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Exact solutions for Fokker-Plank equation of geometric Brownian motion with lie point symmetries

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Abstract In this paper, Lie symmetry analysis is applied to find new solution for Fokker Plank equation of geometric Brownian motion. This analysis classifies the solution format of the Fokker Plank equation.

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1. Introduction

Geometric Brownian motion (GBM) is a very important concept in financial mathematics. This stochastic differential equation is known as Black-Scholes model in financial markets. In this paper we consider a financial market with price process $\{S_t\}_t$ for risky asset.

Let the market be the free of arbitrage possibilities and be specified by the following stochastic differential equation

$$dS_t = \alpha S_t dt + \sigma S_t dW_t,$$

where the drift parameter α and the volatility σ are assumed to be constants. The above SDE is called geometric Brownian motion. Symmetry play a very important role in various fields of nature. As is known to all, Lie method is an effective method and a large number of equation are solved with the aid of this method [2, 4, 8, 9, 11].

There are still many authors using this method to find the exact solutions of any given system of differential equation. There is a lots of literature Lie point symmetry method and its application in differential equation [1, 5, 7, 10, 11, 12]. In the first section the geometric Brownian motion and its Fokker Plank equation are observed. In the next section the Lie point symmetry of differential equation is presented. As

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an application of the cited method, the solution format of the Fokker Plank equation is given in last section.

2. Geometric Brownian Motion

Geometric Brownian motion is one of fundamental building blocks for the modeling of asset prices in a financial market. In fact GBM is one of two natural generalizations of the simplest linear ODE and is as follows [3].

$$dS_t = \alpha S_t dt + \sigma S_t dW_t,$$

where $\{W_t\}_t$ is Brownian motion (Winner process) and $S_0 = s_0$.

Theorem 2.1. *Let $\{X_t\}_t$ be a solution to the stochastic differential equation*

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t,$$

with infinitesimal generator $\mathcal{A}$ given by

$$(\mathcal{A}f)(s, y) = \mu(s, y)\frac{\partial f}{\partial y} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2}(s, y).$$

If the solution $\{X_u\}_{u\in[t,s]}$ has a transition density $p(s, y; t, x)$, then p will satisfy the FokkerPlanck equation

$$\frac{\partial}{\partial t}p(s, y; t, x) = \mathcal{A}^* p(s, y; t, x), \quad (t, x) \in (0, T) \times \mathbb{R},$$

where $p(s, y; t, x) \longrightarrow \delta_y$ as $t \downarrow s$, and

$$(\mathcal{A}^* f)(t, x) = -\frac{\partial}{\partial x}[\mu(t, x)f(t, x)] + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2}(t, x).$$

So obviously the Fokker Plank equation for GBM is as follows:

$$\frac{\partial p}{\partial t} = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 p}{\partial x^2} + (2\sigma^2 - \alpha)x\frac{\partial p}{\partial x} + (\sigma^2 - \alpha)p. \tag{2.1}$$

3. Lie Point Symmetries

Symmetry plays a very important role in various fields of nature. As is known to all, Lie method is an effective method and a large number of equations [8] are solved with the aid of this method. There are still many authors using this method to find the exact solutions [11, 12] of nonlinear differential equations. It is also a powerful tool for finding exact solutions of nonlinear problems. Many examples of applications to physical problems have been demonstrated in a huge number of papers and a lot of excellent books [7, 11, 12, 13]. The general procedure to obtain Lie symmetries of differential equations, and their applications to find analytic solutions of the equations are described in detail in several monographs on the subject (e.g. [8, 11, 12]) and in numerous papers in the literature (e.g. [2, 6, 7, 10]).

Consider a system of DE (PDE or ODE) in the dependent variables $u^\alpha (1 \le \alpha \le m)$ and independent variables $x^i (1 \le i \le n)$ of the form:

$$\Delta^s(x^i, u^\alpha, u_i^\alpha, u_{ij}^\alpha, ...) = 0, \quad 1 \le s \le k, \tag{3.1}$$

where the subscripts denote partial derivatives (e.g. $u_i^\alpha = \partial u^\alpha / \partial x^i$). To determine continuous symmetries of (3.1), it is useful to consider infinitesimal Lie transformations of the form:

$$\tilde{x}^i = x^i + \varepsilon \xi^i + O(\varepsilon^2), \quad \tilde{u}^\alpha = u^\alpha + \varepsilon \eta^\alpha + O(\varepsilon^2), \tag{3.2}$$

that leave the equation system invariant to $O(\varepsilon^2)$. Lie point symmetries correspond to the case where the infinitesimal generators $\xi^i = \xi^i(x^i, u^\alpha)$ and $\eta^\alpha = \eta^\alpha(x^i, u^\alpha)$ depend only on the x^i and the u^α and not on the derivatives or integrals of the u^α. Generalized Lie symmetries are obtained in the case when the transformations (3.2) also depend on the derivatives or integrals of the u^α.

The infinitesimal transformations for the first and second derivatives to $O(\varepsilon^2)$ are given by the prolongation formula:

$$\tilde{u}_i^\alpha = u_i^\alpha + \varepsilon \zeta_i^\alpha, \quad \tilde{u}_{ij}^\alpha = u_{ij}^\alpha + \varepsilon \zeta_{ij}^\alpha, \tag{3.3}$$

where

$$\zeta_i^\alpha = D_i \hat{\eta^\alpha} + \xi^s u_{si}^\alpha, \quad \zeta_{ij}^\alpha = D_i D_j \hat{\eta}^\alpha + \xi^s u_{sij}^\alpha. \tag{3.4}$$

Here

$$\hat{\eta}^\alpha = \eta^\alpha - \xi^s u_s^\alpha, \tag{3.5}$$

corresponds to the canonical Lie transformation for which $\tilde{x}^i = x^i$ and $\tilde{u}^\alpha = u^\alpha + \varepsilon \hat{\eta}^\alpha$. The symbol D_i in (3.4) denotes the total derivative operator with respect to x^i. Similar formula to (3.4) apply for the transformation of the higher order derivatives.

The condition for invariance of the DE system (3.1) to $O(\varepsilon^2)$ under the Lie transformation (3.2) can be expressed in the form:

$$\mathcal{L}_{\mathbf{v}} \Delta^s \equiv \tilde{\mathbf{v}}(\Delta^s) = 0 \quad \text{whenever} \quad \Delta^s = 0, \quad 1 \leq s \leq k, \tag{3.6}$$

where

$$\tilde{\mathbf{v}} = \mathbf{v} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{ij}^\alpha \frac{\partial}{\partial u_{ij}^\alpha} + \cdots, \tag{3.7}$$

is the prolongation of the vector field

$$\mathbf{v} = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \tag{3.8}$$

associated with the infinitesimal transformation (3.2). The symbol $\mathcal{L}_{\mathbf{v}} \Delta^s$ in (3.6) denotes the Lie derivative of Δ^s with respect to the vector field $\mathbf{v}$ (i.e. $\mathcal{L}_{\mathbf{v}} \Delta^s = \frac{\mathrm{d}\Delta^s}{\mathrm{d}\varepsilon}|_{\varepsilon=0}$).

The Lie symmetries of the Fokker Plank equation for GBM (2.1) with variables t, x and p can be found by solving the Lie determining equation (2.1) for the infinitesimal generators of the Lie group. Below we first write down the Lie determining equations that correspond to the point Lie group. the point Lie algebra system is briefly described, and the symmetries are used to obtain some results for the solutions of system (2.1).

The infinitesimal Lie transformations for the system (3.1) are of the form:

$$\tilde{t} = t + \varepsilon \xi^t, \qquad \tilde{x} = x + \varepsilon \xi^x, \qquad \tilde{p} = p + \varepsilon \eta. \tag{3.9}$$

The corresponding canonical symmetry generator $\hat{\eta}$ is given by the formula analogous to (3.5). Thus

$$\hat{\eta} = \eta - \xi^t \eta_t - \xi^x \eta_x, \tag{3.10}$$

relates the canonical symmetry generator $\hat{\eta}$ to η.

The Lie determining equations (3.6) for the infinitesimal generators of the system (2.1) can be written in the form:

$$\begin{aligned}
&\xi^1_x = \xi^1_p = \xi^2_p = \xi^2_{tt} = \xi^1_{ttt} = \eta_{pp} = 0, \qquad (3.11)\\
&\xi^2_x = \frac{x\xi^1_t + 2\xi^2}{2x},\\
&\eta_{tp} = \frac{-2\sigma^2\xi^1_{tt} - 4x\left(\alpha - \frac{\sigma^2}{2}\right)^2\xi^1_t + 8\left(\alpha - \frac{3}{2}\sigma^2\right)\xi^2_t}{8\sigma^2 x},\\
&\eta_{xx} = \frac{-2p(\alpha-\sigma^2)\eta_p + 2x(\alpha - 2\sigma^2)\eta_x + 2p(\alpha-\sigma^2)\xi^1_t + 2\eta_t + 2(\alpha-\sigma^2)\eta_p}{\sigma^2 x^2},\\
&\eta_{xp} = \frac{2x\left(\alpha - \frac{3}{2}\sigma^2\right)\xi^1_t - 4\xi^2_t}{4\sigma^2 x^2},
\end{aligned}$$

for the vector field

$$\mathbf{v} = \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial p}. \tag{3.12}$$

Thus, the general vector field $\mathbf{v}$ in the point Lie algebra corresponding to the transformations (3.2) can be written in the form:

$$\mathbf{v} = \sum_{i=1}^{6} a_i \mathbf{v}_i, \tag{3.13}$$

where the basis vector fields $\{\mathbf{v}_i : 1 \le i \le 6\}$ are

$$\begin{aligned}
&\mathbf{v}_1 = \frac{\partial}{\partial t}, \qquad \mathbf{v}_2 = x\frac{\partial}{\partial x}, \qquad \mathbf{v}_3 = p\frac{\partial}{\partial p},\\
&\mathbf{v}_4 = tx\frac{\partial}{\partial x} - \frac{2\ln(x) + 3t\sigma^2 - 2t\alpha}{2\sigma^2}p\frac{\partial}{\partial p},\\
&\mathbf{v}_5 = t\frac{\partial}{\partial t} + \frac{1}{2}x\ln(x)\frac{\partial}{\partial x} - \frac{\left(\frac{3}{2}\sigma^2 - \alpha\right)\ln(x) + \left(\alpha - \frac{\sigma^2}{2}\right)^2 t}{2\sigma^2}p\frac{\partial}{\partial p},\\
&\mathbf{v}_6 = \frac{t^2}{2}\frac{\partial}{\partial t} + \frac{1}{2}tx\ln(x)\frac{\partial}{\partial x}\\
&\qquad - \frac{\left(\ln^2(x) - 2t\left(\alpha - \frac{3}{2}\sigma^2\right)\ln(x) + \left(\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma^2\right)\right)}{4\sigma^2}p\frac{\partial}{\partial p}.
\end{aligned}$$

The commutator table of the Lie algebra $\mathcal{G}$ spanned by the vector fields $\mathbf{v}_i$'s are given in Table 1. Thus these vector fields make a Lie algebra with respect to Lie bracket.

TABLE 1. Commutators Table of $\mathcal{G}$

$[\mathbf{v}_i, \mathbf{v}_j]$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$
$\mathbf{v}_1$	0	0	0	$\mathbf{v}_2 + \frac{2\alpha-3\sigma^2}{3\sigma^2}\mathbf{v}_3$	$\mathbf{v}_1 - \frac{4\alpha-4\alpha\sigma^2+\sigma^4}{8\sigma^2}\mathbf{v}_3$
$\mathbf{v}_2$	0	0	0	$-\frac{1}{\sigma^2}\mathbf{v}_3$	$\frac{1}{2}\mathbf{v}_2 + \frac{2\alpha-3\sigma^2}{4\sigma^2}\mathbf{v}_3$
$\mathbf{v}_3$	0	0	0	0	0
$\mathbf{v}_4$	$-\mathbf{v}_2 - \frac{2\alpha-3\sigma^2}{3\sigma^2}\mathbf{v}_3$	$\frac{1}{\sigma^2}\mathbf{v}_3$	0	0	$-\frac{1}{2}\mathbf{v}_4$
$\mathbf{v}_5$	$-\mathbf{v}_1 + \frac{4\alpha-4\alpha\sigma^2+\sigma^4}{8\sigma^2}\mathbf{v}_3$	$-\frac{1}{2}\mathbf{v}_2 - \frac{2\alpha-3\sigma^2}{4\sigma^2}\mathbf{v}_3$	0	$\frac{1}{2}\mathbf{v}_4$	0

3.1. **Classification of the Solutions.** The one-parameter groups g_i generated by the $\mathbf{v}_i$ are given in the following list. The entires give the transformed point $\exp(\varepsilon\mathbf{v}_i)(t,x,p) = (\tilde{t},\tilde{x},\tilde{p})$:

$$\begin{aligned}
g_1 &:= \exp(\varepsilon\mathbf{v}_1)(t,x,p) = (t+\varepsilon, x, p),\\
g_2 &:= \exp(\varepsilon\mathbf{v}_2)(t,x,p) = (t, e^{\varepsilon}x, p),\\
g_3 &:= \exp(\varepsilon\mathbf{v}_3)(t,x,p) = (t, x, e^{\varepsilon}p),\\
g_4 &:= \exp(\varepsilon\mathbf{v}_4)(t,x,p) = \left(tx\varepsilon, t, \frac{p}{\sigma^2} + tp\left(\frac{3}{2} - \frac{\alpha}{\sigma^2}\right)\varepsilon + p\right),\\
g_5 &:= \exp(\varepsilon\mathbf{v}_5)(t,x,p) = \left(\frac{1}{2}tx\ln(x)\varepsilon + x, t\varepsilon + t - p\ln(x)\left(\frac{3}{4} - \frac{\alpha}{2\sigma^2}\right)\varepsilon\right.\\
&\qquad \left. + \frac{1}{2}pt\left(\frac{\sigma^2}{4} - \alpha + \frac{\alpha^2}{\sigma^2}\right)\varepsilon + p\right).
\end{aligned}$$

Since each group element g_i is a symmetry, then, if $p = f(t,x)$ is a solution of the equation (2.1), so are the functions:

$$p = f(t-\varepsilon, x), \tag{3.14}$$

$$p = f(t, e^{-\varepsilon}x), \tag{3.15}$$

$$p = e^{\varepsilon}f(t,x), \tag{3.16}$$

$$p = \left[\frac{\varepsilon}{\sigma^2}\ln\frac{x}{1+t\varepsilon} + t\left(\frac{3}{2} - \frac{1}{\sigma^2}\right)\varepsilon + 1\right] f\left(t, \frac{x}{1+t\varepsilon}\right), \tag{3.17}$$

$$\begin{aligned}
p = &\left[\left(\frac{3}{2} - \frac{\alpha}{2\sigma^2}\right)\ln\left(\frac{2(1+\varepsilon)}{(1+\varepsilon+2t)\ln(x)\varepsilon + 2(1+\varepsilon)}\right)\varepsilon\right.\\
&\left. + \frac{1}{2(1+\varepsilon)}t\left(\frac{\sigma^2}{4} - \alpha + \frac{\alpha^2}{\sigma^2}\right)\varepsilon + 1\right] f\left(\frac{t}{1+\varepsilon}, \frac{2x(1+\varepsilon)}{t\ln(x)\varepsilon + 2(1+\varepsilon)}\right).
\end{aligned} \tag{3.18}$$

Here ε is an arbitrary number. Thus the general form of the solutions for the (2.1) are classified in the set of equations (3.14-3.18).

3.2. **Practical Results for the Solutions of GBM Equation.** In this section we give some exact solutions for the GBM equations by using the general form of the solutions (3.14-3.18). This solutions start with a trivial solution. It is noteworthy that the process is same for the any other solution.

1. For $\sigma^2 = \alpha$, consider the trivial solution $u = c$ for the equation (2.1), then considering the equation (3.17) leads us to the solution

$$p = c\left[\frac{\varepsilon}{\sigma^2}\ln\frac{x}{1+t\varepsilon} + t\left(\frac{3}{2}-\frac{1}{\sigma^2}\right)\varepsilon + 1\right]. \tag{3.19}$$

Also the equation (3.18) gives the solution:

$$\begin{aligned} p &= c\Bigg[\left(\frac{3}{2}-\frac{\alpha}{2\sigma^2}\right)\ln\left(\frac{2(1+\varepsilon)}{(1+\varepsilon+2t)\ln(x)\varepsilon+2(1+\varepsilon)}\right)\varepsilon \\ &\quad + \frac{1}{2(1+\varepsilon)}t\left(\frac{\sigma^2}{4}-\alpha+\frac{\alpha^2}{\sigma^2}\right)\varepsilon+1\Bigg]. \end{aligned} \tag{3.20}$$

2. Now consider the solution (3.19), then equations (3.14-3.18) conclude that the following functions are new solutions:

$$\begin{aligned} p &= c\left[\frac{\varepsilon}{\sigma^2}\ln\frac{x}{1+(t-\varepsilon)\varepsilon} + (t-\varepsilon)\left(\frac{3}{2}-\frac{1}{\sigma^2}\right)\varepsilon+1\right], \\ p &= c\left[\frac{\varepsilon}{\sigma^2}\ln\frac{e^{-\varepsilon}x}{1+(t-\varepsilon)\varepsilon} + t\left(\frac{3}{2}-\frac{1}{\sigma^2}\right)\varepsilon+1\right], \\ p &= ce^{\varepsilon}\left[\frac{\varepsilon}{\sigma^2}\ln\frac{x}{1+t\varepsilon} + t\left(\frac{3}{2}-\frac{1}{\sigma^2}\right)\varepsilon+1\right], \\ p &= c\left[\frac{\varepsilon}{\sigma^2}\ln\frac{x}{1+t\varepsilon} + t\left(\frac{3}{2}-\frac{1}{\sigma^2}\right)\varepsilon+1\right]\left[\frac{\varepsilon}{\sigma^2}\ln\frac{x}{(1+t\varepsilon)^2} + t\left(\frac{3}{2}-\frac{1}{\sigma^2}\right)\varepsilon+1\right], \\ p &= c\Big[\Big(\frac{3}{2}-\frac{\alpha}{2\sigma^2}\Big)^2\varepsilon^2\ln\Big(\frac{2(1+\varepsilon)}{(1+\varepsilon+2t)\ln(x)\varepsilon+2(1+\varepsilon)}\Big) \\ &\quad + \frac{1}{2(1+\varepsilon)}t\Big(\frac{\sigma^2}{4}-\alpha+\frac{\alpha^2}{\sigma^2}\Big)\varepsilon+1\Big]\Bigg[\ln\frac{2(1+\varepsilon)}{\left(1+\varepsilon+\frac{2t}{1+\varepsilon}\right)\varepsilon\ln\frac{2x(1+\varepsilon)}{t\ln(x)\varepsilon+2(1+\varepsilon)}+2(1+\varepsilon)} \\ &\quad + \frac{t}{2(1+\varepsilon)^2}\Big(\frac{\sigma^2}{4}-\alpha+\frac{\alpha^2}{\sigma^2}\Big)\varepsilon+1\Bigg] \end{aligned}$$

3. For the last practical case consider the solution (3.20). The functions (3.14-3.16) gives the following new solutions:

$$p = c\Bigg[\left(\frac{3}{2}-\frac{\alpha}{2\sigma^2}\right)\ln\left(\frac{2(1+\varepsilon)}{(1+\varepsilon+2(t-\varepsilon))\ln(x)\varepsilon+2(1+\varepsilon)}\right)\varepsilon + \frac{1}{2(1+\varepsilon)}(t-\varepsilon)\left(\frac{\sigma^2}{4}-\alpha+\frac{\alpha^2}{\sigma^2}\right)\varepsilon+1\Bigg],$$

$$p = c\Bigg[\left(\frac{3}{2}-\frac{\alpha}{2\sigma^2}\right)\ln\left(\frac{2(1+\varepsilon)}{(1+\varepsilon+2t)\ln(e^{-\varepsilon}x)\varepsilon+2(1+\varepsilon)}\right)\varepsilon + \frac{1}{2(1+\varepsilon)}t\left(\frac{\sigma^2}{4}-\alpha+\frac{\alpha^2}{\sigma^2}\right)\varepsilon+1\Bigg],$$

$$p = ce^{\varepsilon}\Bigg[\left(\frac{3}{2}-\frac{\alpha}{2\sigma^2}\right)\ln\left(\frac{2(1+\varepsilon)}{(1+\varepsilon+2t)\ln(e^{-\varepsilon}x)\varepsilon+2(1+\varepsilon)}\right)\varepsilon + \frac{1}{2(1+\varepsilon)}t\left(\frac{\sigma^2}{4}-\alpha+\frac{\alpha^2}{\sigma^2}\right)\varepsilon+1\Bigg].$$

4. Conclusion

The paper considered the Lie algorithm method for finding some exact solutions of GBM equation. These new solutions have been found with one-parameter Lie group of transformations obtained from symmetries. The illustrated method is general and could be applied for all kind of systems.

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