Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 6, No. 2, 2018, pp. 176-185

$\mathcal{L}_2$-transforms for boundary value problems

Arman Aghili
Department of applied mathematics, faculty of mathematical sciences,
University of Guilan, Rasht-Iran, P. O. Box 1841.
E-mail: `arman.aghili@gmail.com`

Abstract In this article, we will show the complex inversion formula for the inversion of the $\mathcal{L}_2$-transform and also some applications of the $\mathcal{L}_2$, and Post-Widder transforms for solving singular integral equation with trigonometric kernel. Finally, we obtained analytic solution for a partial differential equation with non-constant coefficients.

Keywords. Laplace transform, $\mathcal{L}_2$-transform, Post-Widder transform, Singular integral equation.

2010 Mathematics Subject Classification. 26A33, 44A10, 44A20, 35A22.

1. Introduction

The integral transform receives a special attention in the literature because of its different applications and therefore is considered as a standard technique in solving partial differential equations and singular integral equations. The integral transform technique is one of the most useful tools of applied mathematics employed in many branches of science and engineering. The Laplace-type integral transform called the $\mathcal{L}_2$-transform was introduced by Yurekli [5] and denoted as follows

$$\mathcal{L}_2\{f(t);s\} = \int_0^\infty te^{-s^2t^2}f(t)dt. \tag{1.1}$$

Like the Fourier and Laplace transforms, the $\mathcal{L}_2$, is used in a variety of applications. Perhaps the most common usage of the $\mathcal{L}_2$-transform is in the solution of initial value problems. Many problems of mathematical interest lead to the $\mathcal{L}_2$-transform whose inverses are not readily expressed in terms of tabulated functions. For the absence of methods for inversion of the $\mathcal{L}_2$-transform, recently the authors [2, 3], established a simple formula to invert the $\mathcal{L}_2$-transform of a desired function. In this study, we present some new inversion techniques for the $\mathcal{L}_2$-transform and an application of generalized product theorem for solving some singular integral equations and boundary value problems. Constructive examples are also provided.

Example 1.1. Let us verify the following identity

$$\mathcal{L}_2\{\delta(t^{-k}-\lambda);s\} = \frac{e^{-\frac{s^2}{k\lambda\sqrt[k]{\lambda^2}}}}{k\lambda\sqrt[k]{\lambda^2}}, \qquad \lambda > 0,\ k > 1. \tag{1.2}$$

Received: 6 June 2017 ; Accepted: 6 March 2018.

Solution. By definition of the $\mathcal{L}_2$-transform, we have

$$\mathcal{L}_2\{\delta(t^{-k}-\lambda);s\} = \int_0^{+\infty} te^{-s^2t^2}\delta(t^{-k}-\lambda)dt. \tag{1.3}$$

Let us introduce a change of variable $t^{-k}-\lambda=\xi$, then we get

$$\mathcal{L}_2\{\delta(t^{-k}-\lambda);s\} = \int_0^{+\infty} \frac{1}{\sqrt[k]{\xi+\lambda}} e^{\frac{-s^2}{k(\xi+\lambda)\sqrt[k]{\xi+\lambda}}}\delta(\xi)\frac{d\xi}{k(\xi+\lambda)\sqrt[k]{\xi+\lambda}}, \tag{1.4}$$

using elementary property of Dirac delta function, we arrive at

$$\mathcal{L}_2\{\delta(t^{-k}-\lambda);s\} = \frac{e^{\frac{-s^2}{k\lambda\sqrt[k]{\lambda}}}}{k\lambda\sqrt[k]{\lambda^2}}. \tag{1.5}$$

2. Elementary Properties of the $\mathcal{L}_2$-Transform

In this section, we recall some properties of the $\mathcal{L}_2$-transform that will be useful to solve partial differential equations. The real merit of the $\mathcal{L}_2$-transform is revealed by its effect on derivatives. Here we will derive a relation between the $\mathcal{L}_2$-transform of the derivative of the function and the $\mathcal{L}_2$-transform of the function itself. First, we state a Lemma about the $\mathcal{L}_2$-transform of δ-derivatives.

Lemma 2.1. If $f, f', \dots, f^{(n-1)}$ are all continuous functions with a piecewise continuous derivative $f^{(n)}$ on the interval $t \geq 0$ and if all functions are of exponential order $\exp(c^2t^2)$ as $t\to\infty$ for some real constant c then

(1) For $n=1,2,\dots$

$$\begin{aligned}\mathcal{L}_2\{\delta_t^n f(t);s\} = {} & 2^n s^{2n}\mathcal{L}_2\{f(t);s\} - 2^{n-1}s^{2(n-1)}f(0^+) \\ & -2^{n-2}s^{2(n-2)}(\delta_t f)(0^+) - \dots - (\delta_t^{n-1}f)(0^+)\,.\end{aligned} \tag{2.1}$$

(2) For $n=1,2,\dots$

$$\mathcal{L}_2\{t^{2n}f(t);s\} = \frac{(-1)^n}{2^n}\delta_s^n\mathcal{L}_2\{f(t);s\}\,, \tag{2.2}$$

where the differential operators δ_t, δ_t^2, are defined as below

$$\delta_t = \frac{1}{t}\frac{d}{dt}, \qquad \delta_t^2 = \delta_t\delta_t = \frac{1}{t^2}\frac{d^2}{dt^2} - \frac{1}{t^3}\frac{d}{dt}.$$

Proof. See [5].

3. Complex Inversion Formula for the $\mathcal{L}_2$-Transform and Efros's Theorem

Lemma 3.1. Let $F(\sqrt{s})$ be analytic function (assuming that $s=0$ is not a branch point) except at finite number of poles each of which lies to the left hand side of the vertical line Re $s = c$ and if $F(\sqrt{s}) \to 0$ as $s \to \infty$ through the left plane Re $s \leq c$, and

$$\mathcal{L}_2\{f(t); s\} = \int_0^\infty t \exp(-s^2 t^2) f(t) dt = F(s), \tag{3.1}$$

then

$$\mathcal{L}_2^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2F(\sqrt{s}) e^{st^2} ds = \sum_{k=1}^{m} [Res\{2F(\sqrt{s})e^{st^2}\}, s = s_k]. \tag{3.2}$$

Proof. See [1, 2].

Example 3.1. Let us solve the following impulsive differential equation with non-constant coefficients.

$$\frac{1}{t} y^{'}(t) + \lambda y(t) = t^\beta \delta(t - \xi), \qquad y(0) = 0. \tag{3.3}$$

Solution. The impulse function is a useful concept in a wide variety of mathematical physics problems involving the ideas impulsive forces or point sources. By taking the $\mathcal{L}_2$-transform of the above equation term wise, we get

$$\mathcal{L}_2(\delta_t y(t)) + \lambda \mathcal{L}_2(y(t)) = \mathcal{L}_2(t^\beta \delta(t - \xi)). \tag{3.4}$$

Let us assume that $\mathcal{L}_2(y(t)) = Y(s)$, then after evaluation of the $\mathcal{L}_2$-transform each term, we arrive at

$$2s^2 Y(s) + \lambda Y(s) = \xi^{\beta+1} e^{-\xi^2 s^2}, \tag{3.5}$$

solving the above equation, leads to

$$Y(s) = \frac{\xi^{\beta+1} e^{-\xi^2 s^2}}{2s^2 + \lambda}, \tag{3.6}$$

using complex inversion formula for the $\mathcal{L}_2$-transform, we obtain

$$y(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2\left(\frac{\xi^{\beta+1} e^{-\xi^2 s}}{2s + \lambda}\right) e^{st^2} ds, \tag{3.7}$$

at this point, direct application of the second part of the Lemma (2.1) leads to the following solution

$$y(t) = \xi^{\beta+1} e^{(\xi^2 - t^2)\frac{\lambda}{2}}. \tag{3.8}$$

Lemma 3.2. Efros's Theorem for $\mathcal{L}_2$-Transforms

Let $\mathcal{L}_2(f(t)) = F(s)$ and assuming $\Phi(s)$, $q(s)$ are analytic and such that, $\mathcal{L}_2(\Phi(t, \tau)) = \Phi(s)\tau e^{-\tau^2 q^2(s)}$, then we have the following

$$\mathcal{L}_2\left\{\int_0^\infty f(\tau)\Phi(t, \tau) d\tau\right\} = F(q(s))\Phi(s). \tag{3.9}$$

Proof. By definition of the $\mathcal{L}_2$-transform

$$\mathcal{L}_2\{\int_0^\infty f(\tau)\phi(t, \tau) d\tau\} = \int_0^\infty t e^{-s^2 t^2} (\int_0^\infty f(\tau)\phi(t, \tau) d\tau) dt, \tag{3.10}$$

and changing the order of integration we arrive at

$$\int_0^\infty f(\tau)(\int_0^\infty te^{-s^2t^2}\phi(t,\tau)dt)d\tau = \Phi(s)\int_0^\infty f(\tau)\tau e^{-\tau^2 q^2(s)}d\tau = \Phi(s)F(q(s)). \tag{3.11}$$

Example 3.2. Let us solve the following singular integral equation

$$\int_0^\infty \frac{f(\tau)}{\tau^2}\cos(t\tau)d\tau = -t^{2(1-\nu)}. \tag{3.12}$$

Solution. Differentiating with respect to t on the both sides of the above relation, leads to

$$\int_0^\infty \frac{f(\tau)}{\tau}\sin(t\tau)d\tau = 2(1-\nu)t^{1-2\nu}, \tag{3.13}$$

and applying the $\mathcal{L}_2$-transform followed by the generalized product theorem and using the fact that

$$\mathcal{L}_2\{\sin(\tau t)\} = \frac{\pi}{4s^3}\tau e^{-\frac{\tau^2}{4s^2}}, \tag{3.14}$$

and

$$\mathcal{L}_2[x^\nu, x- > s] = \frac{\Gamma(\frac{\nu}{2}+1)}{2s^{\nu+2}}, \tag{3.15}$$

we arrive at

$$F(\frac{1}{2s})\frac{\sqrt{\pi}}{4s^3} = \frac{2(1-\nu)\Gamma(\frac{3-2\nu}{2})}{2s^{3-2\nu}}, \tag{3.16}$$

or

$$F(s) = \frac{(1-\nu)\Gamma(\frac{3-2\nu}{2})}{\sqrt{\pi}4^\nu s^{2\nu}}, \tag{3.17}$$

finally by inversion of the $\mathcal{L}_2$-transform we get

$$f(t) = \frac{(1-\nu)\Gamma(\frac{3-2\nu}{2})}{\sqrt{\pi}4^\nu\Gamma(\nu)}t^{2(\nu-1)}. \tag{3.18}$$

In the next section, we give some illustrative examples and Lemmas related to the $\mathcal{L}_2$, Post-Widder transforms, and complex inversion formula for the Post-Widder transform.

4. Illustrative Lemmas and Examples

Lemma 4.1. By using complex inversion formula for the $\mathcal{L}_2$-transform, we can show that

$$\int_0^\infty \frac{\xi^2 J_1(t\xi)}{t(s^2+\xi^2)}d\xi = \frac{s}{t}K_1(st) = \frac{s}{t}(\frac{st}{2})\int_0^\infty e^{-u-\frac{(st)^2}{4u}}\frac{du}{2u^2}, \tag{4.1}$$

where K_1 is the modified Bessel function of order one, with the above integral representation.

Proof. It is well known that

$$\mathcal{L}_2[\mathcal{L}_2(\phi(x); x->p]p->s] = \frac{1}{2}\mathcal{P}(\phi(x); x->s), \tag{4.2}$$

therefore, the left hand side of (4.1) can be written as follows

$$\mathcal{L}_2[\mathcal{L}_2(\frac{\xi J_1(t\xi)}{t}; \xi->p]p->s] = \frac{1}{2}\mathcal{P}(\frac{\xi J_1(t\xi)}{t}; \xi->s). \tag{4.3}$$

Applying the $\mathcal{L}_2$-transform two times on $(\frac{\xi}{t})^1 J_1(t\xi)$ and using the fact that

$$\mathcal{L}_2\{(\frac{\xi}{t})^1 J_1(t\xi)\} = \frac{e^{-\frac{t^2}{4s^2}}}{2s^4}, \tag{4.4}$$

we get

$$\mathcal{P}(\frac{\xi J_1(t\xi)}{t}; \xi->s) = 2\mathcal{L}_2[\frac{e^{-\frac{t^2}{4p^2}}}{2p^4}]p->s] = \frac{1}{2}\int_0^\infty pe^{-s^2p^2}\frac{e^{-\frac{t^2}{4p^2}}}{2p^4}dp. \tag{4.5}$$

At this point, let us intrduce a change of variable $u = s^2p^2$, after simplifying we obtain

$$\mathcal{P}(\frac{\xi J_1(t\xi)}{t}; \xi->s) = \frac{s}{t}(\frac{st}{2})\int_0^\infty e^{-u-\frac{s^2t^2}{4u}}\frac{du}{2u^2} = \frac{s}{t}K_1(st). \tag{4.6}$$

Lemma 4.2. Show that the following singular integral equation of Post-Widder type

$$\int_0^{+\infty}\frac{ug(u)}{u^2+s^2}du = \frac{1}{\sqrt{as^2+b}}, \tag{4.7}$$

has a formal solution as below

$$g(u) = \frac{4a}{\pi\sqrt{a\,u^2-b}}. \tag{4.8}$$

Proof. By definition of the inverse Post-Widder transform we have

$$g(u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{su^2}\left(\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{e^{ps^2}}{\sqrt{ap+b}}dp\right)_{s->\sqrt{s}} ds, \tag{4.9}$$

introducing the new variable $w = ap + b$ leads to

$$g(u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2e^{su^2}\left(\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{2e^{w(\frac{s^2}{a})-\frac{b}{a}s^2}}{\sqrt{w}}dw\right)_{s->\sqrt{s}} ds \tag{4.10}$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2e^{su^2}\left(\frac{(\frac{s}{a})^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}\right)2e^{-\frac{b}{a}s}ds,$$

hence

$$g(u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2e^{s(u^2-\frac{b}{a})}.\frac{4\sqrt{a}}{\Gamma(\frac{1}{2})\sqrt{s}}ds = \frac{4\sqrt{a}}{\Gamma(\frac{1}{2})}\left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{(u^2-\frac{b}{a})s}}{\sqrt{s}}ds\right), \tag{4.11}$$

therefore, the final solution is as below

$$g(u) = \frac{4\sqrt{a}}{\Gamma(\frac{1}{2})}.\frac{1}{\Gamma(\frac{1}{2})}.\frac{1}{\sqrt{u^2-\frac{b}{a}}} = \frac{4a}{\pi\sqrt{au^2-b}}. \tag{4.12}$$

Lemma 4.3. Show that the following Post-Widder type singular integral equation

$$\int_0^{+\infty}\frac{u\phi(u)}{u^2+s^2}du = \frac{2\ln s}{as^{2k}-\lambda}, \tag{4.13}$$

has a solution as below

$$\phi(u) = \frac{2}{au^{2k}+\lambda}. \tag{4.14}$$

Proof. We may use the following inversion formula for Post-Widder transform [1]

$$\mathcal{P}^{-1}\{F(s)\} = \frac{1}{\pi i}\{F(u^2e^{-i\pi}) - F(u^2e^{i\pi})\}. \tag{4.15}$$

By using the above inversion formula for Post-Widder transform, we obtain

$$\begin{aligned}\mathcal{P}^{-1}\{\tfrac{2\ln s}{as^{2k}-\lambda}\} &= \tfrac{1}{\pi i}\{\tfrac{\ln(u^2e^{-i\pi})}{a(u^2e^{-i\pi})^{2k}-\lambda} - \tfrac{\ln(u^2e^{i\pi})}{a(u^2e^{i\pi})^{2k}-\lambda}\}\\ &= \tfrac{1}{\pi i}\{\tfrac{\ln u - i\pi}{-au^{2k}-\lambda} - \tfrac{\ln u + i\pi}{-au^{2k}-\lambda}\} = \tfrac{2}{au^{2k}+\lambda}.\end{aligned}$$

Example 4.1. Let us solve the following homogeneous singular integral equation

$$\frac{2}{\pi}\int_0^\infty f(\tau)\sin(t\tau)\,d\tau = t^{1-2\nu}. \tag{4.16}$$

Solution. The $\mathcal{L}_2$-transform of the above integral equation, leads to the following

$$F(\frac{1}{2s})\frac{\sqrt{\pi}}{4s^3} = \frac{\Gamma(\frac{1-2\nu}{2}+2)}{2s^{3-2\nu+2}}, \tag{4.17}$$

or

$$F(s) = \frac{\Gamma(\frac{5-2\nu}{2})}{2^{2\nu-1}s^{2\nu}} \quad\rightarrow\quad f(t) = \mathcal{L}_2^{-1}[\frac{\Gamma(\frac{5-2\nu}{2})}{2^{2\nu-1}\Gamma(\nu)s^{\nu}}] \quad\rightarrow\quad f(t) = \frac{2\Gamma(\frac{5-2\nu}{2})}{2^{2\nu-1}\Gamma(\nu)t^{1-2\nu}}.$$

Example 4.2. By using complex inversion formula for $\mathcal{L}_2$-transform, we show that

$$\mathcal{L}_2^{-1}\left[\frac{1}{2s^{2(n+1)}}\exp\frac{\lambda^2}{4s^2}\right] = (\frac{t}{\lambda})^n I_n(\lambda t), \tag{4.18}$$

where $I_n(.)$ is the modified Bessel's function of the first kind of order n.
Solution. Let

$$F(s) = \frac{1}{2s^{2(n+1)}}\exp\left(\frac{\lambda^2}{4s^2}\right), \tag{4.19}$$

then, we have

$$2F(\sqrt{s}) = \frac{1}{s^{n+1}} \exp(\frac{\lambda^2}{4s}). \tag{4.20}$$

Therefore, $s = 0$ is a singular point (essential singularity not branch point). After using the above complex inversion formula, we obtain the original function as following

$$f(t) = Res \left\{ \frac{1}{s^{n+1}} \exp\left(\frac{\lambda^2}{4s}\right) \exp(st^2), \right\} = b_{-1}, \tag{4.21}$$

where, b_{-1} is the coefficient of the term $\frac{1}{s}$ in the Laurent expansion of $2F(\sqrt{s})\exp(st^2)$. Therefore we get the following relation

$$2F(\sqrt{s})\exp(st^2) =$$

$$\frac{1}{s^{n+1}}\left[1 + (st^2) + \frac{(st^2)^2}{2!} + \cdots\right]\left[1 + \frac{1}{4s} + \frac{1}{(4s)^2 2!} + \frac{1}{(4s)^3 3!} + \cdots\right], \tag{4.22}$$

or

$$2F(\sqrt{s})\exp(st^2) =$$

$$\frac{1}{s}\left[1 + (st^2) + .. + \frac{(st^2)^n}{n!} + \frac{(st^2)^{n+1}}{(n+1)!}\cdots\right]\left[\frac{1}{s^n} + \frac{\lambda^2}{4s^{n+1}} + \frac{\lambda^4}{4^2 s^{n+2} 2!} + \frac{\lambda^6}{4^3 s^{n+3} 3!} + \cdots\right],$$

from the above expansion we obtain

$$f(t) = b_{-1} = \left[\frac{1}{n!} + \frac{(\lambda t)^2}{4^1(n+1)!1!} + \frac{(\lambda t)^4}{4^2(n+2)!2!} + \frac{(\lambda t)^6}{4^3(n+3)!3!} + \cdots\right] t^{2n}. \tag{4.23}$$

$$f(t) = \frac{t^n}{\lambda^n}\sum_{k=0}^{\infty}\frac{(\lambda t)^{2k+n}}{2^{2k}(n+k)!k!}, \tag{4.24}$$

by using series expansion for the modified Bessel's functions, we get the following

$$f(t) = b_{-1} = (\frac{t}{\lambda})^n I_n(\lambda t). \tag{4.25}$$

Example 4.3. Let us solve the following singular integral equation with trigonometric kernel

$$\int_0^\infty x\phi(x)\cos\xi x dx = e^\xi Erfc(\sqrt{\xi}). \tag{4.26}$$

Solution. Taking the Laplace transform of both sides of the integral equation with respect to λ, we get

$$\mathcal{L}\{\int_0^\infty x\phi(x)\cos\lambda x dx\} = \frac{1}{(s-1)\sqrt{s}}, \tag{4.27}$$

or, equivalently

$$\int_0^\infty \frac{x}{x^2+s^2}\phi(x)dx = \frac{1}{s(s-1)\sqrt{s}}, \tag{4.28}$$

the left hand side of the above relation is Widder potential transform of $\phi(x)$ [4]. we obtain

$$\mathcal{P}\{\phi(x);s\} = \frac{1}{s(s-1)\sqrt{s}}, \tag{4.29}$$

or

$$\phi(x) = \frac{1}{\pi i}\left(\frac{1}{x^2e^{-i\pi}(x^2e^{-i\pi}-1)\sqrt{x^2e^{-i\pi}}} - \frac{1}{x^2e^{i\pi}(x^2e^{i\pi}-1)\sqrt{x^2e^{i\pi}}}\right), \tag{4.30}$$

after simplifying, we get

$$\phi(x) = \frac{\pi}{x^3(x^2+1)}. \tag{4.31}$$

Lemma 4.4. The following identity holds true

$$\mathcal{L}_2^{-1}\left\{\frac{e^{-x\sqrt{s^2+\lambda}}}{2(s^2+\lambda)}\right\} = \frac{\sqrt{\pi}}{2}e^{-4t^2}Erf\left(\frac{x}{2\sqrt{t^2+t}}\right). \tag{4.32}$$

Proof. By setting $F(s) = \frac{e^{-x\sqrt{s^2+\lambda}}}{2(s^2+\lambda)}$ we have $2F(\sqrt{s}) = \frac{e^{-x\sqrt{s+\lambda}}}{s+\lambda}$. In order to avoid complex integration along complicated key-hole contour, we may use an appropriate integral representation for $e^{-\xi}$ as follows

$$e^{-\xi} = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-\eta^2-\frac{\xi^2}{4\eta^2}}d\eta, \tag{4.33}$$

if we substitute $\xi = x\sqrt{s+\lambda}$ in the above integral, we get

$$\begin{aligned} f(x,t) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{-x\sqrt{s+\lambda}}}{s+\lambda}e^{st^2}ds \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{s+\lambda}\left(\int_0^\infty e^{-\eta^2-\frac{x^2(s+\lambda)}{4\eta^2}}d\eta\right)e^{st^2}ds \\ &= \int_0^\infty e^{-(\eta^2+\lambda t^2)}\left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{-\frac{(x^2-4\eta^2t^2)(s+\lambda)}{4\eta^2}}}{s+\lambda}ds\right)d\eta. \end{aligned}$$

Let us make a change of variable $w = s+\lambda$ to get

$$f(x,t) = \int_0^\infty e^{-(\eta^2+\lambda t^2)}\left(\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{e^{-\frac{(x^2-4\eta^2t^2)w}{4\eta^2}}}{w}dw\right)d\eta.$$

By using the fact that, $\mathcal{L}^{-1}\{\frac{e^{-as}}{s}\} = H(t-a)$ and setting $a = \frac{x^2}{4\eta^2} - t^2$ in the inner integral, we finally get

$$f(x,t) = \int_0^\infty e^{-(\eta^2+\lambda t^2)}H\left(t-\frac{x^2-4\eta^2t^2}{4\eta^2}\right)d\eta = \frac{\sqrt{\pi}}{2}e^{-4t^2}Erf\left(\frac{x}{2\sqrt{t^2+t}}\right). \tag{4.34}$$

In the next section, we implemented the $\mathcal{L}_2$-transform for solving partial differential equation.

5. The $\mathcal{L}_2$-Transforms For Boundary Value Problems

The second order partial differential equations with non-constant coefficients have a number of applications in electrical and mechanical engineering, medical sciences and economics. This section is devoted to the application of such PDEs.

Problem 5.1. Let us solve the following parabolic type partial differential equation

$$\frac{1}{2t}u_t + u_{rr} + \frac{1}{r}u_r = -\lambda^2 u, \qquad 0 < r < 1,\, t > 0, \tag{5.1}$$

with boundary conditions

$$u(1,t) = -\beta \exp(-\lambda^2 t^2), \qquad u(r,0) = 2T_0, |u(r,t)| < M_0,$$

where T_0, M_0 are positive constants.

Solution. Let us take the $\mathcal{L}_2$-transform of the above equation term wise, we get

$$(s^2+\lambda^2)U(r,s) - 0.5u(r,0) + U_{rr}(r,s) + \frac{1}{r}U_r(r,s) = 0, \tag{5.2}$$

or

$$U_{rr} + \frac{1}{r}U_r + (s^2+\lambda^2)U = T_0, \quad U(1,s) = (-\beta)/(s^2+\lambda^2),\ |U(r,s)| < M'. \tag{5.3}$$

The general solution of the transformed equation is given in terms of the Bessel's functions as below

$$U(r,s) = c_1 J_0(r\sqrt{s^2+\lambda^2}) + c_2 Y_0(r\sqrt{s^2+\lambda^2}) + \frac{T_0}{s^2+\lambda^2}, \tag{5.4}$$

since, $Y_0(sr)$ is unbounded as $r \to 0$, we have to choose $c_2 = 0$, thus

$$U(r,s) = c_1 J_0(r\sqrt{s^2+\lambda^2}) + \frac{T_0}{s^2+\lambda^2}, \tag{5.5}$$

from $U(1,s) = -\frac{\beta}{s^2+\lambda^2}$, we find $c_1 = -\frac{T_0+\beta}{(s^2+\lambda^2)J_0(\sqrt{s^2+\lambda^2})}$,
therefore

$$U(r,s) = \frac{T_0}{s^2+\lambda^2} - \frac{(T_0+\beta)J_0(r\sqrt{s^2+\lambda^2})}{(s^2+\lambda^2)J_0(\sqrt{s^2+\lambda^2})}. \tag{5.6}$$

By using complex inversion formula for the $\mathcal{L}_2$-transform, we get

$$u(r,t) = 2T_0 e^{-\lambda^2 t^2} - \frac{2(T_0+\beta)}{2\pi i}\int_{c-i\infty}^{c+\infty} \frac{e^{st^2} J_0(\sqrt{s+\lambda^2}r)}{(s+\lambda^2)J_0(\sqrt{s+\lambda^2})} ds, \tag{5.7}$$

the integrand in the above integral has simple poles at $s+\lambda^2 = \eta_n^2$, $n = 1,2,3,\ldots$ and also at $s+\lambda^2 = 0$ where η_n are simple zeros of Bessel function as $\sqrt{s+\lambda^2} = \eta_1, \eta_2, \ldots, \eta_n, , \ldots$, hence, we deduce that the residue of integrand at $s = -\lambda^2$ is

$$\lim_{s\to-\lambda^2}(s+\lambda^2)\frac{e^{st^2} J_0(\sqrt{s+\lambda^2}r)}{(s+\lambda^2)J_0(\sqrt{s+\lambda^2})} = e^{-\lambda^2 t^2}, \tag{5.8}$$

and also residue of integrand at $s = \eta_n^2 - \lambda^2$ is

$$\lim_{s\to+\eta_n^2-\lambda^2}(s+\lambda^2-\eta_n^2)\frac{e^{st^2}J_0(\sqrt{s+\lambda^2}r)}{(s+\lambda^2)J_0(\sqrt{s+\lambda^2})} =$$
$$= \left(\lim_{s\to+\eta_n^2-\lambda^2}\frac{s+\lambda^2-\eta_n^2}{J_0(\sqrt{s+\lambda^2})}\right)\left(\lim_{s\to+\eta_n^2-\lambda^2}\frac{e^{st^2}J_0(\sqrt{s+\lambda^2}r)}{s+\lambda^2}\right) =$$
$$= \left(\lim_{s\to+\eta_n^2-\lambda^2}\frac{1}{J_0'(\sqrt{s+\lambda^2})\frac{1}{2\sqrt{s+\lambda^2}}}\right)\left(\frac{e^{-\eta_n^2t^2}J_0(\eta_n r)}{\eta_n^2}\right) = \frac{-2e^{-\eta_n^2t^2}J_0(\eta_n r)}{\eta_n J_1(\eta_n)}.$$

Where we have used L'Hospital's rule in evaluating the limit and also the fact that $J_0'(u) = -J_1(u)$, then

$$u(r,t) = 2T_0e^{-\lambda^2t^2} - 2(T_0+\beta)\{e^{-\lambda^2t^2} - \sum_{n=1}^{\infty}2\frac{e^{-\eta_n^2t^2}J_0(\eta_n r)}{\eta_n J_1(\eta_n)}\} = ...$$

$$u(r,t) = 4(T_0+\beta)\sum_{n=1}^{\infty}\frac{e^{-\eta_n^2t^2}J_0(\eta_n r)}{\eta_n J_1(\eta_n)} - 2\beta\lambda e^{-\lambda^2t^2}. \tag{5.9}$$

6. Conclusion

The main purpose of the present study is to extend the application of the $\mathcal{L}_2$-transforms to derive an analytic solution of boundary value problems. We have presented a method for solving singular integral equations and boundary value problem using the $\mathcal{L}_2$-transform and it is hoped that these results and others derived from this be useful to researchers in the various branches of the integral transforms and applied mathematics.

7. Acknowledgments

The author would like to thank the anonymous referee/s and editor/s for careful and thoughtful reading of the manuscript and useful suggestions which helped to improve the presentation of the results.

References

[1] A. Aghili, A. Ansari, *Solving partial fractional differential equations using the $\mathcal{L}_A$- transform*, Asian- European Journal of Mathematics, *3*(2) (2010), 209–220.

[2] A. Aghili, *A note on Fox-singular integral equations and its application*, Int. J. Contemp. Math. Sciences, *2*(28) (2007), 1371–1378

[3] A. Aghili, A. Ansari, and A. Sedghi, *An inversion technique for the $\mathcal{L}_2$- transform with applications*, Int. J. Contemp. Math. Sciences, *2*(28) (2007), 1387–1394.

[4] A. Aghili, A. Ansari, *New method for solving system of P. F. D. E. and fractional evolution disturbance equation of distributed order*, Journal of Interdisciplinary Mathematics, April 2010.

[5] O. Yurekli, I. Sadek, *A Parseval-Goldstein type theorem on the widder potential transform and its applications*, International Journal of Mathematics and Mathematical Sciences, *14*(3) (1991), 517–524.