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A numerical technique for solving a class of 2D variational problems using Legendre spectral method

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Abstract An effective numerical method based on Legendre polynomials is proposed for the solution of a class of variational problems with suitable boundary conditions. The Ritz spectral method is used for finding the approximate solution of the problem. By utilizing the Ritz method, the given nonlinear variational problem reduces to the problem of solving a system of algebraic equations. The advantage of the Ritz method is that it provides greater flexibility in which the boundary conditions are imposed at the end points of the interval. Furthermore, compared with the exact and eigenfunction solutions of the presented problems, the satisfactory results are obtained with low terms of basis elements. The convergence of the method is extensively discussed and finally two illustrative examples are included to demonstrate the validity and applicability of the proposed technique.

Keywords. Ritz method, Legendre polynomials, 2D Variational problems, Eigenfunction method.

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1. Introduction

The calculus of variations investigates methods that permit finding maximal and minimal values of a large number of problems arising in analysis, mechanics and geometry. The problems which consist the investigation of extremum for a function are called variational problems [7, 23]. It began to develop in 1696 and became an independent mathematical discipline with its own methods of investigation after the fundamental works of Euler (1707-1783), whom we may justifiably consider the founder of the calculus of variations [7]. Methods of solving variational problems, i.e. problems involving the investigation of functionals for maxima and minima, are extremely similar to the methods of investigating functions for maxima and minima [7, 8].

The three problems, brachistochrone, geodesics and isoperimetric have played important roles in the development of calculus of variations [6, 7]. In recent years,

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a growing interest has been appeared toward the application of different methods in various types of variational problems, and many new direct and numerical techniques have been introduced in the literature. For more details, we refer readers to [1, 3, 4, 5, 9, 10, 11, 12, 17, 18, 19, 22].
In this paper, we introduce an efficient numerical method to solve variational problems for functionals dependent on two independent variables in the following form

$$\min \quad J[z(x,y)] = \int_0^b \int_0^a F(x,y,z,\frac{\partial z}{\partial x},\frac{\partial z}{\partial y})dxdy, \tag{1.1}$$

with the given boundary conditions

$$z(0,y) = f_1(y), \quad z(a,y) = f_2(y), \quad z(x,0) = g_1(x), \quad z(x,b) = g_2(x), \tag{1.2}$$

where the function F is a three times differentiable function of its arguments. Let us assume that a function $z = z(x,y)$ is being sought that is continuous together with its derivatives up to order two inclusive in the domain D. Furthermore, assume that the values on the boundary Σ of D are given and yield a minimum of the functional (1.1). The Ritz method is one of the most widely used direct variational methods which we use for approximation of the function $z(x,y)$ to minimize the functional (1.1) [7, 8]. According to the Ritz method, we approximate the function $z(x,y)$ with the Legendre polynomial basis and the unknown coefficients such that the function $z(x,y)$ satisfies initial and boundary conditions. Then by substituting the approximate function in the given cost functional, we obtain a nonlinear algebraic equation in terms of the unknown coefficients which should be optimized. By taking the necessary conditions for optimality into account, an algebraic system of equations will appear. The obtained system is solved for finding the unknown coefficients. The main advantage of the Ritz method is that the boundary conditions (1.2) are imposed at the end points of the interval. Moreover, only a small number of bases are needed to obtain a satisfactory result.
The organization of the rest of this paper is as follows. In section 2, we introduce some preliminaries required for our subsequent development. The numerical solution of the problem by Ritz method is presented in section 3. The convergence of the presented method is considered in section 4. To present a clear overview of the method, we select two examples in section 5. We compare the numerical results with analytical solution for Example 5.1 and eigenfunction method for Example 5.2. A conclusion is presented in section 6.

2. Preliminaries

In this section, we briefly present some properties of Legendre orthonormal polynomials, function approximation and then state the Ritz method.

2.1. **Legendre Polynomials.** Let $L_j(r)$ denotes the Legendre polynomial of order j on the interval $[-1,1]$. These polynomials can be derived by the following recessive relation known as Bonnet's recursion formula

$$L_{j+1}(r) = \frac{2j+1}{j+1} r L_j(r) - \frac{j}{j+1} L_{j-1}(r), \; j = 1,2,..$$

where $L_0(r) = 1,\ L_1(r) = r$. By change of variable $x = \frac{2}{a}r - 1$, we can easily obtain the well-known shifted Legendre polynomials on interval $[0, a]$,

$$L_{a,j+1}(x) = \frac{2j+1}{(j+1)a}(2x-a)L_{a,j}(x) - \frac{j}{j+1}L_{a,j-1}(x),\ j = 1, 2, 3, ..$$

$$L_{a,0}(x) = 1,\ L_{a,1}(x) = \frac{2x}{a} - 1.$$

Now define $p_j(x) = \sqrt{\frac{2j+1}{a}}L_{x,j},\ j = 1, 2, ...$ We obtain the orthonormal Legendre polynomials on the interval $[0, a]$, that is,

$$\int_0^a p_i(x)p_j(x)dx = \delta_{i,j},$$

where $\delta_{i,j}$ is the Kronecker delta.
The analytical form of the shifted Legendre polynomials $p_j(x)$ of degree j in the interval $[0, a]$ is defined by [2]

$$p_j(x) = \frac{1}{2^j}\sum_{l=0}^{[\frac{j}{2}]}(-1)^l\binom{j}{l}\binom{2j-2l}{j}s^{j-2l},\ s = \frac{2}{a}x - 1. \tag{2.1}$$

2.2. **Function Approximation.** First, let us introduce a theorem. This theorem exploits the best approximation of a function in an inner product space.

Theorem 2.1. *Let G be an inner product space and $L \neq \emptyset$ be a convex subset which is complete (in the metric induced by the inner product). Then for every given $g \in G$ there exists a unique $h^* \in L$ such that*

$$\min_{h\in L} \|g - h\| = \|g - h^*\|.$$

Proof. [13]. □

Now let $\Delta = [0, a] \times [0, b]$. Suppose that $\{p_i(x)p_j(y)\}_{i,j=0}^{n,m}$ is a subset of $L^2(\Delta)$ out of double product Legendre polynomials. Define

$$\Lambda_{n,m} = \text{Span}\{p_i(x)p_j(y)|0 \le i \le m,\ 0 \le j \le n\},\ m, n \in \mathbb{N}\bigcup\{0\}.$$

Let f be an arbitrary function in $L^2(\Delta)$. Since $\Lambda_{n,m}$ is closed in the complete space $L^2(\Delta)$ so it is complete. Therefore, according to Theorem (2.1), there is a unique best approximation out of $\Lambda_{n,m}$ like $\sigma_{n,m}$ which satisfies in the theorem's condition, that is

$$\forall\ s(x, y) \in \Lambda_{n,m}, \quad \|f - \sigma_{n,m}\|_2 \le \|f - s\|_2,$$

where $\|f\|_2 = \sqrt{< f, f >}$.
Since $\sigma_{n,m} \in \Lambda_{n,m}$ so we can find the real coefficients $a_{ij},\ i = 0, 1, ..., n,\ j = 0, 1, ..., m$ such that

$$\sigma_{n,m}(x, y) \simeq \sum_{i=0}^{n}\sum_{j=0}^{m} a_{ij}p_i(x)p_j(y) = P_n^T A P_m, \tag{2.2}$$

where the shifted Legendre coefficient matrix A, vectors P_n and P_m are given by

$$A = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0m} \\ a_{10} & a_{11} & \cdots & a_{1m} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n0} & a_{n1} & \cdots & a_{nm} \end{bmatrix}, \; P_n = \begin{bmatrix} p_0(x) \\ \vdots \\ \vdots \\ p_n(x) \end{bmatrix}, P_m = \begin{bmatrix} p_0(y) \\ \vdots \\ \vdots \\ p_m(y) \end{bmatrix}$$

and the real coefficients can be specified uniquely as follows [13]:

$$a_{ij} =< p_i(x)p_j(y),\sigma_{n,m}(x,y) >= \int_0^a \int_0^b \sigma_{n,m}(x,y)p_i(x)p_j(y)dxdy,$$
$$i = 0, ..., n, \;\; j = 0, ..., m.$$

2.3. **The Ritz Method.** The underlying idea of the Ritz method is that the values of a functional

$$J[y(x)] = \int_a^b f(x, y, y')dx, \tag{2.3}$$

on the space of all continuously differentiable functions satisfying

$$y(a) = c_0, \;\; y(b) = c_1, \tag{2.4}$$

are considered not on arbitrary admissible curves of a given variational problem but only on all possible linear combinations as a finite series with constant coefficients. In Ritz's method, we seek a solution to the problem of minimization of the functional (2.3), with boundary conditions (2.4), in the form

$$y_n(x) \simeq \sum_{i=1}^{n} c_i \chi_i(x) + \chi_0(x), \tag{2.5}$$

such that $\chi_0(x)$ satisfies (2.4), that is, $\chi_0(a) = c_0, \;\; \chi_0(b) = c_1$. All the remaining functions satisfy the corresponding homogeneous conditions, i.e. in the case at hand, $\chi_i(a) = \chi_i(b) = 0, \;\; i = 1, ..., n$. The c_i are constants. The function $y_n^*(x)$ that minimizes (2.3) on the set of all functions of the form (2.5) is called the nth approximation of the solution by Ritz's method [14].

Now let $\chi_i(x) = w(x)p_i(x)$ then the formula (2.5) can be expressed as

$$y_n(x) \simeq \sum_{i=1}^{n} c_i w(x)p_i(x) + \chi_0(x), \tag{2.6}$$

where $w(x)$ satisfies the homogeneous conditions of the problem and $p_i(x)$ is the Legendre polynomial function.

Consider the following 2-dimensional functional

$$J[y(x,t)] = \int_a^b \int_c^d f(x, t, y, y_x, y_t)dxdt, \tag{2.7}$$

with known initial and boundary conditions. A Ritz approximation to (2.7) is constructed as follows. The approximation y_{mn} is sought in the form of the truncated series

$$y_{nm}(x,t) \simeq \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij} w(x,t) p_i(x) p_j(t) + q(x,t), \tag{2.8}$$

where $p_i(x)$ and $p_j(t)$ are Legendre polynomials. Note that $w(x,t)$ is such that satisfies the homogeneous form of the specified essential boundary conditions. Furthermore $q(x,t)$ satisfies the initial and boundary conditions. It is easy to consider that the approximate solution $y_{mn}(x,t)$ satisfies the initial and boundary conditions [15, 25].

3. Solving Two-Dimensional Variational Problem

Consider the cost functional

$$\min_z \quad J[z(x,y)] = \int_0^b \int_0^a F(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}) dx dy,$$

with the given boundary conditions

$$z(0,y) = f_1(y), \quad z(a,y) = f_2(y), \quad z(x,0) = g_1(x), \quad z(x,b) = g_2(x).$$

According to (2.8), the approximation of function $\hat{z}(x,y)$ by considering the boundary conditions (1.2) is as follows

$$\hat{z}(x,y) \simeq \sum_{i=0}^{n}\sum_{j=0}^{m} c_{ij} xy(x-a)(y-b) p_i(x) p_j(y) + q(x,y), \tag{3.1}$$

such that

$$q(0,y) = f_1(y), \quad q(a,y) = f_2(y), \quad q(x,0) = g_1(x), \quad q(x,b) = g_2(x).$$

The first derivatives of the function $\hat{z}(x,y)$ with respect to x, y are expressed as

$$\begin{aligned} \frac{\partial \hat{z}(x,y)}{\partial x} &\simeq \sum_{i=0}^{n}\sum_{j=0}^{m} c_{ij} y(y-b) p_j(y)[(2x-a)p_i(x) + x(x-a)p_i'(x)] + \frac{\partial q(x,y)}{\partial x}, \\ \frac{\partial \hat{z}(x,y)}{\partial y} &\simeq \sum_{i=0}^{n}\sum_{j=0}^{m} c_{ij} x(x-a) p_i(x)[(2y-b)p_i(y) + y(y-b)p_i'(y)] + \frac{\partial q(x,y)}{\partial y}, \end{aligned} \tag{3.2}$$

respectively. Substituting (3.2) and (3.1) in (1.1), the cost functional $J[\hat{z}(x,y)]$ can be approximated and it becomes a function as $J^* = J[C]$ in nm real unknown variables where

$$C = \begin{bmatrix} c_{00} & c_{01} & \cdots & c_{0m} \\ c_{10} & c_{11} & \cdots & c_{1m} \\ \vdots & \vdots & \cdots & \vdots \\ c_{n0} & c_{n1} & \cdots & c_{nm} \end{bmatrix}.$$

Now the necessary conditions for the minimum of the approximated functional J^* are

$$\frac{\partial J^*}{\partial c_{ij}} = \frac{\partial J[C]}{\partial c_{ij}} = 0, \; i = 1,..,n, \; j = 1,..,m. \tag{3.3}$$

The aforementioned algebraic system can be solved for $c_{ij},\ i = 1, .., n,\ j = 1, .., m$ directly. By determining the matrix C, we can obtain the approximate value of $\hat{z}(x, y)$ from (3.1).

4. Theoretical Analysis

In this section, we will provide theoretical analysis of the proposed method for problem (1.1)-(1.2). We will show that with increase in the number of Legendre polynomials basis m, n in (3.1), the approximate value of the J approaches to its exact value. This fact is shown in Theorem 4.3. Here we construct our function space and present some needed lemmas.
Consider the Banach space $(E(\Delta), ||.||)$ as follows

$$E(\Delta) = \{h(x, y)|h \text{ is continuously differentiable on } \Delta\},$$

$$||h(x, y)|| = ||h(x, y)||_\infty + ||\frac{\partial h(x, y)}{\partial x}||_\infty + ||\frac{\partial h(x, y)}{\partial y}||_\infty.$$

Let E_{mn} be the mn-dimensional linear subspace of the Banach space $(E(\Delta), ||.||)$ spanned by the first mn of the functions $\{p_0(x)p_0(y), ..., p_{m-1}(x)p_{n-1}(y)\}$, that is, the set of all linear combinations of the form

$$\{\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} \alpha_{ij} p_i(x) p_j(y)\},$$

where the coefficients $\alpha_{00}, ..., \alpha_{(m-1)(n-1)}$ are arbitrary real numbers. Then, the restriction of the functional $J[z(x, y)]$ to E_{mn} is a function of mn variables $\alpha_{00}, ..., \alpha_{(m-1)(n-1)}$ as

$$J[\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} \alpha_{ij} p_i(x) p_j(y)]. \tag{4.1}$$

We choose $\alpha_{00}, ..., \alpha_{(m-1)(n-1)}$ in such a way as to minimize (4.1), denoting the minimum value of J by ρ_{mn} and the element of E_{mn} which yields the minimum by z_{mn}.

Lemma 4.1. *The defined functional $J : E(\Delta) \to \mathbb{R}$ in (1.1), is uniformly continuous on the Banach space $(E(\Delta), ||.||)$.*

Proof. $\epsilon > 0$ is given. Let $z \in E(\Delta)$ and $\delta > 0$. Now suppose that $z_1 \in E(\Delta)$ such that

$$||z - z_1|| = ||z - z_1||_\infty + ||\frac{\partial (z - z_1)}{\partial x}||_\infty + ||\frac{\partial (z - z_1)}{\partial y}||_\infty < \delta.$$

Considering our assumptions for the problem (1.1)-(1.2), the function $z(x, y)$ is continuous with its derivatives, hence the function $F(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y})$ is continuous [20] then there exists $\delta(\epsilon) > 0$ such that

$$||F(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}) - F(x, y, z_1, \frac{\partial z_1}{\partial x}, \frac{\partial z_1}{\partial y})||_\infty < \epsilon,$$

provided that $||z - z_1|| < \delta = \delta(\epsilon)$. Therefore we have

$$|J[z(x, y)] - J[z_1(x, y)]| < \epsilon.$$

□

The next Lemma is a result from the Stone-Weierstrass Theorem [21].

Lemma 4.2. *Suppose that* $z(x, y) \in E(\Delta, ||.||)$. *There exists a sequence of polynomial functions* $\{p_i(x)p_j(y)\}_{i,j=0}^{\infty} \in E(\Delta)$ *in which converges to* $z(x, y)$.

The convergence of the proposed method is provided by the following Theorem.

Theorem 4.3. *Let* ρ *be the minimum of the functional* $J[z]$ *on the Banach space* $E(\Delta, ||.||)$. *Then we have*

$$\lim_{(m,n)\to(\infty,\infty)} \rho_{mn} = \rho.$$

Proof. $\epsilon > 0$ is given. By the definition of ρ for any ϵ there exists a $z^* \in E(\Delta, ||.||)$ such that

$$J[z^*] < \rho + \epsilon.$$

Now since $J[z]$ is continuous on $E(\Delta, ||.||)$ then we have

$$|J[z] - J[z^*]| < \epsilon, \tag{4.2}$$

provided that $||z - z^*|| < \delta = \delta(\epsilon)$. According to Lemma 4.2, there is a linear combination ρ_{mn} of the form $\alpha_{00}p_0(x)p_0(y) + ... + \alpha_{(m-1)(n-1)}p_{m-1}(x)p_{n-1}(y)$ such that $||\rho_{mn} - z^*|| < \delta$ for sufficiently large m, n. Therefore, using (4.2), we have

$$\rho < \rho_{mn} = |J[z_{mn}] - J[z^*] + J[z^*]| < |J[z_{mn} - J[z^*]| + |J[z^*]| < \rho + 2\epsilon.$$

Since ϵ is arbitrary, we can conclude that

$$\lim_{(m,n)\to(\infty,\infty)} \rho_{mn} = \lim_{(m,n)\to(\infty,\infty)} J[z_{mn}] = \rho.$$

□

5. Illustrative examples

In this section, we apply our scheme in section 3 for solving two test problems and present the results.

Example 5.1. Find the minimal of the functional

$$\min_{z} \quad J[z(x, y)] = \int_0^1 \int_0^1 ((\frac{\partial z}{\partial x})^2 - (\frac{\partial z}{\partial y})^2)dxdy,$$

with given boundary conditions $z(x, 0) = \sin(\pi x)$, $z(0, y) = z(1, y) = 0$ [24].

Table 1. Absolute Error of $z(x, y)$ for different value of x, y.

x,y	Exact $z(x,y)$	The value of app[†] $z(x,y)$	Abs. Err.[‡] $z(x,y)$
x=y=0.00	0	0	0
x=y=0.25	0.5	0.5	0
x=y=0.50	-2.051010^{-10}	6.856110^{-8}	6.8766110^{-8}
x=y=0.75	-0.5	-0.5	0
x=y=1.00	0	0	0

† The value of approximating for $z(x, y)$
‡ Absolute Error for $z(x, y)$

The exact solution is $z(x, y) = \sin(\pi x)\cos(\pi y)$, so the minimum of the functional is $J = 0$. According to the proposed method in section 3, we approximate the function $z(x, y)$ in terms of the shifted Legendre polynomial basis with respect to $z(x, 0) = \sin(\pi x)$, $z(0, y) = z(1, y) = 0$ as following

$$\hat{z}(x, y) \simeq \sum_{i=0}^{n}\sum_{j=0}^{m} c_{ij}xy(x-1)p_i(x)p_j(y) + \sin(\pi x).$$

Following those mentioned in section 3, the arbitrary approximation of the function $z(x, y)$ can be found. The validity of this numerical scheme has been analyzed by comparison with the exact solution of the problem. In Figure 1, the obtained approximate function $z(x, y)$ by presented method is plotted for $n = m = 6$. In Figure 2, the approximate solution of $z(x, y)$ with $x = 0.5$ and exact solution of the variational problem are plotted together. It is obvious that with increase in the number of the Legendre basis, the approximate values of $z(x, y)$ converge to the exact solution at point $x = 0.5$. Furthermore, Table 1 reports the approximate value of the function $z(x, y)$ for different values of x, y. In this table, the absolute error of the function is reported too. Finally, Table 2 presents the value of performance index J at various value of n, m. We can see that with increase of the number of n, m, the approximate value of J approaches to the exact value of functional, i.e, $J = 0$.

Table 2. Approximate value of functional J^* for Example 5.1.

p.o[†]	n=m=2	n=m=3	n=m=4	n=m=5	n=m=6
J^*	-0.00125768	-0.00125771	-6.256210^{-7}	-6.256210^{-7}	-1.891510^{-10}

† polynomial order

Example 5.2. Consider the following 2D functional

$$\min_{z} \quad J[z(x, t)] = \int_0^1 \int_0^1 \frac{x}{2}[z^2(x, t) + (\frac{\partial z(x, t)}{\partial t})^2]dxdt,$$

with the boundary conditions $z(x, 0) = 1 - x^2$, $z(1, t) = 0$ [16].

FIGURE 1. Approximate solution of $z(x, y)$ for $n = m = 6$ (Example 5.1).

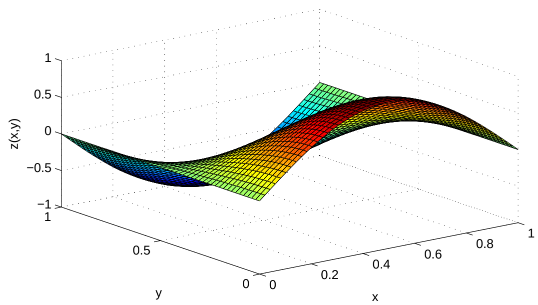


FIGURE 2. Exact and approximate solution of $z(x, y)$ at $x = 0.5$ (Example 5.1).

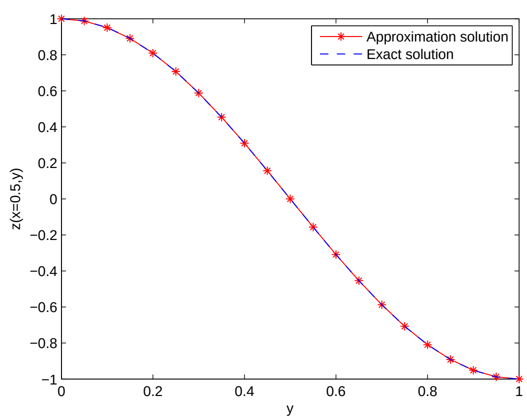


TABLE 3. Approximate value of functional J^* for Example 5.2.

p.o[†]	n=m=2	n=m=3	n=m=4	n=m=5,6
J^*	0.063466638936	0.063466180052	0.063466179667	0.063466179662

[†] polynomial order

Applying the Rizt method, the approximate solution of the state function $z(x, t)$ in terms of the shifted Legendre polynomial basis with respect to $z(x, 0) = 1 - x^2$, $z(1, t) = 0$ is written as

$$\hat{z}(x, t) \simeq t.(x - 1) \sum_{i=0}^{n} \sum_{j=0}^{m} k_{ij} p_i(x) p_j(t) + 1 - x^2. \tag{5.1}$$

Next by implementing the mentioned process in section 3, we obtain the unknown

FIGURE 3. Approximate solution of $z(x,t)$ for $n = m = 6$ (Example 5.2).

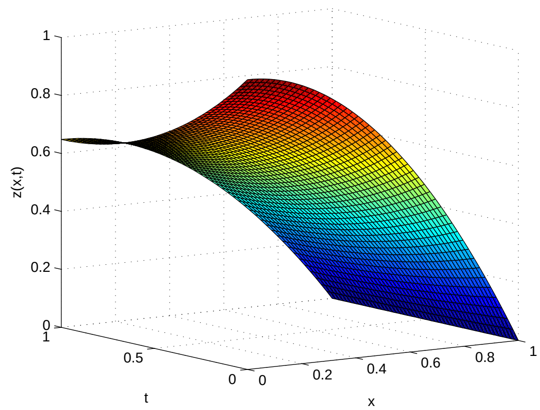
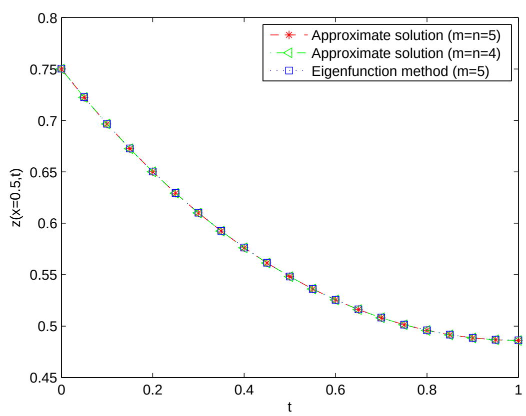


FIGURE 4. Approximate solution of state function $z(x,t)$ for $x = 0.5$ and different value of $m, n(. * . : m = n = 5, - \triangleleft - : m = n = 4)$ and $.\square. :$ state function with the eigenfunction method (Example 5.2).



coefficients k_{ij} in (5.1). Figure 3 shows the surface of the state function $z(x,t)$ by presented method for $m = n = 6$. To verify the validity of our numerical finding results, we utilized the eigenfunction method with specific eigenfunction and found the value of the performance index $J = 0.063$ for $m = 5$. Table 3 presents the obtained numerical results for performance index J at various value of n, m. Obviously we can see that with increase of the number of n, m, the approximate value of J approaches to 0.063. Figure 4 shows the analytical result (for $m = 5$) and the numerical results (for $m = n = 4, m = n = 5$) for the state function $z(x,t)$ at $x = 0.5$. In this figure, it can be considered that the analytical and numerical solutions overlap. We only need a few terms of polynomials to obtain these satisfactory results.

6. Conclusion

In this paper, a numerical solution for a class of variational problems based on Legendre polynomials is established. The suggested method reduces the two variables optimization problem to solving an algebraic system of equations. The approximate polynomial function is of good flexibility in the sense of satisfying the initial and boundary conditions in the proposed method. Theoretical analysis was given to support the suggested method. In addition, only a small number of polynomials order are required to obtain a satisfactory result which the given illustrative examples support this claim.

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