindrical polar coordinates governing the axisymmetric flow, neglecting the body force and also neglecting the variation of viscosity, is given by [26]:
Mass:

$$\frac{\partial}{\partial r}(ru)+r\frac{\partial w}{\partial z}=0 \tag{2.1}$$

Momentum:

$$u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\upsilon(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}+\frac{\partial^2 u}{\partial z^2}) \tag{2.2}$$

$$u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\upsilon(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}) \tag{2.3}$$

where υ, ρ and p are the kinematic viscosity, density and fluid pressure respectively. The velocity field boundary conditions are:

$$r=a \quad : \quad u=0, \quad w=0, \tag{2.4}$$

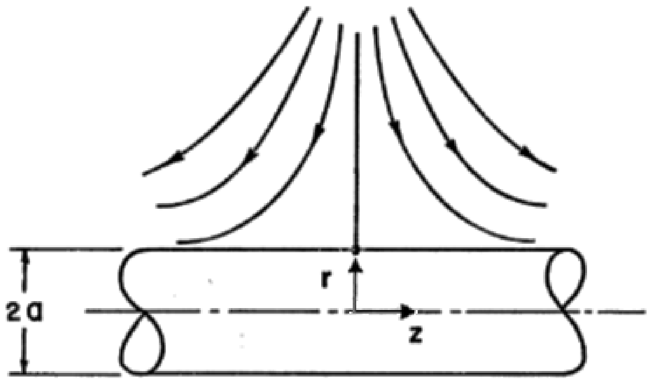
$$r\to\infty \quad : \quad u=-\overline{k}(r-\frac{a^2}{r}), \quad w=2\overline{k}z. \tag{2.5}$$

Here, relation (2.4) is no-slip condition on the circular cylinder wall and the equation (2.5) shows that the viscous flow solution approaches the potential flow solution, as $r\to\infty$ [9].

By considering the similarity transformations in the form:

$$u=-\overline{k}\frac{a}{\sqrt{\eta}}f(\eta), \qquad w=2\overline{k}f^{'}(\eta)z, \qquad \eta=(\frac{r}{a})^2. \tag{2.6}$$

FIGURE 1. Schematic diagram of a coordinate system and flow model



It can be verified that transformations in (2.6) satisfy the conservation of mass equation (2.1) automatically. Insertion of transformations in (2.6) into the governing equations (2.2) and (2.3) yields an ordinary differential equation in term of $f(\eta)$ as following:

$$\eta f^{'''} + f^{'''} + \mathrm{Re}[1 - (f')^2 + f f^{''}] = 0. \tag{2.7}$$

In this equation, $\mathrm{Re} = \frac{\bar{k}a^2}{2\nu}$ is the Reynolds number and primes indicate differentiation with respect to η. The boundary conditions (2.4) and (2.5) become:

$$\begin{cases} \eta = 1 \quad : \quad f = 0\,, \quad f' = 0, \\ \eta \to \infty \quad : \quad f' = 1. \end{cases} \tag{2.8}$$

Relation (2.7) is similar to the one introduced by Wang [31] and were solved by applying the fourth-order Runge-Kutta numerical integration approach. In this study, the relation (2.7) is solved using the rational Chebyshev collocation approach.

3. Shear Stress

For boundary layer flow, the shear stress at the wall surface or the wall skin friction τ_w is calculated as below:

$$\tau_w = \mu \frac{\partial w}{\partial r}\bigg|_{r=a}, \tag{3.1}$$

where μ is the fluid viscosity. Considering the equation (2.6), the surface shear stress would be as the following:

$$\tau_w = \frac{4\mu\bar{k}}{a} z f^{''}(1). \tag{3.2}$$

Therefore, $f''(1)$ is proportional to surface shear stress. Due to their correlation with physical quantities, the $f''(1)$ was obtained in our results.

4. Rational Chebyshev Polynomials

The aim of this work is to apply an effective type of spectral methods called the rational Chebyshev collocation method for solving the problem of boundary value (2.7). The rational Chebyshev polynomials as well as their main properties are presented here [2].

The common Chebyshev polynomial $T_l(\xi)$ is the l-th normalized eigenfunction of the problem of singular Sturm-Liouville:

$$\sqrt{1-\xi^2}\left[\sqrt{1-\xi^2}\, T_l{}'(\xi)\right]' + l^2\, T_l(\xi) = 0\,, \qquad \xi \in (-1,1).$$

The following three-term recurrence relation is also satisfied with the Chebyshev polynomials:

$$T_0\left(\xi\right) = 1, \quad T_1\left(\xi\right) = \xi,$$

$$T_{n+1}\left(\xi\right) = 2\xi T_n\left(\xi\right) - T_{n-1}\left(\xi\right),\ n \geq 1,$$

which in the interval $[-1,1]$ are orthogonal regarding the weight function $\omega(\xi) = \frac{1}{\sqrt{1-\xi^2}}$ i.e.,

$$\int_{-1}^{1} T_i(\xi)\, T_j(\xi)\, \omega(\xi)\, d\xi = \frac{c_i \pi}{2}\delta_{ij},$$

where $c_0 = 2$, $c_i = 1$ for $i \geq 1$ and δ_{ij} is the Kronecker function. As previously stated, it is clear that only for $\xi \in [-1,1]$, the common Chebyshev polynomials would be valid. With respect to problems with semi-infinite domain, a transformation is used to map a semi-infinite interval into a finite domain. New basis sets are obtained by this mapping for the semi-infinite interval [1].

Boyd [3] suggested algebraic mapping as below form:

$$\tau = \frac{L\left(1+\xi\right)}{1-\xi} \qquad \leftrightarrow \qquad \xi = \frac{\tau - L}{\tau + L}, \tag{4.1}$$

where L is constant. For every fixed L, the provided algebraic mapping would map the semi-infinite interval $[0,\infty)$ into $[-1,1]$. Therefore, new basis sets $R_l(\tau)$ would be produced for the semi-infinite interval as the images under the change-of-coordinate of Chebyshev polynomials:

$$R_l\left(\tau\right) = T_l\left(\frac{\tau - L}{\tau + L}\right) = \cos\left(lt\right),\ t = 2\cot^{-1}\left(\sqrt{\frac{\tau}{L}}\right),\ t \in \left[0,\pi\right]. \tag{4.2}$$

Therefore, the rational Chebyshev polynomials $R_n(\tau)$ are defined as the below three-term recurrence equations:

$$R_0\left(\tau\right) = 1\,, \quad R_1\left(\tau\right) = \frac{\tau - L}{\tau + L}\,,$$

$$R_{n+1}\left(\tau\right) = 2\left(\frac{\tau - L}{\tau + L}\right) R_n\left(\tau\right) - R_{n-1}\left(\tau\right),\ n \geq 1. \tag{4.3}$$

It can be seen that $R_l(\tau)$ is the l-th eigenfunction of the problem of singular Sturm-Liouville:

$$(\tau + L)\,\frac{\sqrt{\tau}}{L}\left[(\tau + L)\,\sqrt{\tau}R_l^{'}\left(\tau\right)\right]^{'} + l^2 R_l\left(\tau\right) = 0\,, \quad \tau \in (0, \infty)\,,$$

and rational Chebyshev polynomials would be orthogonal regarding the weight function $\omega(\tau) = \frac{\sqrt{L}}{\sqrt{\tau}(L+\tau)}$ with the orthogonality property, in the interval $[0, \infty)$, as follows:

$$\int_0^\infty R_i\left(\tau\right) R_j\left(\tau\right)\omega\left(\tau\right) d\tau = \frac{c_i \pi}{2}\delta_{ij}, \tag{4.4}$$

where $c_0 = 2$ and $c_i = 1$ for $i \geq 1$.

In this study, basic features of the rational Chebyshev polynomials are introduced [2].

Let $\Omega = [0, \infty)$ and $\omega(\tau)$ be an integrable, non-negative and real valued weight function on the Ω. A normed space $L^2_\omega(\Omega)$, is defined as below:

$$L^2_\omega\left(\Omega\right) = \left\{\upsilon\,|\upsilon \text{ is measurable on } \Omega \text{ and } ||\upsilon||_\omega \leq \infty\right\},$$

where

$$||\upsilon||_\omega = \left(\int_0^\infty |\upsilon\left(\tau\right)|^2 \omega\left(\tau\right) d\tau\right)^{\frac{1}{2}},$$

and $||\,.\,||_\omega$ is the norm induced from the inner product $\langle\,.\,,\,.\,\rangle_\omega$ of the space $L^2_\omega(\Omega)$, i.e.,

$$\langle u, \upsilon\rangle_\omega = \int_0^\infty \upsilon\left(\tau\right) u\left(\tau\right)\omega\left(\tau\right) d\tau.$$

Hence, from the relation of Chebyshev polynomials orthogonality (4.4), it can be concluded that the rational Chebyshev polynomials $R_l(\tau)$ provide for $L^2_\omega(\Omega)$ a set of complete orthogonal basis [17, 22].

For any function $f \in L^2_\omega(\Omega)$, the following expansion can be considered:

$$f\left(\tau\right) = \sum_{i=0}^\infty f_i\; R_i\left(\tau\right), \tag{4.5}$$

with

$$f_i = \frac{\langle f, R_i\rangle_\omega}{\|R_i\|^2_\omega} = \frac{2}{c_i\pi}\int_0^\infty f\left(\tau\right) R_i\left(\tau\right)\omega\left(\tau\right) d\tau,$$

where f_i's are considered as the expansion coefficients relevant to the family $\{R_i\}_{i\geq 0}$.

5. Rational Chebyshev Collocation Method

With respect to any positive integer N, $\Re_N = \text{span}\,\{R_0,\, R_1, \ldots, R_N\}$ is defined and below spectral approximation is considered:

$$f_N\left(\tau\right) = \sum_{k=0}^N f_k R_k\left(\tau\right). \tag{5.1}$$

The principle idea regarding the collocation approach is to achieve the coefficients f_k in a way that in the interior collocation points $\{\tau_j\}_{j=0}^N$ the residual function vanishes. In this approach, to solve the problem (2.7) with boundary conditions (2.8), the below $N+1$ rational Chebyshev-Gauss-Radau points were employed as the collocation points:

$$\tau_j = L\frac{1+\xi_j}{1-\xi_j} \quad , \quad j = 0, 1, \ldots, N, \tag{5.2}$$

where ξ_j's are considered as the $N+1$ points of Chebyshev-Gauss-Radau:

$$\xi_j = -\cos\left(\frac{2j\pi}{2N+1}\right) \quad , \quad j = 0, 1, \ldots, N.$$

Consequently, a system of nonlinear relations with $N+1$ unknowns f_k (the expansion coefficients of $f_k(\tau)$) and $N+1$ equations are produced that by the method of Newton, can be solved numerically.

6. Rcc Method Convergence

To study the rational Chebyshev approach convergence, the orthogonal projection was introduced [1].

Generally, the $L^2_\omega(\Omega)$-Orthogonal projection would be defined as below:

$$P_N : L^2_\omega(\Omega) \to \Re_N \quad \text{by: } \langle P_N f - f, \phi\rangle_\omega = 0\,, \quad \forall \phi \in \Re_N,$$

where $P_N f(\tau) = f_N(\tau)$.

The equation (5.1) indicates that f_N is the orthogonal projection of f upon $\Re_N$ regarding the weighted inner product $\langle\, .\, ,\, .\,\rangle_\omega$.

Now, in order to calculate $\|P_N f - f\|_\omega$, the normed space is defined:

$$H^r_\omega(\Omega) = \Big\{ \upsilon \,|\upsilon \text{ is measurable on } \Omega \text{ and } ||\upsilon||_{r,\omega} < \infty \Big\},$$

where the norm, for the non-negative integer r, is induced by:

$$||\upsilon||_{r,\omega} = \left(\sum_{k=0}^{r} \left\|(\tau+1)^{\frac{r}{2}+k}\,\frac{d^k}{d\tau}\upsilon\right\|^2_\omega\right)^{\frac{1}{2}}.$$

Consequently, the following theorem is presented for the convergence.

Theorem 6.1. *For any $f \in H^r_\omega(I)$ and $f \geq 0$,*

$$||P_N f - f||_\omega \leq cN^{-r}||f||_{r,\omega}.$$

Proof. see [17]. □

From this theorem, it is evident that the approximation of rational Chebyshev is exponentially convergent.

7. Using the Rcc Method for Solving Present Problem

Currently, the rational Chebyshev collocation approach is employed to solve the problem (2.7) with the boundary conditions (2.8).

From the boundary conditions (2.8), it can be observed that the computational domain of the problem (2.7) is semi-infinite interval $[1, \infty]$ and this interval transforms into the $[-1, 1]$ by replacing τ with $\tau - 1$ in mapping equation (4.1) as below form [3]:

$$\tau = L\frac{1+\xi}{1-\xi} + 1 \qquad \leftrightarrow \qquad \xi = \frac{\tau - 1 - L}{\tau - 1 + L}.$$

By this new algebraic mapping, the rational Chebyshev polynomials $R_n(\tau)$ become as the below three-term recurrence equations:

$$R_0(\tau) = 1\,, \quad R_1(\tau) = \frac{\tau - 1 - L}{\tau - 1 + L}\,,$$

$$R_{n+1}(\tau) = 2\left(\frac{\tau - 1 - L}{\tau - 1 + L}\right) R_n(\tau) - R_{n-1}(\tau)\,,\ n \geq 1.$$

Now, $f_N(\tau)$ is substituted on the function f in equation (2.7). Considering the definitions of $R_N(\tau)$ and $f_N(\tau)$, we have $R_i^{'}(\infty) = 0$ for $i = 0, 1, ..., N$, and consequently $f_N^{'}(\infty) = 0$. An extra simple term to the equation (5.1) is added to satisfy the boundary conditions (2.8) and the below approximation would be considered:

$$\tilde{f}_N(\tau) = \tau + \sum_{k=0}^{N} f_k\ R_k(\tau)\,, \tag{7.1}$$

where $\tilde{f}_N'(\infty) = 1$. Therefore, the boundary condition $f'(\infty) = 1$ would be already satisfied. Now, if $f(\tau)$ is replaced with approximate solution $\tilde{f}_N(\tau)$ into the equation (2.7), subsequently the residual function is obtained as below form:

$$\mathrm{Res}(\tau) = \tau \tilde{f}_N^{'''}(\tau) + \tilde{f}_N^{''}(\tau) + \mathrm{Re}\left[1 - (\tilde{f}_N^{'}(\tau))^2 + \tilde{f}_N(\tau)\,\tilde{f}_N^{''}(\tau)\right] = 0. \tag{7.2}$$

As previously mentioned, to obtain the coefficients f_k , the equation (7.2) is equalized to zero at below rational Chebyshev-Gauss-Radau collocation points:

$$\tau_j = L\frac{1+\xi_j}{1-\xi_j} + 1 \quad , \quad j = 0, 1, \ldots, N.$$

Thus, we have:

$$\begin{cases} \mathrm{Res}_N(\tau_j) = 0\,, \quad j = 1, 2, \ldots, N-1, \\ \tilde{f}_N(0) = 0, \\ \tilde{f}_N'(0) = 0. \end{cases} \tag{7.3}$$

System (7.3) includes $N + 1$ nonlinear relations, which is numerically solved using the method of Newton.

C M D E

8. Orders of Convergence

It is very useful if we have the accurate definitions for classifying the convergence rate [3]. We consider an expansion series of function u as below:

$$u(x) = \sum_{n=0}^{\infty} a_n \varphi_n(x),$$

where a_n and φ_n are the coefficients and expansion functions of the series, respectively.

Definition 8.1. The algebraic index of convergence k defined as the largest number for which:

$$\lim_{n\to\infty} |a_n| n^k < \infty, \quad n \gg 1.$$

Alternative Definition: If the coefficients of a series are a_n and if:

$$a_n \sim O[\frac{1}{n^k}], \quad n \gg 1.$$

then k is the algebraic index of convergence.

Definition 8.2. If the algebraic index of convergence k is unbounded -in other words, if the coefficients a_n decrease faster than $1/n^k$ for any finite power of k- then the series is said to have the property of "infinite order", "exponential" or "spectral" convergence.

Alternative Definition: if:

$$a_n \sim O[\exp(-qn^r)], \quad n \gg 1,$$

with q a constant for some $r > 0$, then the series has infinite order or exponential convergence.

The equivalence of the second definition to the first is shown by below expression:

$$\lim_{n\to\infty} n^k \exp(-qn^r) = 0, \quad \text{all } k, \text{all } r > 0.$$

Definition 8.3. The exponential index of convergence r is given by:

$$r = \lim_{n\to\infty} \frac{\log|\log(|a_n|)|}{\log(n)}. \tag{8.1}$$

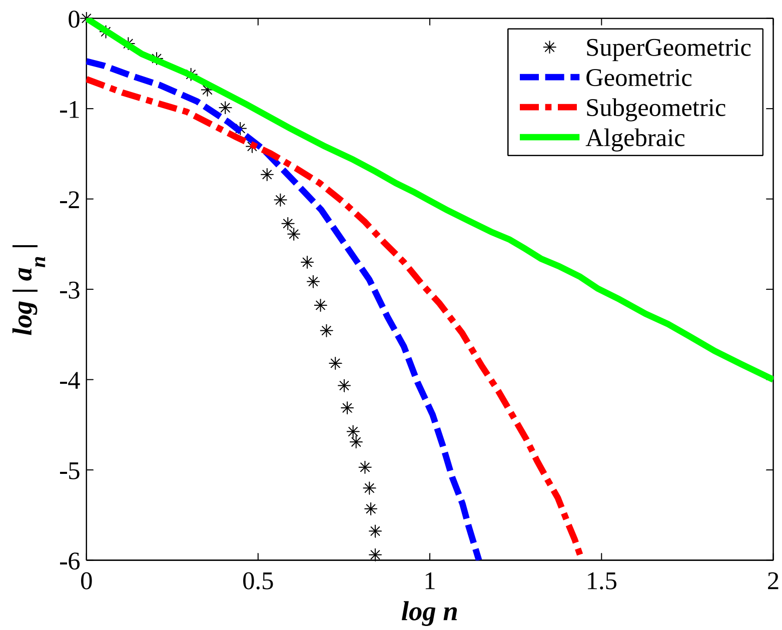
Definition 8.4. (Rates of Exponential Convergence) A series whose coefficients are a_n is said to have the property of supergeometric, geometric or subgeometric convergence depending upon whether:

$$\lim_{n\to\infty} \log(|a_n|)/n = \begin{cases} \infty, & \text{supergeometric,} \\ \text{constant,} & \text{geometric,} \\ 0, & \text{subgeometric.} \end{cases}$$

Alternative Definitions:
1. If $a_n \sim O([\,]\exp\{-(n/j)\log n\})$, convergence is supergeometric.
2. If $a_n \sim O([\,]\exp\{-qn\})$, convergence is geometric.
3. If the exponential index of convergence $r < 1$, then the convergence is subgeometric. (The empty brackets [ ] denote factors that vary more slowly with n than the exponentials.)

For better illustration, presented types of convergence are drawn in Figures 2, 3. On a log-log graph, all types of exponential convergence (Supergeometric, Geometric and Subgeometric) bend away with ever-increasing negative slopes as shown in Figure 2. On a log-linear graph, the coefficients of a Geometrically converging series will asymptote to a straight line as shown in Figure 3. Supergeometric convergence curve develops a more and more negative slope (rather than a constant slope) on a log-linear graph. Subgeometric and algebraic convergence curves bend upward away from the straight line of geometric convergence. Their slopes tend to zero as below.

FIGURE 2. $\log |a_n|$ versus $\log n$ for four rates of convergence.

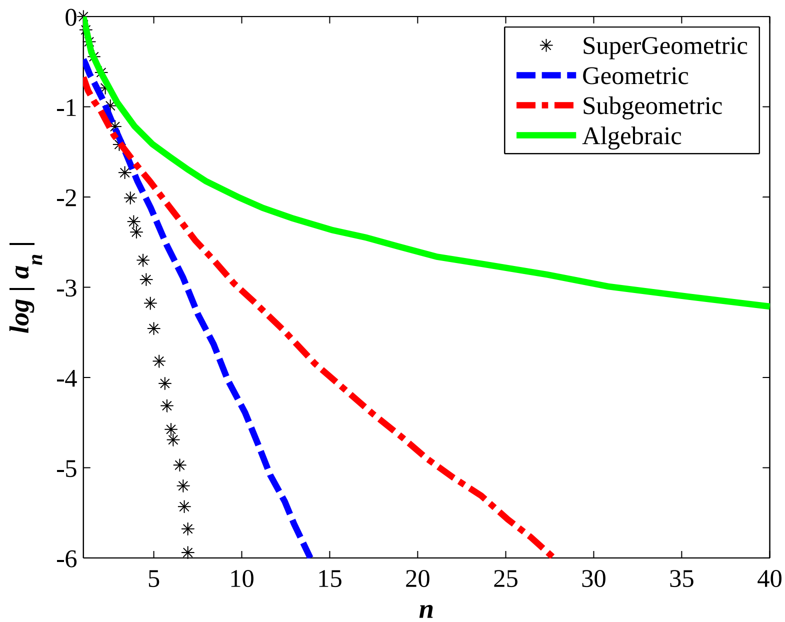


9. Results Presentation

In this section, the rational Chebyshev collocation solution of the relation (2.7) along with the boundary conditions (2.8) for various numbers of the collocation points, N, and by choosing the arbitrary constant map parameter L equal to 2.5, are presented. As mentioned earlier, the $f^{''}(1)$ is proportional to surface shear stress and is a key point of the function. Therefore, it was calculated. Moreover, to check the accuracy of RCC method results, a fourth-order Runge-Kutta technique along with a shooting approach was applied to solve the equation (2.7) and the RCC method is compared by this fourth-order Runge-Kutta technique and available published results of Wang [31] and Gorla [9].

For several values of N and Reynolds number (Re), the $\mathrm{Re}^{-1/2} f^{''}(1)$ approximations calculated by the RCC approach and their absolute errors are indicated in Table

FIGURE 3. $\log|a_n|$ versus n for four rates of convergence.



1. In this Table, the last row gives obtained values by Wang [31] and the errors of the RCC method is calculated with respect to these values. From Table 1, it can be observed that by increasing the number of the collocation points, the absolute values of the errors decrease, showing the rapid convergence and stability of the RCC approach. It is seen that $N \geq 30$ gives good results and the values of the $\mathrm{Re}^{-1/2} f^{''}(1)$ don't change. But we must note that by increasing N, the time of processing is increased too. Therefore, we have used $N = 30$ for next results in this study.

For the verification of accuracy of the RCC approach, a comparison of the presented results with the available published results of Wang [31] and Gorla [9] and calculated by the fourth-order Runge-Kutta method is provided and presented in Tables 2-5 and Figures 4 -6. The results are found in excellent agreement.

In Table 2, the $\mathrm{Re}^{-1/2} f^{''}(1)$ approximations calculated by the RCC approach for $N = 30, L = 2.5$ and several values of Re, have been compared with the fourth-order Runge-Kutta method and those obtained by Wang [31] and Gorla [9]. The present results indicate that the selected value of L is suitable to provide an accurate solution.

Tables 3-5 show the variations of $f(\eta)$, $f^{'}(\eta)$ and $f^{''}(\eta)$ approximated by the method proposed in this study for $N = 30, L = 2.5, \mathrm{Re} = 1$, several values of η and those of Wang [31] and Gorla [9] and the fourth-order Runge-Kutta method. This comparison indicates that the RCC method provides us an approximate solution with a high accuracy level.

TABLE 1. Numerical results of the $\text{Re}^{-1/2}f''(1)$ for $L = 2.5$ and several values of N, Re.

N	**Re = 0.2**		**Re = 1**		**Re = 10**	
	$\mathbf{Re^{-1/2}f''(1)}$	**Error**	$\mathbf{Re^{-1/2}f''(1)}$	**Error**	$\mathbf{Re^{-1/2}f''(1)}$	**Error**
5	1.7420	1.57E-02	1.481356	2.83E-03	1.40503	8.86E-02
10	1.7569	8.00E-04	1.484124	6.10E-05	1.31598	4.50E-04
15	1.7575	2.00E-04	1.484196	1.10E-05	1.31640	3.00E-05
20	1.7575	2.00E-04	1.484182	3.00E-06	1.31643	0
25	1.7576	1.00E-04	1.484182	3.00E-06	1.31643	0
30	1.7576	1.00E-04	1.484183	2.00E-06	1.31643	0
35	1.7576	1.00E-04	1.484183	2.00E-06	1.31643	0
40	1.7576	1.00E-04	1.484183	2.00E-06	1.31643	0
Wang	**1.7577**	——	**1.484185**	——	**1.31643**	——

TABLE 2. Comparison of methods in [9], [31], the fourth-order Range-Kutta and the present method for the values of $\text{Re}^{-1/2}f''(1)$.

Re	**RCC Method**	**Gorla** [9]	**Wang** [31]	**Runge-Kutta**
0.1	1.9463279	1.946369	——	1.946388
0.2	1.7576413	1.7577	1.7577	1.7576
1	1.4841835	1.484185	1.484185	10484184
10	1.3164308	10316427	1.31643	1.31643
100	1.2596526	1.259642	——	1.259642

The comparison between the RCC method and Wang [31] and Gorla [9] have been shown in Figures 4-6. Between the results obtained by the RCC method and Wang [31] and Gorla [9] for all values of η, a very good agreement is seen. It is evident from Figures 4 and 5, that $f(\eta)$ and $f^{'}(\eta)$ obtained by the RCC method agree with the boundary conditions (2.8); so that $f(\eta)$ and $f^{'}(\eta)$ are equal to zero at $\eta = 1$ and as η increases, the $f^{'}(\eta)$ increases to approach 1 at infinity. We know that where $f^{'}(\eta) = 1$, it represents the edge of the boundary layer and it can be seen that in higher Reynolds number, the $f^{'}(\eta)$ approaches to 1 sooner and the thickness of the boundary layer decreases.

The logarithmic graphs of the absolute coefficients $|f_k|$ of the rational Chebyshev functions in the approximate solutions versus $\log k$ and k for $N = 30$ and $L = 2.5$ can be seen in Figures 7 and 8 respectively. The graphs indicate the convergence and stability of the RCC approach. It can be seen from the comparison of Figures 7 and 2 that the series (5.1) has exponential convergence. Also, comparison of Figures 8 and 3 shows the subgeometric convergence of the series.

The exponential index of convergence, r, has been shown in Figure 9 that r is calculated by (8.1). It is seen that $r < 1$ and as previously explained, we conclude that the spectral approximation (5.1) has subgeometric convergence.

TABLE 3. Approximation of the $f(\eta)$ for present method, [9], [31] and Runge-Kutta method (Re= 1, $L = 2.5$).

η	**Gorla** [9]	**Wang** [31]	**Runge-Kutta**	**RCC** ($N = 30$)
1.0	0	0	0	0
1.2	0.02667	0.02667	0.02667	0.02667
1.4	0.09665	0.09665	0.09665	0.09665
1.6	0.19836	0.19836	0.19836	0.19836
1.8	0.32361	0.32361	0.32361	0.32361
2.0	0.46647	0.46647	0.46647	0.46647
3.0	1.32664	1.3266	1.32664	1.32664
4.0	2.2868	2.2867	2.28680	2.28680
5.0	3.27491	3.2748	3.27490	3.27490
6.0	4.27124	4.2712	4.27123	4.27123
7.0	5.27009	5.27	5.27007	5.27007
8.0	6.26973	6.2697	6.26971	6.26969
9.0	7.26963	7.2695	7.26959	7.26957
10.0	8.2696	8.2695	8.26955	8.26953
11.0	9.26961	9.2695	9.26955	9.26952

TABLE 4. Approximation of $f'(\eta)$ for present method, [31] and Runge-Kutta method (Re= 1, $L = 2.5$).

η	**Wang** [31]	**Runge-Kutta**	**RCC** ($N = 30$)
1	0	0	0
1.2	0.25302	0.25303	0.25303
1.4	0.43724	0.43724	0.43724
1.6	0.57315	0.57316	0.57315
1.8	0.67444	0.67445	0.67445
2.0	0.75054	0.75055	0.75055
3.0	0.93068	0.93078	0.93068
4.0	0.97961	0.97961	0.97961
5.0	0.99378	0.99378	0.99378
6.0	0.99805	0.99806	0.99805
7.0	0.99938	0.99938	0.99938
8.0	0.9998	0.99980	0.99980
9.0	0.99993	0.99994	0.99993
10.0	0.99998	0.99998	0.99998
11.0	1	1	0.99999

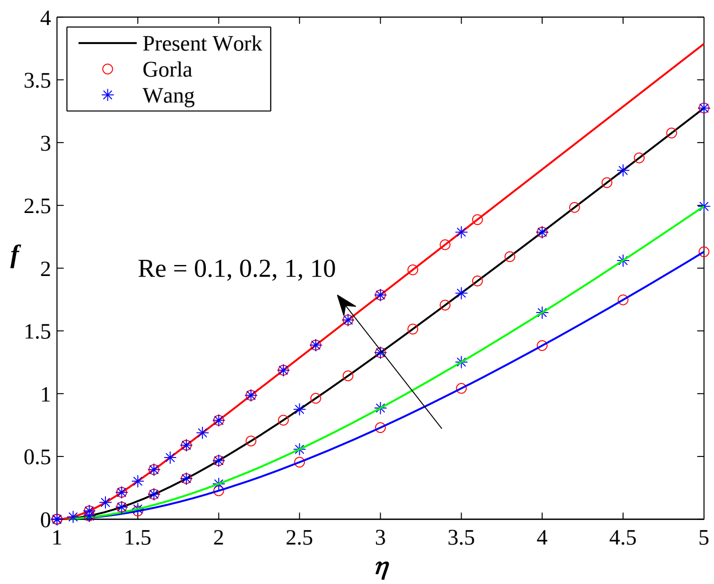
10. CONCLUSION

In this research, an efficient and precise numerical method known as the rational Chebyshev collocation (RCC) approach was applied to solve third-order nonlinear

TABLE 5. Approximation of $f''(\eta)$ for present method, [31] and Runge-Kutta method (Re= 1, $L = 2.5$).

η	**Wang** [31]	**Runge-Kutta**	**RCC** ($N = 30$)
1	1.484185	1.484184	1.484183
1.2	1.07223	1.072232	1.072231
1.4	0.78662	0.786618	0.786617
1.6	0.58369	0.583690	0.583690
1.8	0.43697	0.436973	0.436973
2.0	0.32949	0.329489	0.329489
3.0	0.08647	0.086474	0.086474
4.0	0.02453	0.024532	0.024532
5.0	0.00729	0.007296	0.007295
6.0	0.00224	0.002241	0.002240
7.0	0.0007	0.000705	0.000704
8.0	0.00022	0.000226	0.000225
9.0	0.00007	0.000074	0.000073
10.0	0.00002	0.000025	0.000024
11.0	0	0.000009	0.000008

FIGURE 4. Profiles of the $f(\eta)$ calculated by the RCC method ($N = 30$, $L = 2.5$), Gorla [9] and Wang [31].



differential equation originated from the similarity solution of an axisymmetric stagnation point flow on an infinite stationary cylinder. This method is a strong kind

FIGURE 5. Sample profiles of $f'(\eta)$ calculated by the RCC method ($N = 30$, $L = 2.5$) and Wang [31].

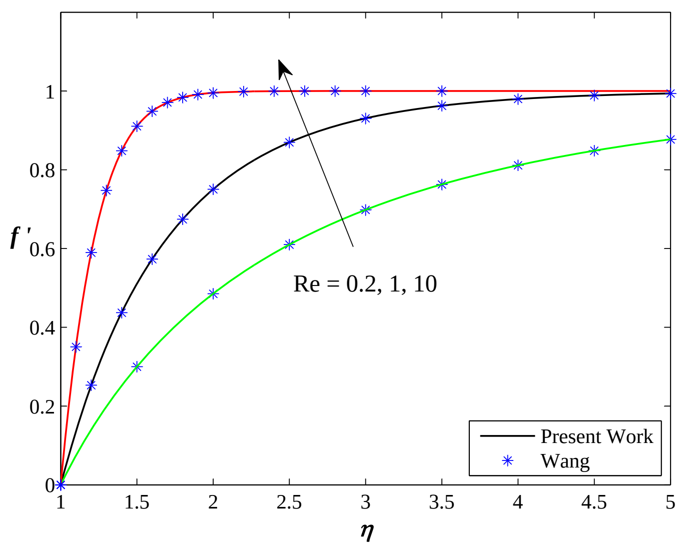


FIGURE 6. Graphs of $f''(\eta)$ calculated by the RCC method ($N = 30$, $L = 2.5$) and Wang [31].

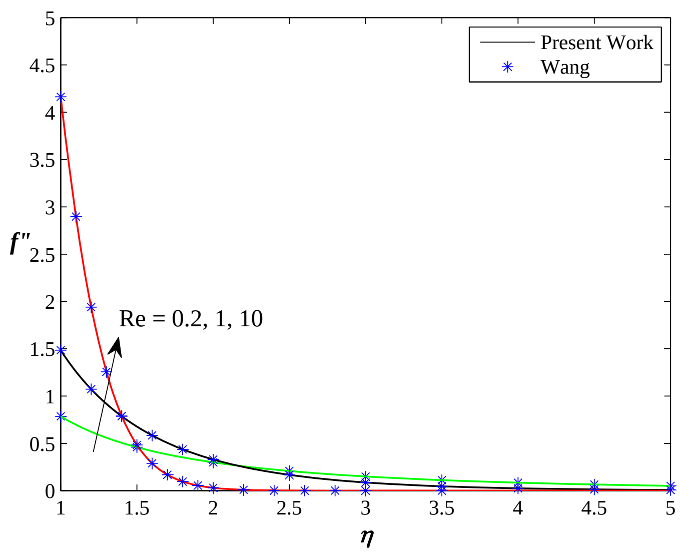


FIGURE 7. Logarithmic graph of absolute coefficients $|f_k|$ of rational Chebyshev functions versus $\log k$ for $N = 30, L = 2.5$.

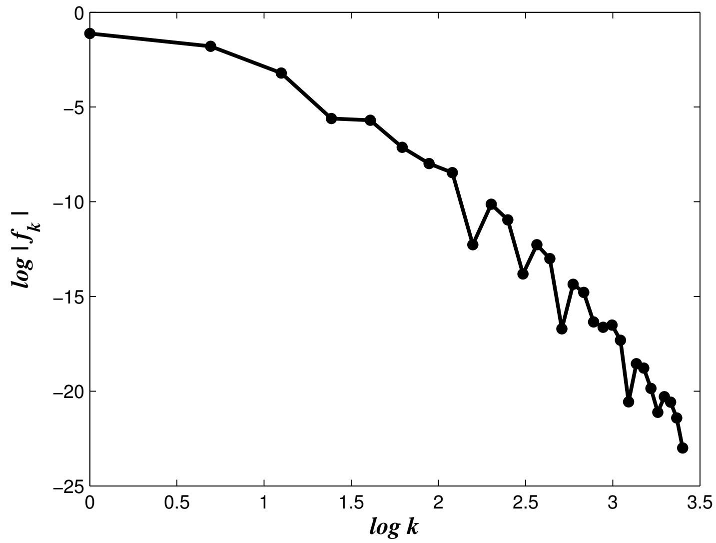


FIGURE 8. Logarithmic graph of absolute coefficients $|f_k|$ of rational Chebyshev functions in the approximate solution for $N = 30, L = 2.5$.

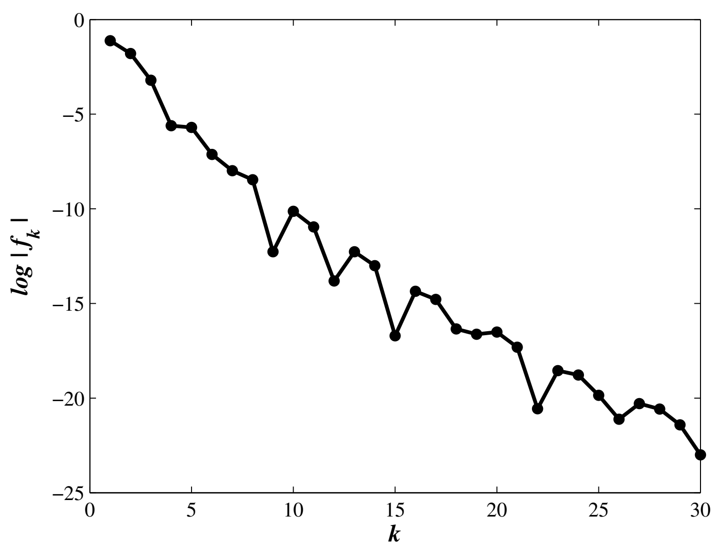
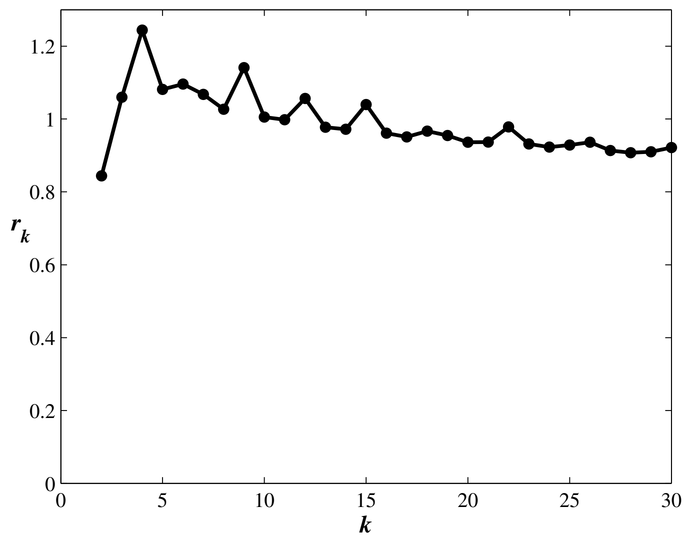


FIGURE 9. The exponential index of convergence r_k versus k for $N = 30$, $L = 2.5$.



of the collocation approach to solve the boundary value problems with semi-infinite domain without truncating them to a finite domain employing, as the basis functions, the rational Chebyshev polynomials. It can be noted that these basis functions would have some benefits: simple to compute, quick convergence and completeness. This approach would decrease the nonlinear ordinary differential equation solution to the solution of a system of algebraic equations.

The comparison between the numerical solutions provided by Gorla [9], Wang [31], the solution of fourth-order Runge-Kutta and approximated by this study, indicates that the RCC approach provides more precise and numerically stable solutions compared to those obtained by other approaches and shows the validity of the current method for problems of boundary value.

References

[1] S. Abbasbandy, H. R. Ghehsareh, and I. Hashim, *An approximate solution of the MHD flow over a non-linear stretching sheet by rational Chebyshev collocation method*, U.P.B. Sci. Bull. Ser. A, *74* (2012), 47–58.

[2] S. Abbasbandy, T. Hayat, H. R. Ghehsareh, and A. Alsaedi, *MHD Falkner-Skan flow of Maxwell fluid by rational Chebyshev collocation method*, Appl. Math. Mech. Engl., *34* (2013), 921–930.

[3] J. P. Boyd, *Chebyshev and Fourier Spectral Methods, 2nd Ed.*, Springer, Berlin, 2000.

[4] J. P. Boyd, *Orthogonal rational functions on a semi-infinite interval*, J. Comput. Phys., *70*(1) (1987), 63–88.

[5] C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang, *Spectral Methods in Fluids Dynamics*, Springer, Berlin, 1990.

[6] G. M. Cunning, A. M. J. Davis, and P. D. Weidman, *Radial stagnation flow on a rotating circular cylinder with uniform transpiration*, J. Eng. Math., *33* (1998), 113–128.

[7] D. Funaro and O. Kavian, *Approximation of some diffusion evolution equations in unbounded domains by Hermite functions*, Math. Comput., *57* (1991), 597–619.
[8] H. R. Ghehsareh, B. Soltanalizadeh, and S. Abbasbandy, *A matrix formulation to the wave equation with non-local boundary condition*, Int. J. Comput. Math., *88* (2011), 1681–1696.
[9] R. S. R. Gorla, *Heat Transfer in an Axisymmetric Stagnation Flow on a Cylinder*, Appl. Sci. Res., *32* (1976) 541–553.
[10] R. S. R. Gorla, *Nonsimilar axisymmetric stagnation flow on a moving cylinder*, Int. J. Eng. Sci., *16* (1978), 392–400.
[11] R. S. R. Gorla, *Transient response behavior of an axisymmetric stagnation flow on a circular cylinder due to a time dependent free stream velocity*, Int. J. Eng. Sci., *16* (1978), 493–502.
[12] R. S. R. Gorla, *Unsteady laminar axisymmetric stagnation flow over a circular cylinder*, Dev. Mech., *9* (1977), 286–288.
[13] R. S. R. Gorla, *Unsteady viscous flow in the vicinity of an axisymmetric stagnation point on a circular cylinder*, Int. J. Eng. Sci., *17* (1979), 87–93.
[14] B. Guo, *Error estimation of Hermite spectral method for nonlinear partial differential equations*, Math. Comput., *68*(227) (1999), 1067–1078.
[15] B. Guo and J. Shen, *Laguerre-Galerkin method for nonlinear partial differential equations on a semi-infinite interval*, Numer. Math., *86* (2000), 635–654.
[16] B. Y. Guo, J. Shen, and Z. Q. Wang, *A rational approximation and its applications to differential equations on the half line*, J. Sci. Comput., *15* (2000), 117–147 .
[17] B. Y. Guo, J. Shen, and Z. Q. Wang, *Chebyshev rational spectral and pseudospectral methods on a semi-infinite interval*, Int. J. Numer. Meth. Eng., *53* (2002), 65–84.
[18] K. Hiemenz, *Die Grenzschicht an einem in den gleichformigen Flussigkeitsstrom eingetauchten geraden Kreiszylinder*, Dingl. Polytech. J., *326* (1911), 321–410.
[19] F. Homann, *Der Einfluss grosser Zahigkeit bei der Stromung um den Zylinder und um die Kugel*, ZAMM-Z. Angew. Math. Me., *16* (1936), 153–164.
[20] L. Howarth, *On the Calculation of Steady Flow in the Boundary Layer Near the Surface of a Cylinder in a Stream*, Aeron. Res. Comm., Rep. Mem., No. 1632, 1934, 1–12.
[21] K. Parand, Z. Delafkar, and F. Bahari Fard, *Rational Chebyshev Tau method for solving natural convection of Darcian fluid about a vertical full cone embedded in porous media whit a prescribed wall temperature*, World Academy of Science, Engineering and Technology, *5*(8) (2011), 1186–1191.
[22] K. Parand and M. Shahini, *Rational Chebyshev pseudospectral approach for solving Thomas-Fermi equation*, Phys. Lett. A, *373* (2009), 210–213.
[23] A. B. Rahimi, *Heat transfer in an axisymmetric stagnation flow on a cylinder at high prandtl numbers using perturbation techniques*, Sci. Iran., *10*(1) (1999), 29–40.
[24] B. Sahoo and F. Labropulu, *Steady Homann flow and heat transfer of an electrically conducting second grade fluid*, Comput. Math. Appl., *63* (2012), 1244–1255.
[25] R. Saleh and A. B. Rahimi, *Axisymmetric stagnation point flow and heat transfer of a viscous fluid on a moving cylinder with time-dependent axial velocity and uniform transpiration*, J. Fluid Eng., T. ASME, *126* (2004), 997–1005.
[26] H. Schlichting, *MHD Boundary layer theory, translate by J. Kestin*, McGraw-Hill, 1968.
[27] J. Shen, *Stable and efficient spectral methods in unbounded domains using Laguerre functions*, SIAM J. Numer. Anal., *38*(4) (2000), 1113–1133.
[28] J. Shen and T. Tang, *High Order Numerical Methods and Algorithms*, Chinese Science Press, Beijing, 2005.
[29] J. Shen, T. Tang, and L. L. Wang, *Spectral Methods, Algorithms, Analyses and Applications*, Springer, Berlin, 2010.
[30] H. S. Takhar, A. J. Chamkha, and G. Nath, *Unsteady axisymmetric stagnation-point flow of a viscous fluid on a cylinder*, Int. J. Eng. Sci., *37* (1999), 1943–1957.
[31] C. Wang, *Axisymmetric stagnation flow on a cylinder*, Q. Appl. Math., *32* (1974), 207–213.
[32] P. D. Weidman and V. Putkaradze, *Axisymmetric stagnation flow obliquely impinging on a circular cylinder*, Eur. J. Mech., B-Fluid, *22* (2003), 123–131.