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An efficient extension of the Chebyshev cardinal functions for differential equations with coordinate derivatives of non-integer order

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Abstract In this study, an effective numerical method for solving fractional differential equations using Chebyshev cardinal functions is presented. The fractional derivative is described in the Caputo sense. An operational matrix of fractional order integration is derived and is utilized to reduce the fractional differential equations to system of algebraic equations. In addition, illustrative examples are presented to demonstrate the efficiency and accuracy of the proposed method.

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1. Introduction

Fractional differential equations (FDEs) have been mentioned by Leibniz in a letter to L'Hospital in 1695. A history of the development of FDEs can be found in [22, 17]. Fractional calculus and FDEs have found applications in several different disciplines. There are several (nonequivalent) definitions of the fractional derivative in widespread use and we choose to focus on one particular form (the so-called Caputo version) in this paper.

In recent years, both mathematicians and physicists have devoted considerable effort to find robust and stable numerical and analytical methods for solving fractional differential equations. Numerical and analytical methods have included the finite difference method [16], Adomian decomposition method [18, 19], variational iteration method [20], homotopy perturbation method [21], homotopy analysis method [8] and other methods.

Recently, the operational matrices of fractional order integration for the Legendre wavelet [24], Chebyshev wavelet [14], Haar wavelet [15] and the second kind Chebyshev wavelet have been developed to solve the FDEs. In this paper, we derive the operational matrix of fractional order integration for Chebyshev cardinal functions

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and apply it to solve the FDEs. This method reduces the FDEs to a system of algebraic equations.

In this paper, we focus on the multi-order fractional differential equations as follows:

$$\begin{cases} D_*^{\alpha_n} y(t) = \sum_{j=1}^{n-1} a_j(t) D_*^{\alpha_j} y(t) + a_0(t) y(t) + f(t), \ m-1 < \alpha_n \leq m, \\ \\ y^{(k)}(0) = c_k, \ k = 0, 1, \ldots, m-1, \ m \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\alpha_1 < \alpha_2 < \ldots < \alpha_{n-1} < \alpha_n$ are constant. Also, $a_j(t) \in C[0,1]$ for $j = 0, 1, \ldots, n-1$, and $f(t) \in C[0,1]$, are known functions. Now, we use the initial conditions to reduce problem (1.1) to a problem with zero initial conditions. Therefore, we define

$$y(t) = \bar{y}(t) + x(t), \tag{1.2}$$

where $\bar{y}(t)$ is a known function that satisfied the initial conditions and $x(t)$ is a new unknown function. Substituting (1.2) in (1.1), we have the following initial-value problem:

$$\begin{cases} D_*^{\alpha_n} x(t) = \sum_{j=1}^{n-1} a_j(t) D_*^{\alpha_j} x(t) + a_0(t) x(t) + g(t), \ m-1 < \alpha_n \leq m, \\ \\ x^{(k)}(0) = 0, \ k = 0, 1, \ldots, m-1, \ m \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $g(t)$ is known function. It is to be noted that $g(t)$ is calculated in terms of $f(t)$ and $y^{(k)}, \ k = 0, 1, \ldots, m-1$.

2. Background on fractional derivatives

There are several definitions for fractional differential equations. These definitions include Grunwald-Letnikov, Riemann-Liouville, Caputo, Weyl, Marchaud, Riesz fractional derivatives, Nishimoto fractional operator, Ji Huan He and Jumarie's definitions. This section is devoted to a description of the operational properties in order to be acquainted with sufficient fractional calculus theory and enable us to follow the solutions of the problems given in this paper. For more details see [25, 5, 11, 27].

2.1. **Preliminaries and notations.**

Definition 2.1. *A real function $y(t)$, $t > 0$, is said to be in the space $C_\alpha, \alpha \in \mathbb{R}$, if there exists a real number $p\,(> \alpha)$, such that $y(t) = t^p y_1(t)$, where $y_1(t) \in C[0, \infty)$, and it is said to be in the space $C_\alpha^m, m \in \mathbb{N} \bigcup \{0\}$, if and only if $y^{(m)}(t) \in C_\alpha$.*

Definition 2.2. *For an arbitrary function $y(t) \in C_\alpha$, $\alpha \in \mathbb{R}$, the Riemann-Liouville fractional integral of order α, is defined as*

$$I^\alpha y(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t \frac{y(\tau)}{(t-\tau)^{1-\alpha}} d\tau, & \alpha \neq 0, \ t > 0, \\ \\ y(t), & \alpha = 0, \end{cases} \tag{2.1}$$

where $\Gamma(\alpha)$ is the well-known Gamma function.

Definition 2.3. *The Riemann-Liouville fractional derivative of $y(t) \in C_\alpha$, of order α is defined as*

$$D^\alpha y(t) = \frac{d^m}{dt^m} I^{m-\alpha} y(t),\ m-1 < \alpha \le m,\ m \in \mathbb{N}. \tag{2.2}$$

Definition 2.4. *The Caputo fractional derivative of $y(t) \in C_\alpha^m$, $m \in \mathbb{N} \bigcup \{0\}$, is defined as*

$$D_*^\alpha y(t) = \begin{cases} I^{m-\alpha} y^{(m)}(t), & m-1 < \alpha < m,\ m \in \mathbb{N}, \\ \\ \dfrac{d^m}{dt^m} y(t), & \alpha = m. \end{cases} \tag{2.3}$$

2.2. **The relation between fractional derivative and fractional integral.**

Theorem 2.5. *Assume that the function $y(t) \in C_\mu, \mu \in \mathbb{R}$, has a fractional derivative of order μ. Then the following relations hold* [23]

$$D^\alpha I^\beta y(t) = \begin{cases} I^{\beta-\alpha} y(t), & \alpha < \beta, \\ y(t), & \alpha = \beta, \\ D^{-\beta+\alpha} y(t), & \alpha > \beta, \end{cases} \tag{2.4}$$

$$I^\alpha D_*^\alpha y(t) = y(t) - \sum_{k=0}^{m-1} y^{(k)}(0^+) \frac{t^k}{k!},\ m-1 < \alpha \le m,\ m \in \mathbb{N}, \tag{2.5}$$

$$D_*^\alpha I^\alpha y(t) = \begin{cases} y(t),\ m-1 < \alpha \le m,\ m \in \mathbb{N}, \\ \\ I^\alpha D_*^\alpha y(t) + y(0),\ 0 < \alpha < 1. \end{cases} \tag{2.6}$$

3. The Chebyshev cardinal functions and operational matrix of the fractional integration

In this section, we describe a brief review of the Chebyshev cardinal functions and give some properties of this functions.

3.1. **Chebyshev cardinal functions.** Chebyshev cardinal functions of order N in $[-1,1]$ are defined as [13, 1]

$$C_j(x) = \frac{T_{N+1}(x)}{T'_{N+1}(x_j)(x-x_j)},\ j = 1, 2, \ldots, N+1, \tag{3.1}$$

where $T_{N+1}(x)$ is the first kind Chebyshev function of order $N+1$ in $[-1,1]$ defined by

$$T_{N+1}(x) = \cos((N+1)\arccos(x)), \tag{3.2}$$

and $x_j,\ j = 1, 2, \ldots, N+1$, are the zeros of $T_{N+1}(x)$ defined by $\cos\left(\frac{(2j-1)\pi}{2N+2}\right),\ j = 1, 2, \ldots, N+1$. We change the variable $t = \frac{(x+1)}{2}$ to use these functions on $[0,1]$. Now any function $f(t)$ on $[0,1]$ can be approximated as

$$f(t) \approx \sum_{j=1}^{N+1} f(t_j) C_j(t) = F^T \Phi_N(t), \tag{3.3}$$

where $t_j,\ j = 1, 2, \dots, N+1$, are the shifted points of $x_j,\ j = 1, 2, \dots, N+1$, and

$$F = [f(t_1), f(t_2), \dots, f(t_{N+1})]^T, \tag{3.4}$$

$$\Phi_N(t) = [C_1(t), C_2(t), \dots, C_{N+1}(t)]^T. \tag{3.5}$$

Note that, the functions $C_j(t)$ are satisfy in the following relation

$$C_j(t_i) = \delta_{ji} = \begin{cases} 1, & j = i, \\ 0, & j \neq i. \end{cases} \tag{3.6}$$

Lemma 3.1. *The functions $C_j(x), j = 1, 2 \dots, N+1$ are orthogonal with respect to $\frac{1}{\sqrt{1-x^2}}$ on $[-1,1]$*

$$\langle C_i(x), C_j(x) \rangle = \int_{-1}^{1} \frac{C_i(x) C_j(x)}{\sqrt{1-x^2}} dx = \begin{cases} \frac{\pi}{N+1}, & j = i, \\ 0, & j \neq i. \end{cases} \tag{3.7}$$

Proof. See [10]. □

Lemma 3.2. *Assume know that $\Phi_N(t)$, be (3.5) and $F = [f_1, f_2, \dots, f_{N+1}]^T$ as the column vectors, then*

$$\Phi_N(t) \Phi_N^T(t) F \approx \bar{F} \Phi_N(t), \tag{3.8}$$

where $\bar{F}$ is a product operational matrix as follows:

$$\bar{F} = diag[f_1, f_2, \dots, f_{N+1}]. \tag{3.9}$$

Proof. See [10]. □

3.2. **Operational matrix of the fractional integration.** The operational matrix of integration and product of the Chebyshev cardinal functions have been derived in [25, 5, 13, 10]. The integration of the vector $\Phi_N(t)$ defined in Eq. (3.5) can be approximated as

$$\int_0^t \Phi_N(\tau) d\tau \approx P \Phi_N(t), \tag{3.10}$$

where P is called the operational matrix of integration for Chebyshev cardinal functions which is an $N+1$ order square matrix. Now, we can derive the operational matrix of fractional order integration for Chebyshev cardinal functions. For this purpose, first we use the method given in [9] to expand $\prod_{j=1, j\neq i}^{N+1} (t - t_j)$ as

$$\prod_{j=1, j\neq i}^{N+1} (t - t_j) = \sum_{k=0}^{N} c_{i,k} t^{N-k}. \tag{3.11}$$

Let

$$s_{i,k} = \sum_{j=1, j\neq i}^{N+1} t_j^k,\ k = 1, 2, \dots, N,\ i = 1, 2, \dots, N+1.$$

Then, the coefficients $c_{i,k}$ is given as follows:

$$c_{i,0} = 1,\ c_{i,k} = -\frac{1}{k}\sum_{s=1}^{k} s_{i,s} c_{i,k-s},\ k = 1, 2, \dots, N,\ i = 1, 2, \dots, N+1. \tag{3.12}$$

Lemma 3.3. *The Riemann-Liouville fractional integration of the vector $\Phi_N(t)$ defined in Eq. (3.5) can be expressed by*

$$I^\alpha \Phi_N(t) \approx P^\alpha \Phi_N(t), \tag{3.13}$$

where matrix P^α is called the operational matrix of fractional order integration for Chebyshev cardinal functions.

Proof. Let

$$I^\alpha \Phi_N(t) = \left[I^\alpha C_1(t), I^\alpha C_2(t), \dots, I^\alpha C_{N+1}(t)\right]^T. \tag{3.14}$$

Using (3.11), we have

$$I^\alpha C_i(t) = \frac{\beta}{T'_{N+1}(t_i)} I^\alpha\Big(\prod_{j=1, j\neq i}^{N+1}(t - t_j)\Big) = \frac{\beta}{T'_{N+1}(t_i)} I^\alpha\Big(\sum_{k=0}^{N} c_{i,k} t^{N-k}\Big)$$
$$= \frac{\beta}{T'_{N+1}(t_i)}\sum_{k=0}^{N} c_{i,k} I^\alpha t^{N-k} = \frac{\beta}{T'_{N+1}(t_i)}\sum_{k=0}^{N} c_{i,k}\frac{\Gamma(N-k+1)}{\Gamma(N-k+\alpha+1)} t^{N-k+\alpha}, \tag{3.15}$$

where $\beta = 2^{2N+1}$. As we know, and using (3.3) any function $I^\alpha C_i(t)$ can be approximated as

$$I^\alpha C_i(t) \approx \sum_{j=1}^{N+1} \theta_{ij} C_j(t), \tag{3.16}$$

where

$$\theta_{ij} = I^\alpha C_i(t_j) = \frac{\beta}{T'_{N+1}(t_i)}\sum_{k=0}^{N} c_{i,k}\frac{\Gamma(N-k+1)}{\Gamma(N-k+\alpha+1)} t_j^{N-k+\alpha}. \tag{3.17}$$

Comparing (3.16) and (3.13), we obtain

$$P^\alpha = \left[\theta_{ij}\right]_{(N+1)\times(N+1)}. \tag{3.18}$$

□

Lemma 3.4. *Suppose $C^T = [c_1, c_2, \dots, c_{N+1}]$ and the vector $\Phi_N(t)$ defined in Eq. (3.5), assume $x(t) = C^T \Phi_N(t)$, then*

$$x^2(t) \approx [c_1^2, c_2^2, \dots, c_{N+1}^2]\Phi_N(t). \tag{3.19}$$

Proof. According to Lemma 3.2, we have

$$\begin{aligned} x^2(t) &= C^T\Phi_N(t)\Phi_N^T(t)C \approx C^T diag[c_1, c_2, \dots, c_{N+1}]\Phi_N(t) \\ &= [c_1^2, c_2^2, \dots, c_{N+1}^2]\Phi_N(t). \end{aligned} \tag{3.20}$$

□

4. Applications to multi-order fractional differential equations

According to (3.3), we can approximate $D_*^{\alpha_n}x(t), g(t)$ and $a_j(t)$, in (1.3) as follows:

$$D_*^{\alpha_n}x(t) \approx C^T\Phi_N(t), \tag{4.1}$$

$$a_j(t) \approx A_j^T\Phi_N(t),\ g(t) \approx G^T\Phi_N(t),\ j = 0, 1, \dots, n-1, \tag{4.2}$$

where coefficients of G, A_j are known column vectors and C is an unknown column vector. From (4.1) and (3.13), we have

$$\begin{aligned} D_*^{\alpha_j}x(t) &= I^{\alpha_n-\alpha_j}D_*^{\alpha_n}x(t) \approx I^{\alpha_n-\alpha_j}(C^T\Phi_N(t)) = C^T I^{\alpha_n-\alpha_j}\Phi_N(t) \\ &\approx C^T P^{\alpha_n-\alpha_j}\Phi_N(t). \end{aligned} \tag{4.3}$$

Now, by substituting (4.1)-(4.2) and (4.3) into (1.3), we obtain

$$\begin{aligned} C^T\Phi_N(t) &= \sum_{j=1}^{n-1} A_j^T\Phi_N(t)\Phi_N^T(t)(P^{\alpha_n-\alpha_j})^T C + A_0^T\Phi_N(t)\Phi_N^T(t)(P^{\alpha_n})^T C \\ &+ G^T\Phi_N(t). \end{aligned} \tag{4.4}$$

If we define

$$F_{\alpha_n-\alpha_j} = (P^{\alpha_n-\alpha_j})^T C,\ F_{\alpha_n} = (P^{\alpha_n})^T C, \tag{4.5}$$

then, according to (4.4) one will set

$$C^T\Phi_N(t) = \sum_{j=1}^{n-1} A_j^T\Phi_N(t)\Phi_N^T(t)F_{\alpha_n-\alpha_j} + A_0^T\Phi_N(t)\Phi_N^T(t)F_{\alpha_n} + G^T\Phi_N(t). \tag{4.6}$$

From (3.8), we obtain

$$\sum_{j=1}^{n-1} A_j^T\bar{F}_{\alpha_n-\alpha_j}\Phi_N(t) + A_0^T\bar{F}_{\alpha_n}\Phi_N(t) + G^T\Phi_N(t) = C^T\Phi_N(t), \tag{4.7}$$

$$(\sum_{j=1}^{n-1} A_j^T\bar{F}_{\alpha_n-\alpha_j} + A_0^T\bar{F}_{\alpha_n} + G^T - C^T)\Phi_N(t) = 0. \tag{4.8}$$

Finally, we have the following system:

$$\sum_{j=1}^{n-1} A_j^T\bar{F}_{\alpha_n-\alpha_j} + A_0^T\bar{F}_{\alpha_n} + G^T - C^T = 0. \tag{4.9}$$

By solving the obtained system included $N+1$ unknowns and $N+1$ equations, we can obtain the vector C. So

$$x(t) = I^{\alpha_n} D_*^{\alpha_n} x(t) \approx C^T I^{\alpha_n} \Phi_N(t) \approx C^T P^{\alpha_n} \Phi_N(t). \tag{4.10}$$

Therefore, from (1.2) we have

$$y(t) \approx \bar{y}(t) + C^T P^{\alpha_n} \Phi_N(t). \tag{4.11}$$

5. Error analysis

In this section, we investigate the error analysis for the method presented in this paper. It is easily verified that, problem (1.3) changes to the following problem

$$D_*^{\alpha_n} x(t) = \sum_{j=1}^{n-1} a_j(t) I^{\alpha_n-\alpha_j} D_*^{\alpha_n} x(t) + a_0(t) I^{\alpha_n} D_*^{\alpha_n} x(t) + g(t), \tag{5.1}$$

where $a_j(t) \in C[0,1]$ for $j = 0, 1, \ldots, n-1$, and $g(t) \in C[0,1]$. If we consider $\alpha_0 = 0$, then we have

$$D_*^{\alpha_n} x(t) = \sum_{j=0}^{n-1} a_j(t) I^{\alpha_n-\alpha_j} D_*^{\alpha_n} x(t) + g(t). \tag{5.2}$$

By taking $u(t) = D_*^{\alpha_n} x(t)$, we obtain the following fractional integral equation

$$u(t) = \sum_{j=0}^{n-1} a_j(t) I^{\alpha_n-\alpha_j} u(t) + g(t). \tag{5.3}$$

According to (3.3), we can approximate $u(t)$ and $g(t)$ as

$$u_N(t) = \sum_{j=1}^{N+1} u(t_j) C_j(t) = C^T \Phi_N(t), \;\; g_N(t) = \sum_{j=1}^{N+1} g(t_j) C_j(t) = G^T \Phi_N(t). \tag{5.4}$$

Then, the problem (5.3) reduces to the following problem:

$$u_N(t) = \sum_{j=0}^{n-1} a_j(t) I^{\alpha_n-\alpha_j} u_N(t) + g_N(t). \tag{5.5}$$

Let ρ stands for the closed interval $[0,1]$, $L^2(\rho)$ is the space of all functions $u : \rho \to \mathbb{R}$ with $\|u\|_{L^2(\rho)} < \infty$, and define $\|u\|_{L^2(\rho)}$ as

$$\|u\|_{L^2(\rho)} = \left(\int_0^1 u(t)^2 \; dt \right)^{\frac{1}{2}}. \tag{5.6}$$

Assume know that $H^m(\rho)$ denotes the Sobolev space [3] of all functions $u(t)$ on ρ such that $u(t)$ and all its weak derivatives up to order m are in $L^2(\rho)$. The norm of $H^m(\rho)$ is defined by

$$\|u\|_{H^m(\rho)} = \left(\sum_{k=0}^{m} \|u^{(k)}(t)\|^2_{L^2(\rho)} \right)^{\frac{1}{2}}. \tag{5.7}$$

Also, we define the semi-norm

$$|u|^2_{H^{m,N}(\rho)} = \sum_{i=min(m,N)}^{N} \|u^{(i)}(t)\|^2_{L^2(\rho)}. \tag{5.8}$$

Now, we consider the following lemmas:

Lemma 5.1. *For all measurable functions $f \geq 0$, the following generalized Hardys inequality*

$$\left(\int_a^b |(Tf)(t)|^q \; w_1(t)dt\right)^{\frac{1}{q}} \leq C \left(\int_a^b |f(t)|^p \; w_2(t)dt\right)^{\frac{1}{p}},$$

holds if and only if

$$\sup_{a<t<b} \left(\int_t^b w_1(t)dt\right)^{\frac{1}{q}} \left(\int_a^t w_2^{1-p'}(t)dt\right)^{\frac{1}{p'}} < \infty, \; p' = \frac{p}{p-1},$$

for $1 < p \leq q < \infty$. Here, T is an operator of the form $(Tf)(t) = \int_a^t k(t,s)f(s)ds$, with $k(t,s)$ a given kernel, w_1, w_2 weight functions, and $-\infty \leq a < b \leq \infty$.

Proof. See [7, 4]. □

Lemma 5.2. *Assume that $u(t) \in H^m(\rho)$ and $\{t_j\}_{j=1}^{N+1}$ represents the $(N+1)$ shifted Chebyshev-Gauss points. Suppose that the approximated solution u_N is given by*

$$u_N(t) = \sum_{j=1}^{N+1} u(t_j)C_j(t),$$

then there exists a constant C independent of N, such that

$$\|u - u_N\|_{L^2(\rho)} \leq CN^{-m}|u|_{H^{m,N}(\rho)}. \tag{5.9}$$

Proof. See [2]. □

Now, we state the main result of this section:

Theorem 5.3. *Assume that the exact solution $u(t)$ of (5.3) is smooth enough and $g(t) \in C(\rho)$. Suppose that the approximate solution $u_N(t)$is given by (5.4), then we have*

$$\| \; e_N(t) \; \|_{L^2(\rho)} \leq MCN^{-m}|u|_{H^{m,N}(\rho)} + C_gN^{-1}|g|_{H^{1,N}(\rho)}, \tag{5.10}$$

where M, C, C_g are constants and $e_N(t) = u_N(t) - u(t),\; e_g(t) = g_N(t) - g(t)$.

Proof. Subtraction of the Eqs. (5.3) and (5.5) yields

$$e_N(t) = \sum_{j=0}^{n-1} a_j(t)I^{\alpha_n - \alpha_j}e_N(t) + e_g(t). \tag{5.11}$$

By taking absolute value, we get

$$|e_N(t)| \leq \sum_{j=0}^{n-1} |a_j(t)| \; |I^{\alpha_n - \alpha_j}e_N(t)| + |e_g(t)|. \tag{5.12}$$

Since $a_j(t) \in C(\rho)$ for $j = 0, 1, \dots, n-1$, we have $\sup_{0\le t\le 1} |a_j(t)| \le l_j$ and consequently

$$|e_N(t)| \le \sum_{j=0}^{n-1} l_j \ |I^{\alpha_n-\alpha_j} e_N(t)| + |e_g(t)|. \tag{5.13}$$

By taking norm $L^2(\rho)$, we obtain

$$\| e_N(t) \|_{L^2(\rho)} \le \sum_{j=0}^{n-1} l_j \ \| I^{\alpha_n-\alpha_j} e_N(t) \|_{L^2(\rho)} + \| e_g(t) \|_{L^2(\rho)} . \tag{5.14}$$

Using generalized Hardys inequality with $k_j(t,s) = (t-s)^{\alpha_n-\alpha_j-1}$, we can write

$$\| e_N(t) \|_{L^2(\rho)} \le \sum_{j=0}^{n-1} \frac{l_j C_j}{\Gamma(\alpha_n-\alpha_j)} \ \| e_N(t) \|_{L^2(\rho)} + \| e_g(t) \|_{L^2(\rho)} . \tag{5.15}$$

Let $M = \sum_{j=0}^{n-1} \frac{l_j C_j}{\Gamma(\alpha_n-\alpha_j)}$. Therefore, we can obtain the following expression

$$\| e_N(t) \|_{L^2(\rho)} \le M \| e_N(t) \|_{L^2(\rho)} \ + \| e_g(t) \|_{L^2(\rho)} . \tag{5.16}$$

Hence (5.9) implies that:

$$\| e_g(t) \|_{L^2(\rho)} \le C_g N^{-1} |g|_{H^{1,N}(\rho)}. \tag{5.17}$$

Finally, following error bound between the exact solution and approximate solution of (5.4) can be established

$$\| e_N(t) \|_{L^2(\rho)} \le MCN^{-m} |u|_{H^{m,N}(\rho)} + C_g N^{-1} |g|_{H^{1,N}(\rho)}. \tag{5.18}$$

□

6. Illustrative examples

In this section, we will use the operational matrix of fractional order integration to solve the fractional differential equations. To demonstrate the performance and efficiency of the present method, we consider three test problems.

Example 6.1. *As the first example, we consider the following fractional differential equation* [15, 28]*:*

$$D_*^2 y(t) + \sin(t) D_*^{\frac{1}{2}} y(t) + t y(t) = f(t), \ t \in [0,1], \tag{6.1}$$

where

$$f(t) = t^9 - t^8 + 56t^6 - 42t^5 + \sin(t) \left(\frac{32768}{6435} t^{\frac{15}{2}} - \frac{2048}{429} t^{\frac{13}{2}} \right), \ y(0) = y'(0) = 0.$$

The exact solution of this equation is $y(t) = t^8 - t^7$. Let

$$D_*^2 y(t) = C^T \phi_N(t), \tag{6.2}$$

together with the initial conditions. Consequently, we have

$$D_*^{\frac{1}{2}} y(t) = I^{\frac{3}{2}} D_*^2 y(t) = C^T P^{\frac{3}{2}} \phi_N(t), \ y(t) = C^T P^2 \phi_N(t). \tag{6.3}$$

TABLE 1. The absolute errors of Example 6.1 with comparison to [15, 28].

t	Present method(N=8)	[28](m=8)	[15](m=8)
0.0625	$1.05447e-007$	$6.8166e-008$	$9.2040e-008$
0.1875	$1.02909e-007$	$6.2025e-006$	$1.2422e-005$
0.3125	$4.93077e-007$	$5.7883e-005$	$4.6183e-007$
0.4375	$5.46689e-006$	$1.9796e-004$	$8.8140e-004$
0.5625	$5.42688e-005$	$4.1838e-004$	$5.7552e-003$
0.6875	$2.79393e-004$	$5.2079e-004$	$2.0001e-002$
0.8125	$9.77621e-004$	$1.1363e-004$	$4.5151e-002$
0.9375	$2.35726e-003$	$3.1379e-003$	$5.9717e-002$

Since $\sin(t), t$ and $f(t)$ are given functions, according to (3.3) we can approximate it, as follows:

$$\begin{cases} \sin(t) = \sum_{j=1}^{N+1} \sin(t_j) C_j(t) = G_1^T \phi_N(t), \\ \\ t = \sum_{j=1}^{N+1} t_j C_j(t) = G_2^T \phi_N(t), \\ \\ f(t) = \sum_{j=1}^{N+1} f(t_j) C_j(t) = F^T \phi_N(t). \end{cases} \tag{6.4}$$

Substituting Eqs. (6.2)-(6.4) into Eq. (6.1), we have

$$C^T \phi_N(t) + G_1^T \phi_N(t) \phi_N^T(t) (P^{\frac{3}{2}})^T C + G_2^T \phi_N(t) \phi_N^T(t) (P^2)^T C = F^T \phi_N(t). \tag{6.5}$$

According to (4.5) and (3.8) we have

$$C^T \phi_N(t) + G_1^T \bar{F}_{\frac{3}{2}} \phi_N(t) + G_2^T \bar{F}_2 \phi_N(t) = F^T \phi_N(t). \tag{6.6}$$

Therefore, Eq. (6.1) has been transformed into a system of algebraic equations. Solving the system of algebraic equations, we can obtain the vector C^T. Then using Eq. (6.3), we get approximation solution of $y(t)$. In Table 1 and Table 2, we compare the absolute error of our results with the absolute error of the results in [15, 28]. By the comparison of the results obtained using the presented method in Table 1 and Table 2 with the methods of [15] and [28], it is easily found that the present approximations are more efficient.

Example 6.2. *Consider the following initial value equation* [8, 12, 26]

$$D_*^\alpha y(t) + y(t) = 0, \ 0 < \alpha \le 2, \ y(0) = 1, \ y^{'}(0) = 0. \tag{6.7}$$

The second initial condition is only for $\alpha > 1$. The exact solution of this equation is as follows [12]

$$y(t) = \sum_{k=0}^{\infty} \frac{(-t)^k}{\Gamma(\alpha k + 1)}.$$

Table 2. The absolute errors of Example 6.1 with comparison to [15, 28].

t	Present method(N=12)	[28](m=16)	[15](m=16)
0.0625	$3.78481e-011$	$5.5745e-010$	$7.4471e-010$
0.1875	$2.64794e-009$	$6.4280e-007$	$8.3627e-007$
0.3125	$2.95275e-007$	$8.1457e-006$	$3.9354e-007$
0.4375	$6.07485e-006$	$3.8396e-005$	$8.1461e-004$
0.5625	$5.37413e-005$	$1.1637e-004$	$5.1642e-003$
0.6875	$2.79598e-004$	$2.9048e-004$	$1.7360e-002$
0.8125	$9.77823e-004$	$6.4027e-004$	$3.8760e-002$
0.9375	$2.35738e-003$	$1.0678e-003$	$5.0103e-002$

Table 3. The absolute errors for $\alpha = 0.85$ with comparison to [26], for Example 6.2.

t	Method[26]		Present method	
	m=5	m=8	N=5	N=8
0.1	$2.0e-003$	$8.0e-004$	$1.14130e-003$	$4.17130e-004$
0.2	$3.0e-003$	$1.2e-003$	$9.50155e-004$	$1.31914e-004$
0.3	$6.2e-004$	$6.6e-004$	$6.41152e-004$	$8.17993e-005$
0.4	$2.9e-003$	$8.0e-004$	$2.89768e-004$	$1.55508e-004$
0.5	$2.0e-003$	$7.5e-004$	$5.66197e-004$	$4.93883e-006$
0.6	$7.2e-004$	$5.9e-004$	$1.04990e-004$	$9.76672e-005$
0.7	$2.5e-003$	$7.6e-004$	$4.20454e-004$	$6.11117e-005$
0.8	$1.3e-003$	$1.8e-004$	$3.13052e-004$	$3.87633e-005$
0.9	$1.5e-003$	$6.2e-004$	$3.05479e-004$	$6.32684e-005$
1	$5.5e-004$	$1.5e-004$	$3.97224e-004$	$6.98429e-005$

Let

$$D_*^\alpha y(t) = C^T \phi_N(t), \tag{6.8}$$

then

$$D_* y(t) = I^{\alpha-1} D_*^\alpha y(t) = C^T P^{\alpha-1} \phi_N(t), \tag{6.9}$$

$$y(t) = C^T P^\alpha \phi_N(t) + y(0) = C^T P^\alpha \phi_N(t) + 1 = C^T P^\alpha \phi_N(t) + e^T \phi_N(t), \tag{6.10}$$

where $e^T = [1, 1, \ldots, 1]$. Submitting Eqs. (6.8)-(6.10) into Eq. (6.7), we have following system of algebraic equations:

$$C^T \phi_N(t) + C^T P^\alpha \phi_N(t) + e^T \phi_N(t) = 0. \tag{6.11}$$

By solving Eq. (6.11) we can find the vector C^T. The absolute error for $\alpha = 0.85$ with comparison to [26] is given in Table 3 on the interval $[0, 1]$. The difference between our results and the results in [26] is obvious. For $\alpha = 1$, the exact solution is given as $y(t) = e^{-t}$.

Example 6.3. *Consider the nonlinear fractional differential equation* [15, 6]

$$aD_*^{2.2}y(t) + bD_*^{\alpha_2}y(t) + cD_*^{\alpha_1}y(t) + ey^3(t) = f(t),\ 0 < \alpha_1 1,\ 1 < \alpha_2 2, \tag{6.12}$$

and

$$f(t) = \frac{2a}{\Gamma(1.8)}t^{0.8} + \frac{2b}{\Gamma(4-\alpha_2)}t^{3-\alpha_2} + \frac{2c}{\Gamma(4-\alpha_1)}t^{3-\alpha_1} + e\frac{t^9}{27},$$

subject to $y(0) = y'(0) = y''(0) = 0$.
The exact solution of this equation is $y(t) = \frac{t^3}{3}$. Let

$$D_*^{2.2}y(t) = C^T\phi_N(t). \tag{6.13}$$

Consequently, we have

$$D_*^{\alpha_2}y(t) = C^TP^{2.2-\alpha_2}\phi_N(t),\ D_*^{\alpha_1}y(t) = C^TP^{2.2-\alpha_1}\phi_N(t), \tag{6.14}$$

$$y(t) = C^TP^{2.2}\phi_N(t), \tag{6.15}$$

$$D_*^{\alpha_1}y(t) = C^TP^{2.2-\alpha_1}\phi_N(t), \tag{6.16}$$

$$y(t) = C^TP^{2.2}\phi_N(t). \tag{6.17}$$

Assume know that

$$C^TP^{2.2} = [a_1, a_2, \dots, a_{N+1}], \tag{6.18}$$

then, according to Lemma 3.4 we have

$$y^3(t) = [a_1^3, a_2^3, \dots, a_{N+1}^3]\phi_N(t). \tag{6.19}$$

Similarly $f(t)$ can be expanded in terms of the Chebyshev cardinal functions as follows:

$$f(t) = \sum_{j=1}^{N+1} f(t_j)C_j(t) = F^T\phi_N(t). \tag{6.20}$$

Substituting Eqs. (6.13)-(6.16) and Eqs. (6.19)-(6.20) into Eq. (6.12), we have

$$\begin{aligned} C^T\phi_N(t) \quad &+ \quad C^TP^{2.2-\alpha_2}\phi_N(t) + C^TP^{2.2-\alpha_1}\phi_N(t) + [a_1^3, a_2^3, \dots, a_{N+1}^3]\phi_N(t) \\ &= \quad F^T\phi_N(t). \end{aligned} \tag{6.21}$$

This is a nonlinear system of algebraic equations. In this example, we chose $a = 1, b = 1, c = 1, e = 1, \alpha_1 = 0.75, \alpha_2 = 1.25$. In Table 4, we compare the absolute error obtained for different values of t by using the present method with $N = 8, N = 16$, and the Haar wavelet operational matrix of the fractional order integration [15] for $m = 32, m = 64$.

Table 4. The absolute errors with comparison to [15], for Example 6.3.

t	Method[15]		Present method	
	m=32	m=64	N=8	N=16
0.1	$6.960e-005$	$1.530e-005$	$1.77358e-006$	$1.62213e-007$
0.2	$1.174e-004$	$3.690e-005$	$3.78743e-006$	$3.62324e-007$
0.3	$1.748e-004$	$5.420e-005$	$5.23244e-006$	$5.69659e-007$
0.4	$2.736e-004$	$5.910e-005$	$7.22167e-006$	$7.32969e-007$
0.5	$3.519e-004$	$9.100e-005$	$9.17622e-006$	$9.09578e-007$
0.6	$3.872e-004$	$8.280e-005$	$1.03997e-005$	$1.04116e-006$
0.7	$3.575e-004$	$1.140e-004$	$1.13764e-005$	$1.17049e-006$
0.8	$3.960e-004$	$1.263e-004$	$1.27935e-005$	$1.36152e-006$
0.9	$5.356e-004$	$1.119e-004$	$1.38150e-005$	$1.27239e-006$

7. Conclusion

In this paper, we derived operational matrix of the fractional integration for Chebyshev cardinal functions, and used this methodology to solve the multi-order fractional differential equations. We transformed the multi-order fractional differential equations into a system of algebraic equations that can be solved easily. Numerical examples were given to show that the proposed method is applicable, efficient and accurate.

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