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A novel technique for a class of singular boundary value problems

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Abstract In this paper, Lagrange interpolation in Chebyshev-Gauss-Lobatto nodes is used to develop a procedure for finding discrete and continuous approximate solutions of a singular boundary value problem. At first, a continuous time optimization problem related to the original singular boundary value problem is proposed. Then, using the Chebyshev-Gauss-Lobatto nodes, we convert the continuous time optimization problem to a discrete time optimization problem. By solving the discrete time optimization problem, we find discrete approximations for the solutions of the main singular boundary value problem. Also, by Lagrange interpolation we obtain a continuous approximation for the solution. The efficiency and reliability of the proposed approach are tested by solving three practical singular boundary value problems.

Keywords. Singular boundary value problem, Chebyshev polynomial, Continuous time optimization problem, Discrete otimization problem.

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1. Introduction

Consider the following singular boundary value problems (SBVPs)

$$y''(t) + \frac{P(t)}{R(t)}y'(t) = f(t, y(t)),\ 0 \le t \le 1, \tag{1.1}$$

subject to the boundary conditions

$$y'(0) = 0,\ \alpha y'(1) + \beta y(1) = \gamma, \tag{1.2}$$

where α, β and γ are nonnegative constants. We assume that $R(0) = 0$, $R(t) \neq 0$ for $t \neq 0$ and $P(0) \neq 0$. Moreover, we suppose that $f(\cdot, \cdot)$ is continuous and $\frac{\partial f}{\partial t}(\cdot, \cdot)$ exists and it is continuous on the domain $[0, 1]$.

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The singular boundary value problem (1.1)-(1.2) arises in a number of applications such as gas dynamics, nuclear physics, chemical reactions, atomic calculations, tumor growth and physiology. For example, SBVP (1.1)-(1.2) with $P(t) = 2$, $R(t) = t$ and

$$f(t, y(t)) = \frac{ny(t)}{y(t) + k},\ n, k > 0,$$

occurs in the modeling of steady state oxygen in a spherical cell with Michaelis-Menten uptake kinetics [25, 26]. In the study of various tumor growth problems [1–4, 6], we deal with the SBVP (1.1)-(1.2) with $P(t) = 0, 1, 2$, $R(t) = t$ and

$$f(t, y(t)) = h(y(t)) + \frac{ny(t)}{y(t) + k},\ n, k > 0.$$

Another case of physical significance is when $P(t) = 2$, $R(t) = t$ and

$$f(t, y(t)) = -le^{-lky(t)},\ l, k > 0,$$

which arises in the study of the distribution of heat sources in human head [12, 15].

The singular boundary value problems have been the central attention to many research works either numerical or analytical [7–9, 15–18, 21–25, 31–33]. The main difficulty of the SBVP (1.1)-(1.2) is that the singularity behavior occurs at $t = 0$. Various efficient numerical techniques have been used to deal with such SBVPs. For instance, Kumar [22] proposed a three-point finite difference method based on uniform mesh for a class of SBVPs. Benko et al. [5] utilized a backward Euler method for the numerical approximation of the solutions of singular second-order differential equations. Kanth and Reddy [19] studied fourth-order finite difference method to solve singular boundary value problems. Moreover, Kanth and Reddy [20] applied cubic spline interpolation method to solve these problems. Caglar and Caglar [7] utilized a method based on cubic B-spline method. Goh et al. [14] treated SBVPs by using quartic B-spline approximation where the values of coefficients are chosen via optimization. Zhang [34] proposed a modified cubic B-spline solution for two point boundary value problems. Moreover, variational iteration method [32, 33], Adomian decomposition method [21], and modified Adomian decomposition method [24] are newly developed approximation methods that are applied to deal with such problems. In some of the mentioned methods the accuracy of the obtained solutions is poor while in the rest, implementation of the proposed approach is hard and time-consuming.

Now, in this paper, a method based on the Lagrange interpolation is proposed to solve the SBVP (1.1)-(1.2). Firstly, a continuous time optimization (CTO) problem related to the main SBVP is proposed. Then, applying the Chebyshev-Gauss-Lobatto (CGL) nodes, we convert the CTO problem to a discrete time optimization problem. The proposed approach is implemented on some numerical examples and the accuracy of the method is compared with some other well-known approaches. Obtained results, show the high accuracy of the method in comparison with the other methods.

The remainder of this paper is organized as follows: in Section 2, the proposed idea is introduced. The accuracy and efficiency of the proposed method are demonstrated by numerical results in Section 3. Finally, Section 4 is devoted to conclusions.

2. The proposed approach

Lagrange interpolation in CGL nodes is important in approximation theory, since the roots of the Chebyshev polynomials of the first kind, also called CGL nodes, are used as nodes in polynomial interpolation and the resulting interpolation polynomial provides an approximation that is close to the polynomial of the best approximation of a continuous function under the maximum norm.

Here, we interpolate the solution in the roots of the Chebyshev polynomials to give the best accuracy in the interpolation of solution. The derivatives of these interpolating polynomials at these points are given exactly by a differentiation matrix. The similar approach was utilized in works [10, 11, 13, 28–30].

In this section, at first, we apply a continuous time optimization (CTO) problem the optimal solution of which is a solution of SBVP (1.1)-(1.2). Thereafter, using the roots of the Chebyshev polynomials, we discretize the state variable of the CTO problem in CGL nodes and obtain a discrete-time optimization (DTO) problem. By solving the DTO problem, we obtain pointwise approximations for the solutions of SBVP (1.1)-(1.2). Moreover, by interpolating, we get continuous approximations for the solutions of the main problem.

According to the assumptions, we can convert the SBVP (1.1)-(1.2) to the following equivalent singular boundary problem

$$R(t)y''(t) + P(t)y'(t) = R(t)f(t, y(t)),\ 0 \leq t \leq 1, \tag{2.1}$$

subject to the boundary conditions

$$y'(0) = 0,\ \alpha y'(1) + \beta y(1) = \gamma. \tag{2.2}$$

In order to solve the SBVP (2.1)-(2.2), we propose the following continuous time optimization (CTO) problem

$$\begin{aligned} &Minimize \quad J = y'(0)^2 + (\alpha y'(1) + \beta y(1) - \gamma)^2 \\ &subject\ to\ \Big\{ R(t)y''(t) + P(t)y'(t) = R(t)f(t, y),\ 0 \leq t \leq 1. \end{aligned} \tag{2.3}$$

It is trivial that if SBVP (2.1)-(2.2) has a solution $y(.)$, then $y(.)$ is an optimal solution for (2.3), and vice versa. Therefore, by solving the CTO problem (2.3), we can find the solution of SBVP (2.1)-(2.2).

Assume that $y_1(t) = y(t)$ and $y_2(t) = y'(t)$. So, the CTO problem (2.3) can be written as follows

$$\begin{aligned} &Minimize \quad J = y_2(0)^2 + (\alpha y_2(1) + \beta y_1(1) - \gamma)^2 \\ &subject\ to\ \begin{cases} y_1'(t) = y_2(t), \\ R(t)y_2'(t) + P(t)y_2(t) = R(t)f(t, y_1(t)),\ t \in [0, 1]. \end{cases} \end{aligned} \tag{2.4}$$

To utilize the roots of Chebyshev polynomials (or CGL nodes), defined on the interval $[-1, 1]$, the transformation $t = \frac{1}{2}(\tau + 1)$ must be used. Moreover, we define

$$\begin{cases} Y_1(\tau) = y_1(\frac{\tau+1}{2}) = y_1(t), \\ Y_2(\tau) = y_2(\frac{\tau+1}{2}) = y_2(t), \\ Y_1'(\tau) = \frac{1}{2}y_1'(t), \\ Y_2'(\tau) = \frac{1}{2}y_2'(t), \\ 0 \leq t \leq 1, -1 \leq \tau \leq 1. \end{cases} \tag{2.5}$$

By this transformation, system (2.4) can be converted to the following equivalent problem

$$\begin{aligned} &Minimize \quad Y_2(-1)^2 + (\alpha Y_2(1) + \beta Y_1(1) - \gamma)^2 \\ &subject\ to \begin{cases} 2Y_1'(\tau) = Y_2(\tau), \\ 2R(\frac{\tau+1}{2})Y_2'(\tau) + P(\frac{\tau+1}{2})Y_2(\tau) = R(\frac{\tau+1}{2})f(\frac{\tau+1}{2}, Y_1(\tau)) \\ -1 \leq \tau \leq 1. \end{cases} \end{aligned} \tag{2.6}$$

To convert the CTO problem (2.6) into a discrete form, the CGL nodes on $[-1, 1]$ are selected as follows

$$\tau_k = cos(\frac{N-k}{N}\pi), \quad k = 0, 1, \dots, N, \tag{2.7}$$

where they are the roots of $(1-\tau^2)\frac{dT_N}{d\tau}$ and $T_j(\tau) = cos(jcos^{-1}(\tau)),\ \tau \in [-1, 1],\ j = 0, 1, \dots, N$, are the Chebyshev polynomials. For interpolating, the following Lagrange polynomials are utilized

$$L_k(\tau) = \frac{2}{N\mu_k}\sum_{j=0}^{N}\frac{1}{\mu_j}T_j(\tau_k)T_j(\tau),\ k = 0, 1, \dots, N,\ \tau \in [-1, 1], \tag{2.8}$$

where $\mu_0 = \mu_N = 2$ and $\mu_k = 1,\ for\ k = 1, 2, \dots, N-1$. Note that $L_k(\tau_k) = 1,\ k = 0, 1, \dots, N$ and $L_k(\tau_j) = 0$, for all $k \neq j$. Now, the Lagrange interpolation for the optimal solution of the CTO problem (2.6) can be defined as follows

$$Y_1(\tau) \simeq Y_1^N(\tau) = \sum_{l=0}^{N} a_l L_l(\tau), \tag{2.9}$$

and

$$Y_2(\tau) \simeq Y_2^N(\tau) = \sum_{l=0}^{N} b_l L_l(\tau), \tag{2.10}$$

where N is a sufficiently big number.
Note that

$$\begin{cases} Y_1(\tau_k) \simeq Y_1^N(\tau_k) = a_k, \\ Y_2(\tau_k) \simeq Y_2^N(\tau_k) = b_k. \end{cases} \tag{2.11}$$

Also,

$$Y_1'(\tau_k) \simeq \sum_{l=0}^{N} a_l D_{lk},\ Y_2'(\tau_k) \simeq \sum_{l=0}^{N} b_l D_{lk},\ k=0,1,\ldots,N, \tag{2.12}$$

where

$$D_{lk} = L_l'(\tau_k) = \begin{cases} \frac{\mu_k}{\mu_l}(-1)^{k+1}\frac{1}{\tau_k-\tau_l}, & if\ k \neq l, \\ -\frac{\tau_k}{2-2\tau_k^2}, & if\ 1 \leq k = l \leq N-1, \\ -\frac{(2N^2+1)}{6}, & if\ k = l = 0, \\ \frac{2N^2+1}{6}, & if\ k = l = N. \end{cases} \tag{2.13}$$

For details of the above relations, we refer to [10, 11]. Now, using relations (2.11) and (2.12), we approximate the CTO problem (2.6) by the following discrete-time optimization (DTO) problem

$$\begin{aligned} &Minimize \quad J_N = (b_0)^2 + (\alpha b_N + \beta a_N - \gamma)^2 \\ &subject\ to \begin{cases} 2\sum_{l=0}^{N} a_l D_{lk} = b_k, \\ 2R(\frac{\tau_k+1}{2})\sum_{l=0}^{N} b_l D_{lk} + P(\frac{\tau_k+1}{2})b_k = R(\frac{\tau_k+1}{2})f(\frac{\tau_k+1}{2}, a_k), \\ k=0,1,\ldots,N, \end{cases} \end{aligned} \tag{2.14}$$

where N is a sufficiently big number. We solve the DTO problem (2.14) by using nonlinear programming techniques and obtain discrete approximations for the solution of the SBVP (2.1)-(2.2) as $y(t_k) \simeq a_k^*$, and $y'(t_k) \simeq b_k^*, k = 0,1,\ldots,N$ where $t_k = \frac{1}{2}(\tau_k+1)$ and $\tau_k, k = 0,1,\ldots,N$ are the CGL nodes. Moreover, by Lagrange interpolation, we get continuous approximations as

$$\begin{cases} y^*(t) \simeq \sum_{l=0}^{N} a_l^* L_l(2t-1),\ 0 \leq t \leq 1, \\ y^{*\prime}(t) \simeq \sum_{l=0}^{N} b_l^* L_l(2t-1),\ 0 \leq t \leq 1. \end{cases} \tag{2.15}$$

3. Numerical results

To show the efficiency of our proposed method, we implement it on three different singular boundary value problems arising in real applications.

Example 3.1. Consider the following singular boundary value problem

$$\begin{aligned} &y''(t) + \frac{2}{t}y'(t) - 4y(t) = -2, \\ &y'(0) = 0,\ y(1) = 5.5. \end{aligned}$$

This problem has an analytical solution as follows

$$y(t) = 0.5 + 5\frac{\sinh 2t}{t \sinh 2}.$$

We solve the corresponding discrete time optimization problem (2.14) for this problem, by assumption $N = 10$ and by using FMINCON function in *Matlab* software. The approximate solutions a_k^* and b_k^* for $k = 0,1,2,\ldots,N$ are given in Table 1. In Table 2, the approximate and exact values of $y(t)$, for $t = 0, 0.1, 0.2, \ldots, 1$ are given. As

TABLE 1. The coefficients a_k^* and b_k^*, for $k = 0, 1, \ldots, 10$ in Example 3.1.

k	a_k^*	b_k^*
$k=0$	3.257205647713493	−0.000000000005190
$k=1$	3.258306577657286	0.089986385716817
$k=2$	3.273997529038257	0.352335065355567
$k=3$	3.335956270708754	0.770660636841323
$k=4$	3.481911583461575	1.331807681443218
$k=5$	3.740271368317843	2.028638545710036
$k=6$	4.114911798188221	2.844501234351804
$k=7$	4.570834213294062	3.723787979506456
$k=8$	5.027192123499485	4.548599892238480
$k=9$	5.371247426061933	5.150743926293068
$k=10$	5.499999999999999	5.373147207272525

TABLE 2. The values of $y(t)$ for $t = 0, 0.1, \ldots, 1$ in Example 3.1.

t	Approximate solution	Exact solution
0.0	3.257205647713493	3.257205647717832
0.1	3.275623816473727	3.275623816476181
0.2	3.331321581289175	3.331321581291895
0.3	3.425641420573009	3.425641420564865
0.4	3.560863537311638	3.560863537324633
0.5	3.740271368317845	3.740271368319426
0.6	3.968246145139263	3.968246145128546
0.7	4.250393467673129	4.250393467685512
0.8	4.593705860687058	4.593705860688228
0.9	5.006766424282846	5.006766424282001
1.0	5.499999999999999	5.500000000000000

it can be seen in this table, the obtained values are very near to the exact values. Also, the graphs of discrete and continuous approximate solutions for $y(\cdot)$ and $y'(\cdot)$ are presented in Figure 1. The absolute error of the presented method, i.e. the absolute difference between the approximate solution and the analytical solution, is compared with other numerical methods in Table 3. The numerical methods selected for comparison are the higher order finite difference method (HFDM) [19], the cubic B-spline method (CBSM) [7], the quartic B-spline method (QBSM) [14] and the modified cubic B-spline method (MCBSM) [34]. As one clearly observes, the absolute error of our method is less than those of the other ones and therefore by the proposed method we can obtain a better approximate solution for this SBVP. The absolute error of our proposed method for different values of N, is given in Figure 2 which shows that when we increase the number of nodes, i.e. N, the absolute error tends to zero.

Example 3.2. Consider the following singular second-order boundary value problem

$$y''(t) + \frac{1}{t}y'(t) = (\frac{8}{8-t^2})^2,$$
$$y'(0) = 0, y(1) = 0.$$

FIGURE 1. Discrete and continuous approximate solutions $y(.)$ and $y'(.)$ in Example 3.1.

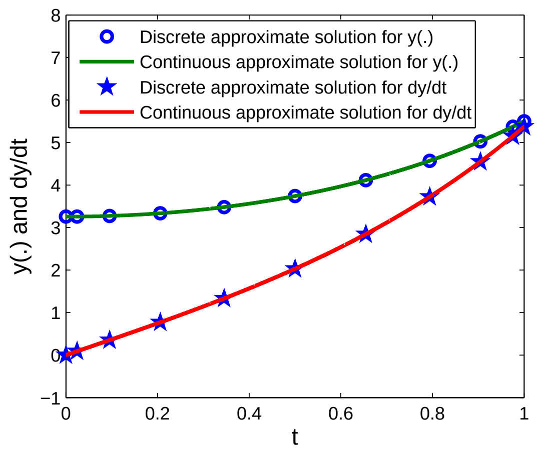
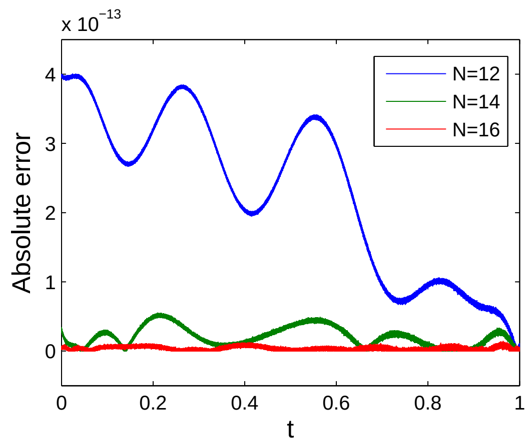


FIGURE 2. Absolute error of our method for Example 3.1.



The exact solution of this problem is $y(t) = 2\ln(\frac{7}{8-t^2})$. We solve the corresponding discrete time optimization problem (2.14), by assumption $N = 10$. The approximate solutions a_k^* and b_k^* for $k = 0, 1, 2, \ldots, N$ are given in Table 4. In Table 5, the values of $y(t)$, $t = 0, 0.1, 0.2, \ldots, 1$ obtained by the proposed method and the exact values of $y(t)$ for $t = 0, 0.1, \ldots, 1$ are displayed. Also, graphs of discrete and continuous approximate solutions for $y(\cdot)$ and $y'(\cdot)$ are shown in Figure 3. The absolute error of the proposed method is compared with those of four other numerical methods [8, 14, 19, 34] in Table 6. As one clearly observes, the absolute error of the proposed

TABLE 3. Comparison of absolute error for Example 3.1.

t	Our method for $N = 10$	HFDM [19] for $N = 20$	CBSM [7] for $N = 20$	QBSM [14] for $N = 20$	MCBSM [34] for $N = 20$
0.0	4.3×10^{-12}	6.16×10^{-7}	2.97×10^{-4}	5.26×10^{-8}	9.23×10^{-6}
0.1	2.4×10^{-12}	6.13×10^{-7}	2.95×10^{-4}	5.26×10^{-8}	3.84×10^{-7}
0.2	2.7×10^{-12}	6.03×10^{-7}	2.92×10^{-4}	5.25×10^{-8}	1.73×10^{-7}
0.3	8.1×10^{-12}	5.58×10^{-7}	2.85×10^{-4}	5.21×10^{-8}	1.05×10^{-7}
0.4	1.3×10^{-11}	5.54×10^{-7}	2.75×10^{-4}	5.12×10^{-8}	7.19×10^{-8}
0.5	1.5×10^{-12}	5.14×10^{-7}	2.58×10^{-4}	4.94×10^{-8}	5.24×10^{-8}
0.6	1.1×10^{-11}	4.59×10^{-7}	2.36×10^{-4}	4.62×10^{-8}	3.89×10^{-8}
0.7	1.3×10^{-11}	3.85×10^{-7}	2.02×10^{-4}	4.08×10^{-8}	2.83×10^{-8}
0.8	1.1×10^{-12}	2.89×10^{-7}	1.54×10^{-4}	3.22×10^{-8}	1.88×10^{-8}
0.9	0.8×10^{-12}	1.63×10^{-7}	8.96×10^{-4}	1.92×10^{-8}	9.62×10^{-9}

TABLE 4. The coefficients a_k^* and b_k^*, for $k = 0, 1, \ldots, 10$ in Example 3.2.

k	a_k^*	b_k^*
$k = 0$	-0.267062785263904	-0.000000000011761
$k = 1$	-0.264561221454872	0.050062578414913
$k = 2$	-0.257037701610039	0.100502512383910
$k = 3$	-0.244435265441268	0.151706700500273
$k = 4$	-0.226657370635746	0.204081632707703
$k = 5$	-0.203565388624767	0.258064515999351
$k = 6$	-0.174974908232746	0.314136125706599
$k = 7$	-0.140650633462087	0.372836218429264
$k = 8$	-0.100299567373873	0.434782608635710
$k = 9$	-0.053562045354685	0.500695410324417
$k = 10$	-0.000000000000000	0.571428571428639

TABLE 5. The values of $y(t)$ for $t = 0, 0.1, \ldots, 1$ in Example 3.2.

t	Approximate solution	Exact solution
0.0	-0.267062785263904	-0.267062785249045
0.1	-0.266913063117316	-0.264561221445740
0.2	-0.264781828282564	-0.257037701601957
0.3	-0.256414426219371	-0.244435265448499
0.4	-0.236996828378616	-0.226657370614006
0.5	-0.203565388624767	-0.203565388619885
0.6	-0.156993433644465	-0.174974908246232
0.7	-0.102943063651232	-0.140650633441460
0.8	-0.051297726089961	-0.100299567370943
0.9	-0.013765268361832	-0.053562045355284
1.0	-0.000000000000000	0.000000000000000

method is less than those of the other ones. In Figure 4, the absolute error of our method for different values of N is given.

FIGURE 3. Discrete and continuous approximate solutions $y(.)$ and $y'(.)$ in Example 3.2.

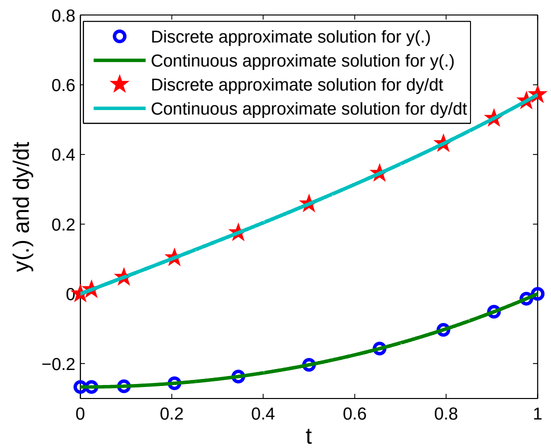
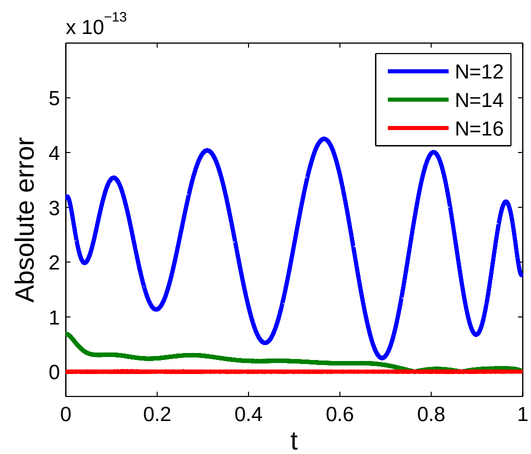


FIGURE 4. Absolute error of our method for Example 3.2.



Example 3.3. Consider the following singular boundary value problem

$$\begin{aligned} &y''(t) + \frac{1}{t}y'(t) = -e^{y(t)}, \\ &y'(0) = 0, y(1) = 0. \end{aligned}$$

The exact solution of this problem is $y(t) = 2\ln(\frac{4-2\sqrt{2}}{(3-2\sqrt{2})x^2+1})$. The discrete approximate solutions a_k* and b_k^* for $N = 10$ and $k = 0, 1, 2, \ldots, N$ are given in Table 7. Moreover, the obtained values of $y(t)$, for $t = 0, 0.1, \ldots, 10$ in comparison with the

Table 6. Comparison of absolute error for Example 3.2.

t	Our method for $N = 10$	HFDM [19] for $N = 20$	CBSM [8] for $N = 20$	QBSM [14] for $N = 20$	MCBSM [34] for $N = 20$
0.0	1.5×10^{-11}	9.22×10^{-5}	2.72×10^{-5}	9.40×10^{-9}	2.57×10^{-8}
0.1	9.1×10^{-12}	9.04×10^{-5}	2.69×10^{-5}	9.29×10^{-9}	1.52×10^{-8}
0.2	8.1×10^{-12}	8.61×10^{-5}	2.63×10^{-5}	9.15×10^{-9}	1.08×10^{-8}
0.3	7.2×10^{-12}	8.15×10^{-5}	2.53×10^{-5}	8.92×10^{-9}	8.13×10^{-9}
0.4	2.7×10^{-11}	7.68×10^{-5}	2.38×10^{-5}	8.56×10^{-9}	6.23×10^{-9}
0.5	4.9×10^{-12}	7.17×10^{-5}	2.18×10^{-5}	8.05×10^{-9}	4.74×10^{-9}
0.6	1.3×10^{-11}	6.61×10^{-5}	1.92×10^{-5}	7.31×10^{-9}	3.49×10^{-9}
0.7	2.1×10^{-11}	5.92×10^{-5}	1.59×10^{-5}	6.28×10^{-9}	2.43×10^{-9}
0.8	1.1×10^{-12}	4.97×10^{-5}	1.17×10^{-5}	4.85×10^{-9}	1.50×10^{-9}
0.9	6.0×10^{-13}	3.43×10^{-5}	6.51×10^{-6}	2.83×10^{-9}	6.92×10^{-10}

Table 7. The coefficients a_k^* and b_k^*, for $k = 0, 1, \dots, 10$ in Example 3.3.

k	a_k^*	b_k^*
$k = 0$	0.316694367638146	−0.000000000005154
$k = 1$	0.316488879825641	−0.016793022896101
$k = 2$	0.313567794632539	−0.065432636956595
$k = 3$	0.302170315439319	−0.140426247984659
$k = 4$	0.276148766251959	−0.232349437772314
$k = 5$	0.232696783871571	−0.329032487934742
$k = 6$	0.174848254821172	−0.418429617917738
$k = 7$	0.111334880657558	−0.491673886340597
$k = 8$	0.053989034172848	−0.544346623506062
$k = 9$	0.014210173554678	−0.575525891968148
$k = 10$	−0.000000000000000	−0.585786437625405

exact values are presented in Table 8. In Figure 5, the discrete and continuous approximate solutions for $y(\cdot)$ and $y'(\cdot)$ are shown. The absolute error of the proposed method is compared with those of two other numerical methods [21, 27] in Table 9. We can see that, the absolute error of the proposed method is less than those of the other ones and therefore by our method we can obtain a better approximate solution for this SBVP. The absolute error of our proposed method for different values of N is given in Figure 6.

4. Conclusions

We presented a new approach for solving nonlinear singular boundary value problems. In the proposed approach, we utilized the roots of Chebyshev to convert the given SBVP to a discrete time optimization problem. By solving the obtained discrete programming problem, we found approximate solutions for the original SBVP. The numerical results showed that the presented approach is an applicable technique and yields numerical solutions in very good agreement with the existing exact solutions. Moreover, our approach has much better results than those of the other methods for solving SBVPs.

TABLE 8. The values of $y(t)$ for $t = 0, 0.1, \ldots, 1$ in Example 3.3.

t	Approximate solution	Exact solution
0.0	0.316694367638146	0.316694367640750
0.1	0.313265850492726	0.313265850498063
0.2	0.303015422830425	0.303015422832300
0.3	0.286047265315019	0.286047265304854
0.4	0.262531127440249	0.262531127456033
0.5	0.232696783871571	0.232696783873834
0.6	0.196826805704263	0.196826805692954
0.7	0.155248106671408	0.155248106682756
0.8	0.108322763442638	0.108322763444465
0.9	0.056438602470276	0.056438602469236
1.0	0.000000000000000	0.000000000000000

FIGURE 5. Discrete and continuous approximate solutions $y(.)$ and $y'(.)$ in Example 3.3.

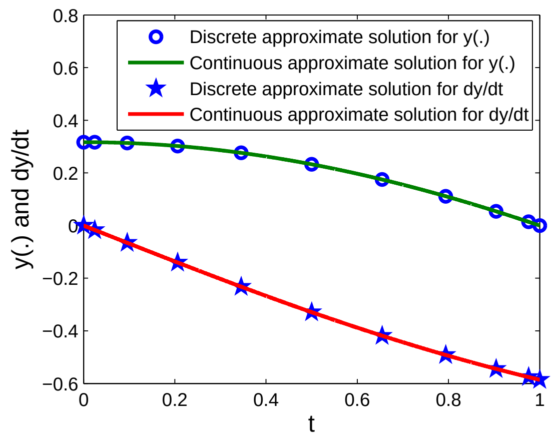
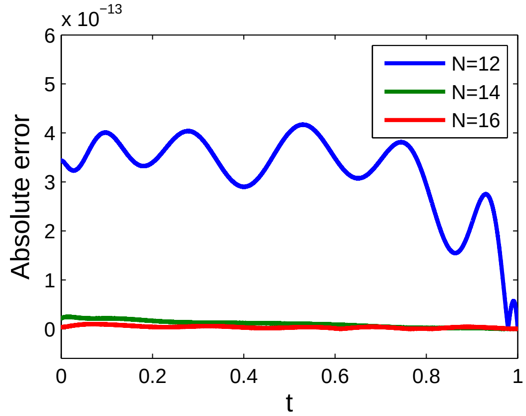


TABLE 9. Comparison of absolute error of Example 3.3.

t	Our method for $N = 10$	Approach in [21] for $N = 20$	Approach in [27] for $N = 10$	Approach in [27] for $N = 14$
0.0	2.6×10^{-12}	2.00×10^{-6}	3.77×10^{-8}	6.72×10^{-8}
0.1	5.3×10^{-12}	1.99×10^{-6}	1.05×10^{-7}	6.69×10^{-8}
0.2	1.8×10^{-12}	1.97×10^{-6}	6.33×10^{-9}	7.87×10^{-9}
0.3	1.0×10^{-12}	1.94×10^{-6}	5.91×10^{-8}	6.92×10^{-9}
0.4	1.5×10^{-11}	1.83×10^{-6}	2.12×10^{-7}	2.87×10^{-8}
0.5	2.2×10^{-12}	1.78×10^{-6}	1.00×10^{-8}	7.40×10^{-10}
0.6	1.1×10^{-11}	1.67×10^{-6}	5.36×10^{-7}	6.32×10^{-8}
0.7	1.1×10^{-11}	1.34×10^{-6}	4.25×10^{-8}	6.95×10^{-8}
0.8	1.8×10^{-12}	9.20×10^{-7}	8.32×10^{-7}	3.38×10^{-9}
0.9	1.04×10^{-12}	4.57×10^{-7}	4.67×10^{-8}	7.85×10^{-8}

FIGURE 6. Absolute error of our method for Example 3.3.



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