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A new method based on fourth kind Chebyshev wavelets to a fractional–order model of HIV infection of CD4$^+$T cells

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Abstract This paper deals with the application of fourth kind Chebyshev wavelets (FKCW) in solving numerically a model of HIV infection of CD4$^+$T cells involving Caputo fractional derivative. The present problem is a system of nonlinear fractional differential equations. The goal is to approximate the solution in the form of FKCW truncated series. To do this, an operational matrix of fractional integration is constructed for these wavelets. This matrix transforms the problem to a nonlinear system of algebraic equations. Solving the new system, enables one to identify the unknown coefficients of expansion. Numerical results are compared with other existing methods to illustrate the applicability of the method.

Keywords. Fourth kind Chebyshev wavelets, HIV model, Caputo derivative.
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1. Introduction

The human immunodeficiency virus (HIV) is a virus that attacks the immune system which is our body's natural defense against the diseases. This virus destroys a type of white blood cells in the immune system called CD4$^+$T and makes copies of itself inside these cells. AIDS is a status which the immune system is too weak to fight off infection.

The dynamics of CD4$^+$T cells and HIV interaction are formulated by the following system of ordinary differential equations [27]

$$\begin{cases} \frac{dT}{dt} = p - \alpha T + rT\left(1 - \frac{T+I}{T_{\max}}\right) - kVT, \\ \frac{dI}{dt} = kVT - \beta I, \\ \frac{dV}{dt} = N\beta I - \gamma V, \end{cases} \tag{1.1}$$

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with the initial conditions

$$T(0) = T_0, \;\; I(0) = I_0, \;\; V(0) = V_0. \tag{1.2}$$

The descriptions of variables and parameters of (1.1) are listed in Table 1.

TABLE 1. Variables and Parameters

Variables and parameters	Descriptions	Units
$T(t)$	Uninfected CD4$^+$T cells	cells/mm^3
$I(t)$	Infected CD4$^+$T cells	cells/mm^3
$V(t)$	HIV virus particles	virions/mm^3
p	Generation rate of $T(t)$	day^{-1} mm^{-3}
α	Natural death rate of CD4$^+$T cells	day^{-1}
r	Growth rate of CD4$^+$T cells	day^{-1}
$T_{\max}$	Maximal of CD4$^+$T cells	mm^3/cells
k	Infection rate of CD4$^+$T cells	day^{-1}
β	Blanket death rate of $I(t)$	day^{-1}
N	Production rate of virus particles	virions/cell
γ	Death rate of free virus	day^{-1}

The mathematical model (1.1) represents the changes of infected CD4$^+$T cells, uninfected CD4$^+$T cells and virus particles numbers in a period of time. In medical sciences, the computation of these variables are vital to measure the progression of disease and to get a better cure. Accordingly, the numerical solution of the HIV model is a considerable task of mathematicians. The methods existing in the literature to solve (1.1) with the initial conditions (1.2) include as follows: Laplace Adomian decomposition method [20], Multi –step Laplace Adomian decomposition method [5], Variational iteration method [16], Differential transform method [25], Homotopy perturbation method [17], Bessel collocation method [31], Legendre wavelets method [26] and Taylor's series method [11].

The important point to note here is that biological systems have memory and aftereffects. In the models interpreted by ordinary differential equations with integer-order, such effects are neglected. Because of the relation of fractional calculus to the systems with memory, fractional models are more suitable for biological systems. Thus, by replacing the integer-order derivative in (1.1) with Caputo fractional derivative, the following model is established

$$\begin{cases} D_*^{\nu_1} T = p - \alpha T + rT\left(1 - \frac{T+I}{T_{\max}}\right) - kVT, \\ D_*^{\nu_2} I = kVT - \beta I, \\ D_*^{\nu_3} V = N\beta I - \gamma V, \end{cases} \tag{1.3}$$

with the same initial conditions (1.2), where $0 < \nu_1, \; \nu_2, \; \nu_3 \leq 1$ and D_* denotes the Caputo fractional derivative. For $\nu_1 = \nu_2 = \nu_3 = 1$, (1.3) reduces to the classical model (1.1). As for prerequisites, the reader is expected to be familiar with the basic concepts of Riemann–Liouville fractional integral and Caputo fractional derivative.

See [4, 21].

For solving the fractional model (1.3), Gökdogan et al. have presented a multi–step differential transform method [7], Haq et al. have used Adomian decomposition method [8] and also Gandomani et al. have proposed Müntz–Legendre polynomials collocation method [6].

From recent decades up to now, wavelet methods have been accounted as powerfool tools with diverse applications in science and technology. Especially, wavelets generated by the dilation and translation of orthogonal polynomials have been considered more than other ones in finding the solution of various types of differential and integral equations. The useful characteristics of wavelets can be summarized as follows: (a) Wavelet methods are appropriate for the computer programming. (b) The operational matrix achieved by wavelets exhibits sparsity which is computationally very fast. (c) By applying wavelets, it is possible to obtain a good approximation of a function by using only a few coefficients. (d) For a large number of the basis functions, most of the wavelet coefficients vanish. (e) The solution is multi–resolution [2, 3].

As we know, there are four kinds of Chebyshev polynomials. All of them are used widely in approximation theory. They are orthogonal on $[-1, 1]$ with respect to different weight functions [15]. The operational matrices of the fractional integration related to first and second kind Chebyshev wavelets have been constructed to the numerical solution of fractional differential equations in [13] and [28], respectively. Recently, Zhou et al. have been gained an operational matrix of fractional integration of third kind Chebyshev wavelets for solving fractional convection–diffusion equation [32]. FKCW have been paid less attention by authors and no serious attempt has been made to extend FKCW for solving fractional problems. Motivated by the mentioned works, our main aim is to develope the applications of FKCW in approximation of the solution of (1.3). For this purpose, using collocation method, an operational matrix of fractional integration for FKCW is first fabricated. This matrix converts (1.3) to a nonlinear system of algebraic equations. After solving the new system by any standard iterative scheme, the solution of problem is acquired in terms of FKCW. Numerical results of the method are compared with some recent methods. Since the exact solution of (1.3) is unknown, residual errors of (1.3) are evaluated to indicate the efficiency of method.

The remainder of this paper proceeds as follows: In section 2, FKCW and some properties of them are described. Uniform convergence and error estimation of FKCW expansion are also investigated. Section 3 implements the numerical method for solving (1.3). In section 4, numerical and graphical results of applying the present method are announced. Finally, concluding remarks are drawn in section 5.

2. Some properties of FKCW

2.1. **Wavelets and FKCW.** Wavelets are a class of functions used to localize a given function in the dilation and translation. The dilation parameter a and translation

parameter b of a mother wavelet ψ define the following continuous wavelets

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right), \quad a \in \mathbb{R}^+, \ b \in \mathbb{R}. \tag{2.1}$$

Now, suppose $a_0 > 1$ and $b_0 > 0$ are fixed and take $a = a_0^{-k}$ and $b = na_0^{-k}b_0$ such that $k, \ n \in \mathbb{N}$. Instead of using the family of wavelets (2.1), we use the family of wavelets indexed by $\mathbb{N}$, named the discrete wavelets,

$$\psi_{k,n}(x) = a_0^{\frac{k}{2}}\psi\left(a_0^k x - nb_0\right), \quad k, \ n \in \mathbb{N}. \tag{2.2}$$

The family of (2.2) constitutes an orthogonal basis on $L^2(\mathbb{R})$ and by choosing $a_0 = 2$ and $b_0 = 1$, this basis will be an orthonormal basis on $L^2(\mathbb{R})$ [14, 18].

FKCW are defined on $[0,1]$ for $n = 1, ..., 2^{k-1}$, $k \in \mathbb{N}$, $m = 0, ..., M-1$, $M \in \mathbb{N}$ as

$$\psi_{n,m}(t) = \begin{cases} \sqrt{\frac{1}{\pi}}2^{\frac{k}{2}}W_m\left(2^k t - 2n + 1\right), & \frac{n-1}{2^{k-1}} \leq t < \frac{n}{2^{k-1}}, \\ 0, & otherwise, \end{cases} \tag{2.3}$$

in which W_m are Chebyshev polynomials of fourth kind of degree $m = 0, 1, \ldots$ that are orthogonal with respect to the weight function $\omega(t) = \sqrt{\frac{1-t}{1+t}}$ on $[-1,1]$ and for $m \in \mathbb{N}$ can be determined by the following recurrence formula [15]

$$W_0(t) = 1, \quad W_1(t) = 2t + 1, \quad W_{m+1}(t) = 2tW_m(t) - W_{m-1}(t).$$

The coefficient $\sqrt{\frac{1}{\pi}}$ in (2.3) is for normality. The dilation and translation parameters are $a = 2^{-k}$ and $b = (2n-1)2^{-k}$, respectively. Also, $k \in \mathbb{N}$ denotes the level of resolution.

It is worth mentioning that FKCW defined in (2.3) constitute an orthonormal basis on $L^2[0,1]$ with respect to the weight function $\omega_n(t) = \omega\left(2^k t - 2n + 1\right)$.

2.2. **Function approximation.** Any function $f(t) \in L^2_\omega[0,1]$ can be expanded at the level $k \in \mathbb{N}$ as

$$f(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} f_{n,m}\psi_{n,m}(t), \tag{2.4}$$

where

$$f_{n,m} = \langle f(t), \psi_{n,m}(t)\rangle_{L^2_\omega[0,1]} = \int_0^1 f(t)\psi_{n,m}(t)\omega_n(t)dt,$$

in which $\langle ., . \rangle_{L^2_\omega[0,1]}$ denotes the inner product in $L^2_\omega[0,1]$.

Usually, the infinite series in (2.4) is truncated and written in the following

$$f(t) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} f_{n,m}\psi_{n,m}(t) = \mathbf{F}^T\mathbf{\Psi}(t), \tag{2.5}$$

which approximates $f(t)$ as a finite linear combination of FKCW. In (2.5), $\mathbf{F}$ and $\mathbf{\Psi}(t)$ are column vectors with $\hat{m} = 2^{k-1}M$ entries given by

$$\mathbf{F} = \left[f_{1,0}, \ldots, f_{1,M-1}, \ldots, f_{2^{k-1},0}, \ldots, f_{2^{k-1},M-1}\right]^T,$$
$$\mathbf{\Psi}(t) = \left[\psi_{1,0}(t), \ldots, \psi_{1,M-1}(t), \ldots, \psi_{2^{k-1},0}(t), \ldots, \psi_{2^{k-1},M-1}(t)\right]^T.$$

The following proposition ensures the uniform convergence of FKCW expansion.

Proposition 2.1. *Assume $f(t)$ be a continuous function and $R = \sup|f''(t)|$ on $[0,1]$. Then, it can be expanded as an infinite sum of FKCW and the series converges uniformly to $f(t)$, that is*

$$f(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} f_{n,m}\psi_{n,m}(t).$$

Moreover, for $m > 1$,

$$|f_{n,m}| < \frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{R}{n^{\frac{5}{2}}(m-1)^2}.$$

Proof. The coefficients of FKCW expansion of $f(t)$ are defined as

$$f_{n,m} = \langle f(t), \psi_{n,m}(t)\rangle_{L^2_\omega[0,1]} = \int_0^1 f(t)\psi_{n,m}(t)\omega_n(t)dt$$
$$= \sqrt{\frac{1}{\pi}}2^{\frac{k}{2}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(t)W_m\left(2^k t - 2n + 1\right)\omega\left(2^k t - 2n + 1\right)dt.$$

Let $x = 2^k t - 2n + 1$. Thus,

$$f_{n,m} = \sqrt{\frac{1}{\pi}}2^{-\frac{k}{2}} \int_{-1}^{1} f\left(\frac{x + 2n - 1}{2^k}\right) W_m(x)\omega(x)dx.$$

Putting $x = \cos\theta$ entails

$$f_{n,m} = \sqrt{\frac{1}{\pi}}2^{-\frac{k}{2}} \int_0^{\pi} f\left(\frac{\cos\theta + 2n - 1}{2^k}\right) \frac{\sin\left(m + \frac{1}{2}\right)\theta}{\sin\frac{1}{2}\theta}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\sin\theta d\theta.$$

Knowing $\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\sin\theta = 2\sin^2\frac{\theta}{2}$ yields

$$f_{n,m} = 2\sqrt{\frac{1}{\pi}}2^{-\frac{k}{2}} \int_0^{\pi} f\left(\frac{\cos\theta + 2n - 1}{2^k}\right) \sin\left(m + \frac{1}{2}\right)\theta \sin\frac{1}{2}\theta d\theta$$
$$= \sqrt{\frac{1}{\pi}}2^{-\frac{k}{2}} \int_0^{\pi} f\left(\frac{\cos\theta + 2n - 1}{2^k}\right) \left(\cos m\theta - \cos(m+1)\theta\right) d\theta.$$

With the aid of integration by parts technique, it follows

$$f_{n,m} = \sqrt{\frac{1}{\pi}}2^{-\frac{3k}{2}}\frac{1}{m} \int_0^{\pi} f'\left(\frac{\cos\theta + 2n - 1}{2^k}\right) \sin m\theta \sin\theta d\theta$$
$$- \sqrt{\frac{1}{\pi}}2^{-\frac{3k}{2}}\frac{1}{m+1} \int_0^{\pi} f'\left(\frac{\cos\theta + 2n - 1}{2^k}\right) \sin(m+1)\theta \sin\theta d\theta. \tag{2.6}$$

Let $\zeta_1 = \frac{1}{m}\int_0^\pi f'\left(\frac{\cos\theta+2n-1}{2^k}\right)\sin m\theta\sin\theta d\theta$. Then,

$$\begin{aligned}\zeta_1 &= \frac{1}{2m}\int_0^\pi f'\left(\frac{\cos\theta+2n-1}{2^k}\right)(\cos(m-1)\theta-\cos(m+1)\theta)\,d\theta\\ &= \frac{2^{-k}}{2m}\int_0^\pi f''\left(\frac{\cos\theta+2n-1}{2^k}\right)\left(\frac{\sin(m-1)\theta}{m-1}-\frac{\sin(m+1)\theta}{m+1}\right)\sin\theta d\theta.\end{aligned}$$

Obviously,

$$\begin{aligned}|\zeta_1| &\le \frac{1}{2m}2^{-k}R\int_0^\pi\left|\frac{\sin(m-1)\theta}{m-1}-\frac{\sin(m+1)\theta}{m+1}\right||\sin\theta|\,d\theta\\ &\le \frac{1}{2m}2^{-k}(\pi R)\left(\frac{1}{m-1}+\frac{1}{m+1}\right)\\ &< \frac{2^{-k}R\pi}{(m-1)^2}.\end{aligned}$$

Let $\zeta_2 = \frac{1}{m+1}\int_0^\pi f'\left(\frac{\cos\theta+2n-1}{2^k}\right)\sin(m+1)\theta\sin\theta d\theta$. In a similar way,

$$|\zeta_2| \le \frac{2^{-k}R\pi}{m(m+1)} < \frac{2^{-k}R\pi}{(m-1)^2}.$$

Now, for $m>1$,

$$|f_{n,m}| \le \sqrt{\frac{1}{\pi}}2^{-\frac{3k}{2}}(|\zeta_1|+|\zeta_2|) < \sqrt{\frac{1}{\pi}}2^{-\frac{3k}{2}}\frac{2\times 2^{-k}R\pi}{(m-1)^2} = \frac{2R\sqrt{\pi}}{(m-1)^2}2^{-\frac{5k}{2}}.$$

Since $n\le 2^{k-1}$, it is clear that $2^{-\frac{5k}{2}} \le (2n)^{-\frac{5}{2}}$. Hence,

$$|f_{n,m}| < \frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{R}{n^{\frac{5}{2}}(m-1)^2}.$$

If $m=1$, according to (2.6),

$$|f_{n,1}| \le \sqrt{\frac{1}{\pi}}2^{-\frac{3k}{2}}\left(\pi+\frac{\pi}{2}\right)\sup|f'(t)| \le \frac{3}{4}\sqrt{\frac{\pi}{2}}\frac{\sup|f'(t)|}{n^{\frac{3}{2}}}.$$

So, $\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{\infty} f_{n,m}$ is absolutely convergent and the infinite sum of FKCW converges to $f(t)$ uniformly. □

The proposition below seeks an upper bound for the error estimation of FKCW expansion.

Proposition 2.2. *Under the assumptions of Proposition (2.1), assume the truncated series* $f_{k,M}(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} f_{n,m}\psi_{n,m}$ *be the FKCW approach of* $f(t)$ *at the level* k *for a given* M*. Then,*

$$\|f(t)-f_{k,M}(t)\| < \left(\frac{\pi}{8}R^2\sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}\frac{1}{n^5(m-1)^4}\right)^{\frac{1}{2}}.$$

Proof. Using the orthonormality of $\psi_{n,m}$ and Proposition 2.1, the following relations are held.

$$\begin{aligned}\|f(t)-f_{k,M}(t)\|^2 &= \int_0^1 \left|\sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty} f_{n,m}\psi_{n,m}(t)\right|^2 \omega_n(t)dt\\ &= \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}|f_{n,m}|^2\\ &< \frac{\pi}{8}R^2\sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}\frac{1}{n^5(m-1)^4}.\end{aligned}$$

□

Intuitively, if $f(t)=\frac{1}{\pi^2}\sin(\pi t)$ then $R=1$ and Proposition 2.2 reveals

$$\|f(t)-f_{1,3}(t)\|<0.1798,\ \|f(t)-f_{1,7}(t)\|<0.0278,\ \|f(t)-f_{1,13}(t)\|<0.0093.$$

Thus, we deduce at the level k for large enough value of M, $f_{k,M}\to f(t)$.

For a fixed M, we affirm

$$f(t)=\sum_{n=1}^{\infty}\sum_{m=0}^{M-1} f_{n,m}\psi_{n,m}(t).$$

Likely, one can prove for a given M by increasing k the accuracy of the approximation $\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} f_{n,m}\psi_{n,m}(t)$ is improvable.

2.3. **Operational matrix of fractional integration (OMFI).** OMFI of a family of wavelets can be obtained with or without Block-Pulse functions (BPF). For obtaining OMFI of Bernoulli wavelets, Keshavarz et al. expanded these wavelets into Bernoulli polynomials [10]. Then, the authors specified a relation for OMFI of Bernoulli wavelets for $k=2$ and arbitrary M. Moreover, Rong et al. formed Jacobi wavelets based on the explicit formula of Jacobi polynomials [24]. Then, they found an OMFI for Jacobi wavelets. In both of these works, the designed operational matrices contain many entries in the form of summation. Thus, although OMFI works efficiently but for not so small values of k and M, it has large computational effort.

In [1, 13, 19, 22, 23, 28–30, 32], OMFI has been constructed for Haar, first kind Chebyshev, Shannon, Legendre, Gegenbauer, second kind Chebyshev, Euler, CAS and third kind Chebyshev wavelets, respectively by using BPF. The main idea in all of these references is that the corresponding wavelets family is first expanded into BPF. Then, OMFI of BPF is used for calculating the OMFI of the wavelet. The resulting OMFI has a simple formula and is easy to perform for every k and M. Motivated by the works mentioned above, we intend to determine OMFI of FKCW using BPF.

Initially, it is necessary to review some relevant materials of BPF [9].

2.3.1. *BPF.*

Definition 2.3. The $\hat{m}$-set of BPF on $[0,1]$ is defined in the following

$$b_i(t) = \begin{cases} 1, & \frac{i-1}{\hat{m}} \leq t < \frac{i}{\hat{m}}, \\ 0, & otherwise, \end{cases}$$

where $i = 1, \ldots, \hat{m}$.

Remark 2.4. BPF are disjoint and orthogonal, namely

- $b_i(t)b_j(t) = \begin{cases} 0, & i \neq j, \\ b_i(t), & i = j, \end{cases}$
- $\int_0^1 b_i(t)b_j(t)dt = \begin{cases} 0, & i \neq j, \\ \frac{1}{\hat{m}}, & i = j. \end{cases}$

The function $f(t) \in L^2[0,1]$ can be written as

$$f(t) \approx \sum_{i=1}^{\hat{m}} f_i b_i(t) = \mathbf{f}^T \mathbf{B}_{\hat{m}}(t),$$

where $\mathbf{f} = [f_1, f_2, \ldots, f_{\hat{m}}]^T$, $\mathbf{B}_{\hat{m}}(t) = [b_1(t), b_2(t), \ldots, b_{\hat{m}}(t)]^T$, for $i = 1, \ldots, \hat{m}$

$$f_i = \hat{m} \int_0^1 f(t) b_i(t) dt.$$

Definition 2.5. The tensor product of two vectors $\mathbf{f}_{\hat{m}} = [f_i]$ and $\mathbf{g}_{\hat{m}} = [g_i]$ is defined as

$$\mathbf{f} \otimes \mathbf{g} = (f_i \times g_i)_{\hat{m}}.$$

Similarly, for two matrices $\mathbf{A} = [a_{i,j}]$ and $\mathbf{B} = [b_{i,j}]$ of $\hat{m} \times \hat{m}$

$$\mathbf{A} \otimes \mathbf{B} = (a_{i,j} \times b_{i,j})_{\hat{m} \times \hat{m}}.$$

The Lemma below will be needed in the next section.

Lemma 2.6. *Let the functions $f(t)$, $g(t) \in L^2[0,1]$ be expanded into BPF, that is $f(t) = \mathbf{f}^T \mathbf{B}_{\hat{m}}(t)$ and $g(t) = \mathbf{g}^T \mathbf{B}_{\hat{m}}(t)$. Then*

$$f(t)g(t) = \left(\mathbf{f}^T \otimes \mathbf{g}^T\right) \mathbf{B}_{\hat{m}}(t).$$

Proof.

$$\begin{aligned} f(t)g(t) &= \mathbf{f}^T \mathbf{B}_{\hat{m}}(t) \mathbf{B}_{\hat{m}}^T(t) \mathbf{g} = f_1 g_1 b_1(t) + f_2 g_2 b_2(t) + \ldots + f_{\hat{m}} g_{\hat{m}} b_{\hat{m}}(t) \\ &= \left(\mathbf{f}^T \otimes \mathbf{g}^T\right) \mathbf{B}_{\hat{m}}(t). \end{aligned}$$

□

For the sake of simplicity, we rewrite (2.5) as

$$f(t) \approx \sum_{i=1}^{\hat{m}} f_i \psi_i(t) = \mathbf{F}_{\hat{m}}^T \mathbf{\Psi}_{\hat{m}}(t),$$

where $f_i = f_{n,m},\ \psi_i = \psi_{n,m}$. The index i is determined by $i = (n-1)M + m + 1$, and $\hat{m} = 2^{k-1}M$. By this notation, we get

$$\mathbf{F}_{\hat{m}} = \left[f_1, \ldots, f_M|, f_{M+1}, \ldots, f_{2M}|, \ldots, |f_{(2^{k-1}-1)M+1}, \ldots, f_{\hat{m}}\right]^T,$$
$$\mathbf{\Psi}_{\hat{m}}(t) = \left[\psi_1, \ldots, \psi_M|, \psi_{M+1}, \ldots, \psi_{2M}|, \ldots, |\psi_{(2^{k-1}-1)M+1}, \ldots, \psi_{\hat{m}}\right]^T.$$

Taking the collocation points $t_i = \frac{2i-1}{2\hat{m}},\ i = 1, \ldots, \hat{m}$, we define the $\hat{m} \times \hat{m}$ FKCW matrix $\mathbf{\Phi}_{\hat{m}\times\hat{m}}$ as

$$\mathbf{\Phi}_{\hat{m}\times\hat{m}} = \left[\mathbf{\Psi}_{\hat{m}}(t_1), \mathbf{\Psi}_{\hat{m}}(t_2), \ldots, \mathbf{\Psi}_{\hat{m}}(t_{\hat{m}})\right]. \tag{2.7}$$

2.3.2. *Operational matrix.* From [12], OMFI of order ν of the BPF vector $\mathbf{B}_{\hat{m}}(t)$ is offered as

$$I^{\nu}\mathbf{B}_{\hat{m}}(t) \approx \boldsymbol{\mathcal{F}}^{\nu}_{\hat{m}\times\hat{m}}\mathbf{B}_{\hat{m}}(t), \tag{2.8}$$

where

$$\boldsymbol{\mathcal{F}}^{\nu}_{\hat{m}\times\hat{m}} = \frac{1}{\hat{m}^{\nu}}\frac{1}{\Gamma(\nu+2)}\begin{bmatrix} 1 & \xi_1 & \xi_2 & \ldots & \xi_{\hat{m}-1} \\ 0 & 1 & \xi_1 & \ldots & \xi_{\hat{m}-2} \\ 0 & 0 & 1 & \ldots & \xi_{\hat{m}-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix},$$

in which $\xi_i = (i+1)^{\nu+1} - 2i^{\nu+1} + (i-1)^{\nu+1},\ i = 1, \ldots, \hat{m}-1$.

Let $\mathbf{P}^{\nu}_{\hat{m}\times\hat{m}}$ be the OMFI of FKCW, that is

$$I^{\nu}\mathbf{\Psi}_{\hat{m}}(t) \approx \mathbf{P}^{\nu}_{\hat{m}\times\hat{m}}\mathbf{\Psi}_{\hat{m}}(t), \tag{2.9}$$

Now, for $t \in \left\{\frac{2i-1}{2\hat{m}} | i = 1, \ldots, \hat{m}\right\}$, using (2.7) implies that

$$\mathbf{\Psi}_{\hat{m}}(t) = \mathbf{\Phi}_{\hat{m}\times\hat{m}}\mathbf{B}_{\hat{m}}(t). \tag{2.10}$$

By (2.8) and (2.10), one can write

$$I^{\nu}\mathbf{\Psi}_{\hat{m}}(t) = I^{\nu}\mathbf{\Phi}_{\hat{m}\times\hat{m}}\mathbf{B}_{\hat{m}}(t) = \mathbf{\Phi}_{\hat{m}\times\hat{m}}I^{\nu}\mathbf{B}_{\hat{m}}(t) \approx \mathbf{\Phi}_{\hat{m}\times\hat{m}}\boldsymbol{\mathcal{F}}^{\nu}_{\hat{m}\times\hat{m}}\mathbf{B}_{\hat{m}}(t). \tag{2.11}$$

Also, with the aid of (2.10), we get

$$\mathbf{B}_{\hat{m}}(t) = \mathbf{\Phi}^{-1}_{\hat{m}\times\hat{m}}\mathbf{\Psi}_{\hat{m}}(t). \tag{2.12}$$

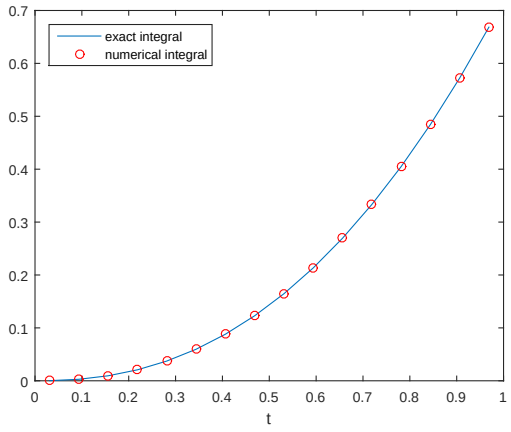
Consequently, from (2.11) and (2.12), it follows

$$I^{\nu}\mathbf{\Psi}_{\hat{m}}(t) \approx \mathbf{\Phi}_{\hat{m}\times\hat{m}}\boldsymbol{\mathcal{F}}^{\nu}_{\hat{m}\times\hat{m}}\mathbf{\Phi}^{-1}_{\hat{m}\times\hat{m}}\mathbf{\Psi}_{\hat{m}}(t). \tag{2.13}$$

Comparison (2.13) with (2.9) results in

$$\mathbf{P}^{\nu}_{\hat{m}\times\hat{m}} \approx \mathbf{\Phi}_{\hat{m}\times\hat{m}}\boldsymbol{\mathcal{F}}^{\nu}_{\hat{m}\times\hat{m}}\mathbf{\Phi}^{-1}_{\hat{m}\times\hat{m}},$$

FIGURE 1. $\frac{1}{3}$ order integration of t^2



which is independent of $\mathbf{B}_{\hat{m}}(t)$.

Particularly, for $k = 2,\ M = 3,\ \nu = 0.7$, OMFI of FKCW is computed as

$$\mathbf{P}^{0.7}_{6\times 6} = \begin{bmatrix} 0.2329 & 0.1683 & -0.0142 & 0.5558 & -0.0602 & 0.0116 \\ 0.0416 & 0.1574 & 0.0913 & 0.6457 & -0.0943 & 0.0211 \\ -0.0460 & -0.1031 & 0.1638 & 0.2089 & -0.0537 & 0.0145 \\ 0 & 0 & 0 & 0.2329 & 0.1683 & -0.0142 \\ 0 & 0 & 0 & 0.0416 & 0.1574 & 0.0913 \\ 0 & 0 & 0 & -0.0460 & -0.1031 & 0.1638 \end{bmatrix}.$$

One should mention that CPU time of MATLAB software to calculate $\mathbf{P}^{0.7}_{\hat{m}\times\hat{m}}$ for the values ($k = 4, M = 5$ *or* $\hat{m} = 40$), ($k = 6, M = 7$ *or* $\hat{m} = 224$), ($k = 8, M = 9$ *or* $\hat{m} = 1152$) and ($k = 10, M = 11$ *or* $\hat{m} = 5632$) is 0.0042s, 0.1170s, 3.2529s and 152.1935s, respectively. This means that OMFI of FKCW is computationally fast.

For example, consider $f(t) = t^2$. The fractional integration of order ν of f is given by

$$I^{\nu} f(t) = \frac{\Gamma(3)}{\Gamma(\nu+3)} t^{\nu+2}.$$

The correctness of OMFI are shown in Figure 1 for $\nu = \frac{1}{3},\ k = 3,\ M = 4$ or $\hat{m} = 16$.

3. IMPLEMENTATION OF NUMERICAL METHOD

First, let us rewrite the fractional–order model of HIV infection (1.3) as

$$\begin{cases} D^{\nu_1}_* T = p + (r-\alpha)T - \frac{r}{T_{\max}} T^2 - \frac{r}{T_{\max}} TI - kVT, \\ D^{\nu_2}_* I = kVT - \beta I, \\ D^{\nu_3}_* V = N\beta I - \gamma V. \end{cases} \tag{3.1}$$

Put

$$\begin{cases} D_*^{\nu_1} T(t) \approx \mathbf{C}^T \mathbf{\Psi}_{\hat{m}}(t), \\ D_*^{\nu_2} I(t) \approx \mathbf{D}^T \mathbf{\Psi}_{\hat{m}}(t), \\ D_*^{\nu_3} V(t) \approx \mathbf{K}^T \mathbf{\Psi}_{\hat{m}}(t), \end{cases} \tag{3.2}$$

in which $\mathbf{C} = [c_1, \ldots, c_{\hat{m}}]^T, \mathbf{D} = [d_1, \ldots, d_{\hat{m}}]^T$, $\mathbf{K} = [k_1, \ldots, k_{\hat{m}}]^T$. Integrating of fractional–order, we obtain

$$\begin{cases} T(t) = I^{\nu_1} D_*^{\nu_1} T(t) + T_0 \approx \mathbf{C}^T \mathbf{P}^{\nu_1}_{\hat{m}\times\hat{m}} \mathbf{\Psi}_{\hat{m}}(t) + T_0, \\ I(t) = I^{\nu_2} D_*^{\nu_2} I(t) + I_0 \approx \mathbf{D}^T \mathbf{P}^{\nu_2}_{\hat{m}\times\hat{m}} \mathbf{\Psi}_{\hat{m}}(t) + I_0, \\ V(t) = I^{\nu_3} D_*^{\nu_3} V(t) + V_0 \approx \mathbf{K}^T \mathbf{P}^{\nu_3}_{\hat{m}\times\hat{m}} \mathbf{\Psi}_{\hat{m}}(t) + V_0, \end{cases} \tag{3.3}$$

where $I^{(\cdot)}$ is the Riemann–Liouville fractional integral operator. Collocating (3.3) at the points $t \in \left\{ \frac{2i-1}{2\hat{m}} | i = 1, \ldots, \hat{m} \right\}$ and using (2.10) lead to

$$\begin{cases} T(t) \approx \mathbf{C}^T \mathbf{P}^{\nu_1}_{\hat{m}\times\hat{m}} \mathbf{\Phi}_{\hat{m}\times\hat{m}} \mathbf{B}_{\hat{m}}(t) + [T_0, \ldots, T_0]_{1\times\hat{m}} \mathbf{B}_{\hat{m}}(t), \\ I(t) \approx \mathbf{D}^T \mathbf{P}^{\nu_2}_{\hat{m}\times\hat{m}} \mathbf{\Phi}_{\hat{m}\times\hat{m}} \mathbf{B}_{\hat{m}}(t) + [I_0, \ldots, I_0]_{1\times\hat{m}} \mathbf{B}_{\hat{m}}(t), \\ V(t) \approx \mathbf{K}^T \mathbf{P}^{\nu_3}_{\hat{m}\times\hat{m}} \mathbf{\Phi}_{\hat{m}\times\hat{m}} \mathbf{B}_{\hat{m}}(t) + [V_0, \ldots, V_0]_{1\times\hat{m}} \mathbf{B}_{\hat{m}}(t). \end{cases} \tag{3.4}$$

Let $\mathbf{C}^T \mathbf{P}^{\nu_1}_{\hat{m}\times\hat{m}} \mathbf{\Phi}_{\hat{m}\times\hat{m}} = [a_1, \ldots, a_{\hat{m}}]$. Then, the nonlinear terms of (3.1) can be expressed as

$$\begin{aligned} T^2(t) = &\left[a_1^2, \ldots, a_{\hat{m}}^2\right] \mathbf{B}_{\hat{m}}(t) + 2\left[T_0 a_1, \ldots, T_0 a_{\hat{m}}\right] \mathbf{B}_{\hat{m}}(t) \\ &+ \left[T_0^2, \ldots, T_0^2\right]_{1\times\hat{m}} \mathbf{B}_{\hat{m}}(t), \end{aligned} \tag{3.5}$$

$$\begin{aligned} T(t)I(t) = &\left([a_1, \ldots, a_{\hat{m}}] \otimes \mathbf{D}^T \mathbf{P}^{\nu_2}_{\hat{m}\times\hat{m}} \mathbf{\Phi}_{\hat{m}\times\hat{m}}\right) \mathbf{B}_{\hat{m}}(t) \\ &+ \left([T_0, \ldots, T_0]_{1\times\hat{m}} \otimes \mathbf{D}^T \mathbf{P}^{\nu_2}_{\hat{m}\times\hat{m}} \mathbf{\Phi}_{\hat{m}\times\hat{m}}\right) \mathbf{B}_{\hat{m}}(t) \\ &+ [T_0 I_0, \ldots, T_0 I_0]_{1\times\hat{m}} \mathbf{B}_{\hat{m}}(t) + [I_0 a_1, \ldots, I_0 a_{\hat{m}}] \mathbf{B}_{\hat{m}}(t), \end{aligned} \tag{3.6}$$

$$\begin{aligned} V(t)T(t) = &\left([a_1, \ldots, a_{\hat{m}}] \otimes \mathbf{K}^T \mathbf{P}^{\nu_3}_{\hat{m}\times\hat{m}} \mathbf{\Phi}_{\hat{m}\times\hat{m}}\right) \mathbf{B}_{\hat{m}}(t) \\ &+ \left([T_0, \ldots, T_0]_{1\times\hat{m}} \otimes \mathbf{K}^T \mathbf{P}^{\nu_3}_{\hat{m}\times\hat{m}} \mathbf{\Phi}_{\hat{m}\times\hat{m}}\right) \mathbf{B}_{\hat{m}}(t) \\ &+ [T_0 V_0, \ldots, T_0 V_0]_{1\times\hat{m}} \mathbf{B}_{\hat{m}}(t) + [V_0 a_1, \ldots, V_0 a_{\hat{m}}] \mathbf{B}_{\hat{m}}(t). \end{aligned} \tag{3.7}$$

Substituting (3.2), (3.4), (3.5), (3.6) and (3.7) into (3.1) and dispersing t at the collocation points, the following nonlinear system of $3\hat{m}$ algebraic equations for $3\hat{m}$

unknowns is achieved

$$\begin{cases} \mathbf{C}^T\mathbf{\Phi}_{\hat{m}\times\hat{m}} = [p,\ldots,p]_{1\times\hat{m}} + (r-\alpha)\,[a_1+T_0,\ldots,a_{\hat{m}}+T_0] \\ -\frac{r}{T_{\max}}\left[a_1^2+2T_0a_1+T_0^2,\ldots,a_{\hat{m}}^2+2T_0a_{\hat{m}}+T_0^2\right] \\ -\frac{r}{T_{\max}}\left([a_1,\ldots,a_{\hat{m}}]\otimes\mathbf{D}^T\mathbf{P}^{\nu_2}_{\hat{m}\times\hat{m}}\mathbf{\Phi}_{\hat{m}\times\hat{m}} + [I_0a_1,\ldots,I_0a_{\hat{m}}]\right) \\ -\frac{r}{T_{\max}}\left([T_0,\ldots,T_0]_{1\times\hat{m}}\otimes\mathbf{D}^T\mathbf{P}^{\nu_2}_{\hat{m}\times\hat{m}}\mathbf{\Phi}_{\hat{m}\times\hat{m}} + [T_0I_0,\ldots,T_0I_0]_{1\times\hat{m}}\right) \\ -k\left([a_1,\ldots,a_{\hat{m}}]\otimes\mathbf{K}^T\mathbf{P}^{\nu_3}_{\hat{m}\times\hat{m}}\mathbf{\Phi}_{\hat{m}\times\hat{m}} + [V_0a_1,\ldots,V_0a_{\hat{m}}]\right) \\ -k\left([T_0,\ldots,T_0]_{1\times\hat{m}}\otimes\mathbf{K}^T\mathbf{P}^{\nu_3}_{\hat{m}\times\hat{m}}\mathbf{\Phi}_{\hat{m}\times\hat{m}} + [T_0V_0,\ldots,T_0V_0]_{1\times\hat{m}}\right), \\ \mathbf{D}^T\mathbf{\Phi}_{\hat{m}\times\hat{m}} = k\left([a_1,\ldots,a_{\hat{m}}]\otimes\mathbf{K}^T\mathbf{P}^{\nu_3}_{\hat{m}\times\hat{m}}\mathbf{\Phi}_{\hat{m}\times\hat{m}} + [V_0a_1,\ldots,V_0a_{\hat{m}}]\right) \\ +k\left([T_0,\ldots,T_0]_{1\times\hat{m}}\otimes\mathbf{K}^T\mathbf{P}^{\nu_3}_{\hat{m}\times\hat{m}}\mathbf{\Phi}_{\hat{m}\times\hat{m}} + [T_0V_0,\ldots,T_0V_0]_{1\times\hat{m}}\right) \\ -\beta\left(\mathbf{D}^T\mathbf{P}^{\nu_2}_{\hat{m}\times\hat{m}}\mathbf{\Phi}_{\hat{m}\times\hat{m}} + [I_0,\ldots,I_0]_{1\times\hat{m}}\right), \\ \mathbf{K}^T\mathbf{\Phi}_{\hat{m}\times\hat{m}} = \left(N\beta\mathbf{D}^T\mathbf{P}^{\nu_2}_{\hat{m}\times\hat{m}} - \gamma\mathbf{K}^T\mathbf{P}^{\nu_3}_{\hat{m}\times\hat{m}}\right)\mathbf{\Phi}_{\hat{m}\times\hat{m}} \\ +\,[N\beta I_0-\gamma V_0,\ldots,N\beta I_0-\gamma V_0]_{1\times\hat{m}}, \end{cases} \tag{3.8}$$

which is solved by Newton-Raphson method or *fsolve* function of MATLAB and MAPLE softwares. Substituting $c_i,\ d_i$ and $k_i,\ i=1,\ldots,\hat{m}$ into (3.3), $T(t),\ I(t)$ and $V(t)$ are obtained.

4. Numerical example

Firstly, it is notable that we perform all of the computations by MATLAB R2015a software on a 64-bit PC with 2.20 GHz processor and 8 GB memory.
Consider that $\nu_1=\nu_2=\nu_3=\nu$. Also, the initial values and parameters of the system (1.3) explained in Table 1 are given as follows

$$\begin{aligned} &T_0=0.1,\ I_0=0,\ V_0=0.1,\ p=0.1,\ \alpha=0.02,\ \beta=0.3,\\ &r=3,\ \gamma=2.4,\ k=0.00027,\ T_{\max}=1500,\ N=10. \end{aligned}$$

Let $k=3,\ M=4$. After solving the nonlinear system (3.8) and substituting the obtained coefficients, $c_i,\ d_i$ and $k_i,\ i=1,\ldots,\hat{m}$ into (3.3), the solution of (1.3) is specified.

Tables 2, 3 and 4 compare the numerical results of FKCW in the case $\nu=1$ with those results of [17, 20, 31] and fourth–order Runge–Kutta (RK4) method.

Table 5 is devoted to the maximum differences of RK4 method from FKCW method (for $k=3,\ M=4$ or $\hat{m}=16$) and Variational iteration method (16 terms) in the case $\nu=1$.
These amounts for Variational iteration method are as follows

$$\begin{cases} L_\infty(T_{VIM}) = \max|T_{RK}-T_{VIM}|, \\ L_\infty(I_{VIM}) = \max|I_{RK}-I_{VIM}|, \\ L_\infty(V_{VIM}) = \max|V_{RK}-V_{VIM}|, \end{cases}$$

and for the present wavelet method are defined in the following

$$\begin{cases} L_\infty(T_{WM}) = \max|T_{RK} - T_{WM}|, \\ L_\infty(I_{WM}) = \max|I_{RK} - I_{WM}|, \\ L_\infty(V_{WM}) = \max|V_{RK} - V_{WM}|. \end{cases}$$

The results of Table 5 expose that in comparison with VIM the present method is in a better agreement with RK4 method.

Figures 2(A), 3(A) and 4(A) demonstrate the behaviour of solution for some values of ν. It is understandable that when ν tends to 1 the solution of fractional model (1.3) approaches to the solution of classical model (1.1). Figures 2(B), 3(B) and 4(B) also conclude the present method for $\nu = 1$ is well adapted to RK4 method.

Tables 6, 7 and 8 analogy the present method to the method of [6] for $\nu = 0.90,\ 0.95,\ 0.98$. It is visible that the results of two methods are close to each other.

We can test the accuracy of FKCW method easily by using the residual errors of $T(t)$, $I(t)$ and $V(t)$ for the system (1.3). For each $t \in [0, 1]$, these errors are defined as

$$\begin{cases} E(T(t)) = D_*^{\nu_1} T - p + \alpha T - rT\left(1 - \frac{T+I}{T_{\max}}\right) + kVT \approx 0, \\ E(I(t)) = D_*^{\nu_2} I - kVT + \beta I \approx 0, \\ E(V(t)) = D_*^{\nu_3} V - N\beta I + \gamma V \approx 0. \end{cases}$$

Table 9 calculates absolute residual errors for $\nu = 0.75$, $k = 2$ and $M = 6,\ 9$.

Figures 5, 6 and 7 portray $E(T(t))$, $E(I(t))$ and $E(V(t))$, respectively for $k = 3$, $M = 4$ or $(\hat{m} = 16)$ in the case $\nu = 0.98$ by means of present method.

Tables 10, 11 and 12 compare absolute residual errors of 1st, 2nd and 3rd kind Chebyshev wavelets with present method for $\hat{m} = 16$ and $\nu = 0.98$.

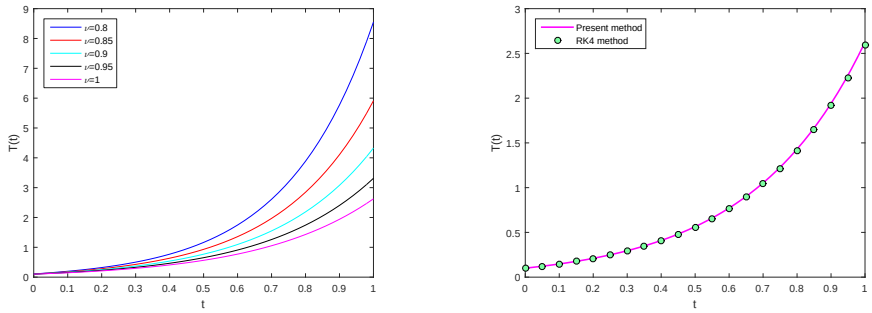
TABLE 2. $T(t)$ solutions for $\nu = 1$

t	Method of [20]	Method of [31]	VIM [17]	RK4	Present method
0.2	0.2088072731	0.2038616561	0.2088073214	0.2088080833	0.2102858565
0.4	0.4061052625	0.3803309335	0.4061346587	0.4062405393	0.4096883861
0.6	0.7611467713	0.6954623767	0.7624530350	0.7644238890	0.7720687308
0.8	1.3773198590	1.2759624442	1.3978805880	1.4140468310	1.4304307811
1.0	2.3291697610	2.3832277428	2.5067466690	2.5915948020	2.6253702046

TABLE 3. $I(t)$ solutions for $\nu = 1$

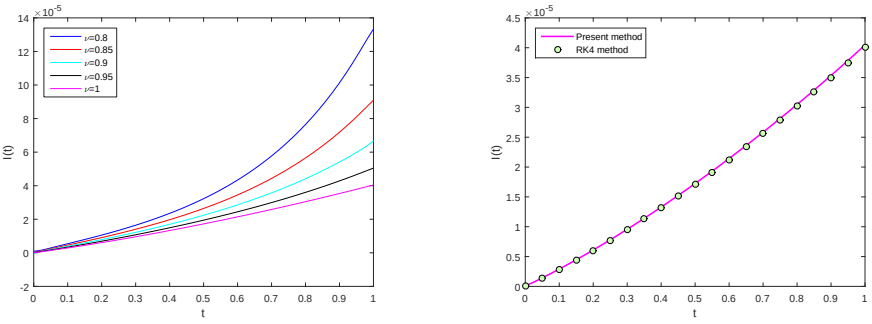
t	Method of [20]	Method of [31]	VIM [17]	RK4	Present method
0.2	0.60327073e-5	0.62478721e-5	0.60326344e-5	0.60327070e-5	0.60995473e-5
0.4	0.13159162e-4	0.12935522e-4	0.13148785e-4	0.13158910e-4	0.13289401e-4
0.6	0.21268369e-4	0.20352672e-4	0.21014172e-4	0.21232982e-4	0.21433580e-4
0.8	0.30069187e-4	0.28373021e-4	0.27951305e-4	0.30242702e-4	0.30482363e-4
1.0	0.39873654e-4	0.36908424e-4	0.24315623e-4	0.40333219e-4	0.40436484e-4

FIGURE 2. The numerical behaviour of $T(t)$



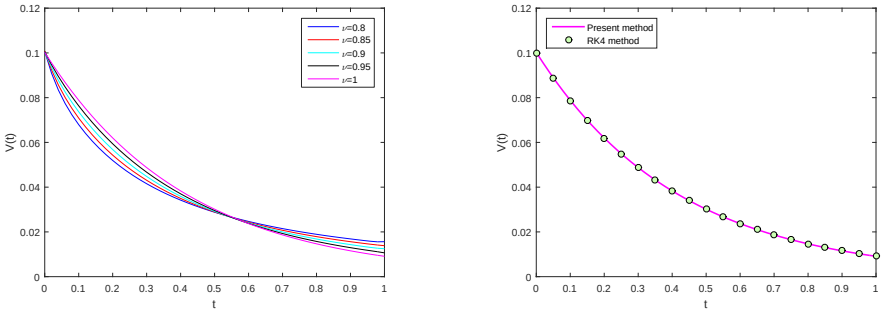
(A) Solutions of $T(t)$ (B) $T(t)$ for $\nu = 1$

FIGURE 3. The numerical behaviour of $I(t)$



(A) Solutions of $I(t)$ (B) $I(t)$ for $\nu = 1$

FIGURE 4. The numerical behaviour of $V(t)$



(A) Solutions of $V(t)$ (B) $V(t)$ for $\nu = 1$

FIGURE 5. Residual error of $T(t)$ for $\hat{m} = 16,\ \nu = 0.98$

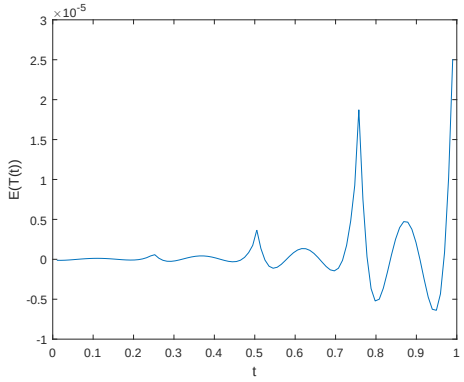


FIGURE 6. Residual error of $I(t)$ for $\hat{m} = 16,\ \nu = 0.98$

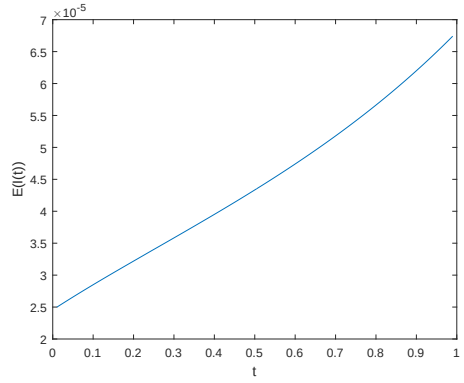


FIGURE 7. Residual error of $V(t)$ for $\hat{m} = 16,\ \nu = 0.98$

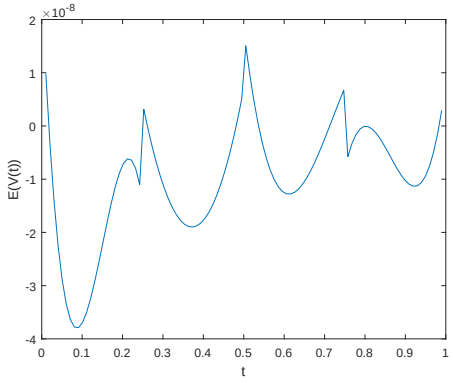


TABLE 4. $V(t)$ solutions for $\nu = 1$

t	Method of [20]	Method of [31]	VIM [17]	RK4	Present method
0.2	0.0618799603	0.0618799186	0.0618799531	0.0618798433	0.0619988576
0.4	0.0383132488	0.0382949349	0.0383082013	0.0382948879	0.0383340405
0.6	0.0243917435	0.0237043186	0.0239202926	0.0237045501	0.0237076863
0.8	0.0099672189	0.0146795698	0.0162170455	0.0146803638	0.0146692185
1.0	0.0033050764	0.0237043186	0.0160841871	0.0091008450	0.0091033490

TABLE 5. Maximum differences of RK4 method from VIM and present wavelet method for $\nu = 1$

$L_\infty(T_{VIM})$	$L_\infty(T_{WM})$	$L_\infty(I_{VIM})$	$L_\infty(I_{WM})$	$L_\infty(V_{VIM})$	$L_\infty(V_{WM})$
8.48e-2	3.34e-2	1.60e-5	1.03e-7	7.00e-3	2.50e-6

TABLE 6. Solution for $\nu = 0.90$

t	T(t)		I(t)		V(t)	
	[6]	Ours	[6]	Ours	[6]	Ours
0.2	0.2501158	0.2527680	7.7692349e-6	7.8959056e-6	0.0567019	0.0568077
0.4	0.5309877	0.5377599	1.6753726e-5	1.6999140e-5	0.0358789	0.0358936
0.6	1.0784967	1.0948600	2.8164952e-5	2.8582639e-5	0.0240146	0.0240055
0.8	2.1489168	2.1869990	4.3464569e-5	4.4150437e-5	0.0168397	0.0168260
1.0	4.2409152	4.3272263	6.5030221e-5	6.6704335e-5	0.0123158	0.0124822

TABLE 7. Solution for $\nu = 0.95$

t	T(t)		I(t)		V(t)	
	[6]	Ours	[6]	Ours	[6]	Ours
0.2	0.2272514	0.2292073	6.8290042e-6	6.9203613e-6	0.0592643	0.0593793
0.4	0.4606339	0.4653858	1.4746964e-5	1.4924381e-5	0.0369869	0.0370138
0.6	0.8979687	0.9089264	2.4155672e-5	2.4445824e-5	0.0237983	0.0237942
0.8	1.8178109	1.7431956	3.5585428e-5	3.6030591e-5	0.0157587	0.0157459
1.0	3.2590105	3.3131521	4.9905893e-5	5.0489617e-5	0.0107509	0.0106363

5. CONCLUDING REMARKS

Throughout this paper, the generalization of conventional HIV infection model was inspected as a system of nonlinear fractional differential equations. By using the FKCW collocation method, we provided an effective procedure with high accuracy to the numerical solution of the generalized model. The important advantages of the method are categorized as follows:

TABLE 8. Solution for $\nu = 0.98$

t	T(t)		I(t)		V(t)	
	[6]	Ours	[6]	Ours	[6]	Ours
0.2	0.2157344	0.2173827	6.3363901e-6	6.4120048e-6	0.0608295	0.0609482
0.4	0.4264249	0.4303295	1.3753198e-5	1.3900743e-5	0.0377472	0.0377825
0.6	0.8133480	0.8221360	2.2294605e-5	2.2532082e-5	0.0237253	0.0237253
0.8	1.5242915	1.5434025	3.2111872e-5	3.2463529e-5	0.0151098	0.0150975
1.0	2.8300579	2.8705419	4.3511235e-5	4.4043761e-5	0.0097708	0.0096023

TABLE 9. Absolute residual errors for $\nu = 0.75$ and $k = 2$

t	$\lvert E(T(t))\rvert$		$\lvert E(I(t))\rvert$		$\lvert E(V(t))\rvert$	
	$M = 6$	$M = 9$	$M = 6$	$M = 9$	$M = 6$	$M = 9$
0.2	1.78e-7	1.89e-8	4.60e-5	4.50e-5	4.70e-8	2.95e-8
0.4	1.29e-6	1.01e-7	8.06e-5	7.85e-5	8.60e-9	1.41e-8
0.6	6.60e-5	1.82e-7	1.48e-4	1.43e-4	1.12e-7	3.23e-8
0.8	1.09e-5	4.62e-8	2.82e-4	2.71e-4	3.40e-8	1.65e-8
1.0	1.83e-3	8.28e-6	5.56e-4	5.32e-4	5.61e-7	7.33e-7

TABLE 10. $\lvert E(T(t))\rvert$ for $\hat{m} = 16$ and $\nu = 0.98$

t	1st CW	2nd CW	3rd CW	4th CW (Present)
0.1	1.394145363e-7	2.853561788e-9	1.483582065e-7	1.173094767e-7
0.3	2.125257259e-7	3.386933558e-7	2.179188310e-7	2.162348578e-7
0.5	2.110107749e-6	2.065419183e-6	1.935604056e-6	2.334686191e-6
0.7	1.614663088e-6	1.159239888e-6	1.623875887e-6	1.405070119e-6
0.9	1.972559666e-6	1.630945443e-6	2.145168218e-6	1.920314006e-6

TABLE 11. $\lvert E(I(t))\rvert$ for $\hat{m} = 16$ and $\nu = 0.98$

t	1st CW	2nd CW	3rd CW	4th CW (Present)
0.1	2.849599731e-5	2.849598901e-5	2.849599720e-5	2.849599563e-5
0.3	3.582931357e-5	3.582930290e-5	3.582931139e-5	3.582931343e-5
0.5	4.333970193e-5	4.333967383e-5	4.333969282e-5	4.333969529e-5
0.7	5.179770982e-5	5.179773022e-5	5.179770952e-5	5.179770946e-5
0.9	6.200333814e-5	6.200332757e-5	6.200333231e-5	6.200334232e-5

- The calculations are in the matrix form. This manner of representation makes the computer programming simple and convenient. In the other word, this method is computer–oriented.
- For small values of k and M, the method has acceptable accuracy.

TABLE 12. $|E(V(t))|$ for $\hat{m} = 16$ and $\nu = 0.98$

t	1st CW	2nd CW	3rd CW	4th CW (Present)
0.1	3.239707247e-8	3.162182990e-8	2.144213279e-8	3.705880902e-8
0.3	1.020724609e-8	9.810029955e-9	7.175736574e-9	1.095296066e-8
0.5	1.286560594e-8	1.330563966e-8	1.288493966e-8	7.083119397e-9
0.7	4.910431575e-10	2.140398553e-9	3.394702824e-9	2.464652142e-9
0.9	8.491690354e-9	2.724232653e-8	7.752388443e-9	1.028396356e-8

- Due to the operational matrix is detected after discretization, the computation of it is done once and is used to solve fractional–order system of ordinary differential equations as well as integer–order one.
- The operational matrix simplifies the problem to a solvable system of algebraic equations.

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REFERENCES

[1] Y. Chen, M. Yi., and C. Yu, *Error analysis for numerical solution of fractional differential equation by Haar wavelets method*, J. Comput. Sci., *3*(5) (2012), 367–373.
[2] C. K. Chui, *An Introduction to Wavelets*, Academic Press, 1992.
[3] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, 1992.
[4] K. Diethelm, *The analysis of Fractional Differential Equations*, Springer–Verlag, 2010.
[5] N. Dogan, *Numerical treatment of the model for HIV infection of $CD4^+$ T cells by using multistep Laplace adomian decomposition method*, Discrete Dyn. Nat. Soc., Article ID 976352 (2012), 11 pages.
[6] M. R. Gandomani and M. T. Kajani, *Numerical solution of a fractional order model of HIV infection of $CD4^+$ T cells using Müntz-Legendre polynomials*, Int. J. Bioautomation, *20*(2) (2016), 193–204.
[7] A. Gökdogan, A. Yildirim, and M. Merdan, *Solving a fractional-order model of HIV infection of $CD4^+$ T cells*, Math. Comput. Model., *54*(9–10) (2011), 2132–2138.
[8] F. Haq, K. Shah, G. U. Rahman, and M. Shahzad, Numerical analysis of fractional order model of HIV-1 infection of $CD4^+$T-cells, Comput. Methods Differ. Equ., *5*(1) (2017), 1–11.
[9] Z. Jiang and W. Schaufelberger, *Block Pulse Functions and Their Applications in Control Systems*, Springer-Verlag, 1992.
[10] E. Keshavarz, Y. Ordokhani, and M. Razzaghi, *Bernoulli wavelet operational matrix of fractional order integration and its applications in solving the fractional order differential equations*, Appl. Math. Model., *38*(24) (2014), 6038–6051.
[11] M. Khalid, M. Sultana, F. Zaidi, and F. S. Khan, *A Numerical Solution of a Model for HIV Infection $CD4^+$ T cells*, Int. J. Inno. Sci. Res., *16*(1) (2015), 79–85.
[12] A. Kilicman and Z. A. A. Al Zhour, *Kronecker operational matrices for fractional calculus and some applications*, Appl. Math. Comp., *187*(1) (2007), 250–265.
[13] Y. Li, *Solving a nonlinear fractional differential equation using Chebyshev wavelets*, Commun. Nonlinear Sci. Numer. Simul., *15*(9) (2010), 2284–2292.
[14] S. Mallat, *A Wavelet Tour of Signal Processing*, Academic Press, 2000.
[15] J. C. Mason and D. C. Handscomb, *Chebyshev Polynomials*, CRC Press, 2003.

[16] M. Merdan, *Homotopy perturbation method for solving a model for hiv infection of $CD4^+$ T cells*, Istanb. Commerce Uni. J. Sci., *12*(6) (2007), 39–52.
[17] M. Merdan, A. Gökdogan, and A. Yildirim, *On the numerical solution of the model for HIV infection of $CD4^+$ T cells*, Comput. Math. Appl., *62*(1) (2011), 118-123.
[18] M. Misiti, Y. Misiti, G. Oppenheim, and J. M. Poggi, *Wavelets and their Applications*, Wiley Press, 2007.
[19] K. Nouri and N. B. Siavashani, *Application of Shannon wavelet for solving boundary value problems of fractional differential equations*, Wavelets linear algebr., *1*(1) (2014), 33–42.
[20] M. Y. Ongun, *The Laplace Adomian Decomposition Method for solving a model for HIV infection of $CD4^+$ T cells*, Math. Comput. Model., *53*(5–6) (2011), 597–603.
[21] I. Podlubny, *Fractional Differential Equations*, Academic Press, 1999.
[22] M. U. Rehman and R. A. Khan, *The Legendre wavelet method for solving fractional differential equations*, Commun Nonlinear Sci Numer Simulat, *16*(11) (2011), 4163–4173.
[23] M. U. Rehman and U. Saeed, *Gegenbauer wavelets operational matrix method for fractional differential equations*, J. Korean Math. Soc., *52*(5) (2015), 1069–1096.
[24] L. J. Rong and P. Chang, *Jacobi wavelet operational matrix of fractional integration for solving fractional integro-differential equation*, J. Phys. Conf. Ser, (2016), 14 pages. DOI 10.1088/1742-6596/693/1/012002.
[25] V. K. Srivastava, M. K. Awasthi, and S. Kumar, *Numerical approximation for HIV infection of $CD4^+$ T cells mathematical model*, Ain Shams Eng. J., *5*(2) (2014), 625–629.
[26] S. G. Venkatesh, S. R. Balachandar, S. K. Ayyaswamy, and K. Balasubramanian, *A new approach for solving a model for HIV infection of $CD4^+$ T cells arising in mathematical chemistry using wavelets*, J. Math. Chem., *54*(5) (2016), 1072–1082.
[27] L. Wang and M. Y. Li, *Mathematical analysis of the global dynamics of a model for HIV infection of $CD4^+$ T cells*, Math. Biosci., *200*(1) (2006), 44–57.
[28] Y. Wang and Q. Fan, *The second kind Chebyshev wavelet method for solving fractional differential equations*, Appl. Math. Comput., *218*(17) (2012), 8592–8601.
[29] Y. Wang and L. Zhu, *Solving nonlinear Volterra integro-differential equations of fractional order by using Euler wavelet method*, Adv. Differ. Equ., (2017), 16 pages. DOI 10.1186/s13662-017-1085-6.
[30] M. Yi and J. Huang, *CAS wavelet method for solving the fractional integro-differential equation with a weakly singular kernel*, Int. J. Comput. Math., *92*(8) (2015), 1715–1728.
[31] S. Yüzbasi, *A numerical approach to solve the model for HIV infection of $CD4^+$ T cells*, Appl. Math. Model., *36*(12) (2012), 5876–5890.
[32] F. Zhou and X. Xu, *The third kind Chebyshev wavelets collocation method for solving the time-fractional convection diffusion equations with variable coefficients*, Appl. Math. Comput., *280*(20) (2016), 11–29.