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European and American put valuation via a high-order semi-discretization scheme

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Abstract Put options are commonly used in the stock market to protect against the decline of the price of a stock below a specified price. On the other hand, finite difference approach is a well-known and well-resulted numerical scheme for financial differential equations. As such in this work, a new spatial discretization based on finite difference semi-discretization procedure with high order of accuracy is constructed for the problem of European and American put options. Several numerical experiments are also worked out.

Keywords. Option pricing, Numerical scheme, Black-Scholes PDE, Semi-discretization, Put option.

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1. Introductory notes

Partial differential equations (PDEs) of parabolic type are contributory for many applications in financial mathematics. In practice, the solution to most parabolic PDEs can be calculated numerically by using some computational procedures. In most cases of financial applications, this procedure happens to be some variation of the well-resulted finite difference (FD) method [1].

Financial securities have become essential tools for corporations and investors. As an instance, options can be used to hedge assets and portfolios in order to run the risk due to the changes in stock prices see e.g. [20, 21].

It is known that in finance [15], a put or put option is a stock market device which gives the owner a right, but not the obligation, to sell an asset, at a specified price (the strike), by a predetermined date which is known as the expiry or maturity to a given party. Furthermore, the problem of pricing a European or American option can be cast as a PDE. Using the FD approach [4], this problem can be recast as a set of ordinary differential equations (ODEs) or an initial value problem which should be priced as fast as possible via computational methods.

The famous Black-Scholes model is a convenient way to calculate the price of an option. There is an enriched literature regarding the numerical solution of Black-Scholes PDE in finance using different strategies [13]. The FD approach is easy to handle and normally produce results of reasonable accuracy and that is why there

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is a special focus on this type of schemes. The technique of FD for solving financial PDEs is a popular approach in option pricing (see e.g. [16] and the references cited therein).

In 1973, Fisher Black and Myron Scholes explored the hypothesis of geometric Brownian motion [2]

$$\frac{dS}{S} = \nu dt + \sigma dW(t), \tag{1.1}$$

as the stock model where ν is the drift term of the stock and σ is the volatility and $W(t)$ is the standard Wiener process.

It is remarked that (1.1) follows from the no-arbitrage assumption, a keystone of the Black-Scholes theory, which says there are no risk-free gains available in an efficient market.

Consider a general option value $V(S,t)$. The Taylor-expansion reads [8]:

$$dV = \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial t} dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} dS^2 + \cdots . \tag{1.2}$$

From equation (1.1), it follows that

$$dS^2 = (\nu S dt + \sigma S dW(t))^2 = \nu^2 S^2 dt^2 + 2\nu\sigma S^2 dW(t) dt + \sigma^2 S^2 dW(t)^2. \tag{1.3}$$

Then, by applying Itô's Lemma [15] and the assumption that $dW(t)^2 \equiv dt$ with probability 1, equation (1.3) reads to the leading order $dS^2 \equiv \sigma^2 S^2 dt$. Inserting this into (1.2) yields in

$$dV = \sigma S \frac{\partial V}{\partial S} dW(t) + \left(\nu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt. \tag{1.4}$$

Further calculations and eliminating the main existence of randomness will finally simplify to the famous Black-Scholes PDE as comes next

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{1.5}$$

where r is the risk-free interest rate.

A European option can be exercised only at the expiry date whereas an American option has the additional feature that exercise is permitted at any time during the life of the option. American derivatives are getting popular trading instruments in present-day financial markets. On the other hand, pricing an American option is more complicated since at each time step we have to determine not only the option value but also whether or not it should be exercised.

One of the methods to solve (1.5) computationally is via Monte-Carlo simulations. The disadvantage with a Monte-Carlo method is that it converges slowly (the statistical error for a standard Monte Carlo method is proportional to $N^{1/2}$, where N is the number of simulations. Hence, we need to do a huge amount of simulations to reduce the statistical error.

This work proposes an accurate scheme for the European and American put option pricing problem. Our technique is based on a sixth-order semi-discretization and thus has the advantage of computing sufficiently accurate option prices using relatively coarse mesh sizes. We provide numerical shreds of evidence that for approximately the same step sizes, our scheme is more accurate than the existing ones. Although the sixth order FD approximations for the spatial grid nodes sounds complicated in practice, its implementation would be straightforward by incorporating the use of matrices.

To illustrate further, the notion of differentiation matrices for FD discretization along with the pseudo-spectral notion are applied to simplify the procedure. The resulting technique is fast and exhibits a high rate of convergence and is successful in computing accurate option prices.

The rest of this work is unfolded as follows. Section 2 starts by explaining the pseudo-spectral notion for the FD approximations of the sixth-order using the differentiation matrices. Next, we discuss its use for financial equations in Section 3. The whole procedure of solving the European/American put option is reduced to solving a system of ODEs with a time discretization. The proposed method is based upon the so-called semi-discretization method (SD) which is also known as the method of lines [12]. Notice that this work could be considered as an extension over the paper [11]. Section 4 deals with the American case by applying the penalty approach. The numerical results of the presented approach are brought forward in Section 5. At last, some concluding remarks and research lines for future models in finance are made known.

2. A semi-discretization for European option

The FD method has been studied and used to solve financial equations [6]. It is known that the explicit schemes for the untransformed Black-Scholes equation have convergence and stability conditions, whilst the fully implicit and Crank-Nicolson schemes are unconditionally convergent and stable, spurious oscillations may yet occur in the solution depending on the way of implementation and the spatial discretization order [6]. Technically speaking, the accuracy of the explicit and fully implicit Euler schemes is of first order, and the Crank-Nicolson's accuracy is of second order.

In this section and to tackle the model formally, consider the following initial-boundary version of the Black-Scholes PDE (1.5):

$$\frac{\partial V}{\partial t} = \left(r - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} - rS\frac{\partial}{\partial S}\right) V, \tag{2.1}$$

$$V(0,t) = E\exp\{-r(T-t)\}, \tag{2.2}$$

$$\lim_{S\to\infty} V(S,t) = 0, \tag{2.3}$$

$$V(S,T) = \max\{E - S, 0\}, \tag{2.4}$$

where T is the maturity time and E is the exercise (strike) price.

It is stated that (2.1) is a linear PDE with non-constant coefficients and non-homogeneous boundary conditions and non-differentiable or even discontinuous final conditions, while the asset price follows the log-normal distribution and arises from (1.1).

In the semi-discretization technique for solving (2.1), the spatial variables are those to which the discretization is done. Subsequently, the temporal variable is the one left in the ODE system to be integrated. Actually, it is known that the numerical semidiscretization is a technique for solving PDEs by discretizing in all but one dimension and then integrating the semi-discrete problem as a system of ODEs.

In general, an FD approximation to the value of a derivative of a function at a point x_i in its domain relies on a suitable combination of sampled function values at nearby points. Let us take into account a full grid of n points, $x = \{x_1, \ldots, x_n\}$ for approximating the spatial variable, i.e., S when valuing (2.1). Usually, this set of nodes is considered to be equally spaced for the FD technique. The underlying formalism used to construct these approximation formulas is known as the calculus of FD [17].

The difference formulas for the FD approximations can simply be derived from Taylor's formula when the under consideration function is quite differentiable and smooth, e.g., $f(x_{i+1}) = f(x_i) + hf'(x_i) + \frac{1}{2}h^2 f''(\xi_i),\ x_i < \xi_i < x_{i+1}$, which provides an error estimate as well. Our main intention is to attain as high as possible of accuracy in a short approach of FD. As such, one may suggest in using a very refined step size while the second-order FD discretization is taken into account. The problem of this strategy, as is well-known, is that it will mostly ruin the whole numerical procedure due to the appearance of round-off errors.

The complication is that round-off errors due to the finite precision of numbers stored in the computer have now begun to affect the computation [10]. This highlights the inherent difficulty with numerical differentiation: FD formulas inevitably require dividing very small quantities, and so round-off inaccuracies may produce noticeable numerical errors. Thus, while they typically produce reasonably good approximations to the derivatives for moderately small step sizes, achieving high accuracy requires switching to higher-precision computer arithmetic.

To remedy this and provide a new accurate numerical scheme for put option, we rely on the use of much more accurate approximation of the spatial discretizations. So, more sample points are employed. Generally speaking, formulas for any given derivative with an asymptotic error of any chosen order can be derived from the Taylor formulas as long as a sufficient number of sample points are used.

Another way is via polynomial interpolation because the Taylor formulas are exact for polynomials of sufficiently low order. It is not complicated to show that the FD approximations are equivalent to the derivatives of interpolating polynomials [10].

We here take into account the sixth order FD approximations as follows:

$$V'(x_i) = \frac{-V(x_{i-3}) + 9V(x_{i-2}) - 45V(x_{i-1}) + 45V(x_{i+1}) - 9V(x_{i+2}) + V(x_{i+3})}{60h} + \mathcal{O}(h^6), \tag{2.5}$$

$$V'(x_i) = \frac{10V(x_{i-6}) - 72V(x_{i-5}) + 225V(x_{i-4}) - 400V(x_{i-3}) + 450V(x_{i-2}) - 360V(x_{i-1}) + 147V(x_i)}{60h} + \mathcal{O}(h^6), \tag{2.6}$$

$$V'(x_i) = \frac{-2V(x_{i-5}) + 15V(x_{i-4}) - 50V(x_{i-3}) + 100V(x_{i-2}) - 150V(x_{i-1}) + 10V(x_{i+1}) + 77V(x_i)}{60h} + \mathcal{O}(h^6), \tag{2.7}$$

$$V'(x_i) = \frac{V(x_{i-4}) - 8V(x_{i-3}) + 30V(x_{i-2}) - 80V(x_{i-1}) + 24V(x_{i+1}) - 2V(x_{i+2}) + 35V(x_i)}{60h} + \mathcal{O}(h^6), \tag{2.8}$$

$$V'(x_i) = \frac{2V(x_{i-2}) - 24V(x_{i-1}) + 80V(x_{i+1}) - 30V(x_{i+2}) + 8V(x_{i+3}) - V(x_{i+4}) - 35V(x_i)}{60h} + \mathcal{O}(h^6), \tag{2.9}$$

$$V'(x_i) = \frac{-10V(x_{i-1}) + 150V(x_{i+1}) - 100V(x_{i+2}) + 50V(x_{i+3}) - 15V(x_{i+4}) + 2V(x_{i+5}) - 77V(x_i)}{60h} + \mathcal{O}(h^6), \tag{2.10}$$

$$V'(x_i) = \frac{360V(x_{i+1}) - 450V(x_{i+2}) + 400V(x_{i+3}) - 225V(x_{i+4}) + 72V(x_{i+5}) - 10V(x_{i+6}) - 147V(x_i)}{60h} + \mathcal{O}(h^6). \tag{2.11}$$

In the discretization process, the difference formula (2.5) is used for the center nodes and at the boundary nodes, the appropriate difference formula is taken into consideration to have a unified sixth order approximation for the first derivative.

The PDE (2.1) needs the approximation of the second derivative as well. This could be deduced in what follows:

$$V''(x_i) = \frac{2V(x_{i-3}) - 27V(x_{i-2}) + 270V(x_{i-1}) + 270V(x_{i+1}) - 27V(x_{i+2}) + 2V(x_{i+3}) - 490V(x_i)}{180h^2} + \mathcal{O}(h^6), \tag{2.12}$$

$$V''(x_i) = \frac{-126V(x_{i-7}) + 1019V(x_{i-6}) - 3618V(x_{i-5}) + 7380V(x_{i-4}) - 9490V(x_{i-3}) + 7911V(x_{i-2}) - 4014V(x_{i-1}) + 938V(x_i)}{180h^2} + \mathcal{O}(h^6), \tag{2.13}$$

$$V''(x_i) = \frac{11V(x_{i-6}) - 90V(x_{i-5}) + 324V(x_{i-4}) - 670V(x_{i-3}) + 855V(x_{i-2}) - 486V(x_{i-1}) + 126V(x_{i+1}) - 70V(x_i)}{180h^2} + \mathcal{O}(h^6), \tag{2.14}$$

$$V''(x_i) = \frac{-2V(x_{i-5}) + 16V(x_{i-4}) - 54V(x_{i-3}) + 85V(x_{i-2}) + 130V(x_{i-1}) + 214V(x_{i+1}) - 11V(x_{i+2}) - 378V(x_i)}{180h^2} + \mathcal{O}(h^6), \tag{2.15}$$

$$V''(x_i) = \frac{-11V(x_{i-2}) + 214V(x_{i-1}) + 130V(x_{i+1}) + 85V(x_{i+2}) - 54V(x_{i+3}) + 16V(x_{i+4}) - 2V(x_{i+5}) - 378V(x_i)}{180h^2} + \mathcal{O}(h^6), \tag{2.16}$$

$$V''(x_i) = \frac{126V(x_{i-1}) - 486V(x_{i+1}) + 855V(x_{i+2}) - 670V(x_{i+3}) + 324V(x_{i+4}) - 90V(x_{i+5}) + 11V(x_{i+6}) - 70V(x_i)}{180h^2} + \mathcal{O}(h^6), \tag{2.17}$$

$$V''(x_i) = \frac{-4014V(x_{i+1}) + 7911V(x_{i+2}) - 9490V(x_{i+3}) + 7380V(x_{i+4}) - 3618V(x_{i+5}) + 1019V(x_{i+6}) - 126V(x_{i+7}) + 938V(x_i)}{180h^2} + \mathcal{O}(h^6). \tag{2.18}$$

Although such discretizations are useful in producing accurate approximations, another problem would happen which is in the implementation of the proposed FD variant for European and American put option pricing.

In fact, these long difference formulas would make the whole process of the semi-discretization tedious in case of a raw implementation. Accordingly, we are interested in pursuing such approximations in a matrix format to make the whole process as easier and unified as possible. Due to this, the notion of differentiation matrices is now applied for the spatial discretization with sixth order accuracy.

For both computational and theoretical purposes, it is often convenient to collect all the weights for the approximations at the grid points in a differentiation matrix [7].

The differentiation matrix maps a vector of the function values $Y = [V(x_1), \ldots, V(x_n)]^*$, at the collocation points to a vector Y' defined as follows [19]:

$$Y' = \sum_{k=1}^{n} D_{j,k} V(x_k) = DY. \tag{2.19}$$

Mainly, the derivative of order p for the function $V(x)$ can be expressed by

$$Y^{(p)}(x) = D^p Y. \tag{2.20}$$

The elements in the differentiation matrices are closely linked to the terms in Lagrange's interpolation formula. In our cases, the differentiation matrices are indefinite matrices.

From now on, the FD approximation for the spatial discretizations is considered based on the concept of differentiation matrix (2.19). In fact, we may use the Lagrange's interpolation of the sixth degree passing through the seven nearby equally-spaced nodes to construct the differentiation matrix.

Note that the differentiation matrices constructed using the highest possible degree are dense and may not be useful. But using a pseudo-spectral approach [19], i.e., differentiation matrices of the sixth order, we will not face with the Runge phenomenon and are sparse by increasing the number of nodes.

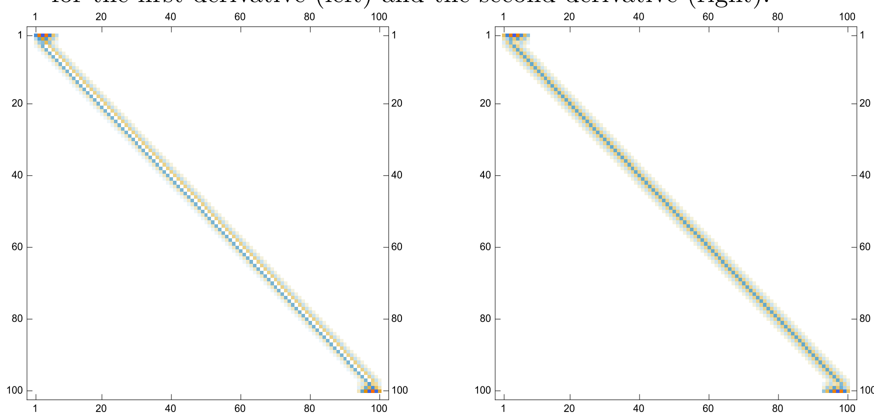
As an illustration for the case of $n = 11$ with the step size h, the differentiation matrix for the first and second derivatives are (respectively) given by

$$\mathcal{D} \simeq \begin{pmatrix}
-\frac{49}{20h} & \frac{6}{h} & -\frac{15}{2h} & \frac{20}{3h} & -\frac{15}{4h} & \frac{6}{5h} & -\frac{1}{6h} & 0 & 0 & 0 & 0 \\
-\frac{1}{6h} & -\frac{77}{60h} & \frac{5}{2h} & -\frac{5}{3h} & \frac{5}{6h} & -\frac{1}{4h} & \frac{1}{30h} & 0 & 0 & 0 & 0 \\
\frac{1}{30h} & -\frac{2}{5h} & -\frac{7}{12h} & \frac{4}{3h} & -\frac{1}{2h} & \frac{2}{15h} & -\frac{1}{60h} & 0 & 0 & 0 & 0 \\
-\frac{1}{60h} & \frac{3}{20h} & -\frac{3}{4h} & 0 & \frac{3}{4h} & -\frac{3}{20h} & \frac{1}{60h} & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{60h} & \frac{3}{20h} & -\frac{3}{4h} & 0 & \frac{3}{4h} & -\frac{3}{20h} & \frac{1}{60h} & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{60h} & \frac{3}{20h} & -\frac{3}{4h} & 0 & \frac{3}{4h} & -\frac{3}{20h} & \frac{1}{60h} & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{60h} & \frac{3}{20h} & -\frac{3}{4h} & 0 & \frac{3}{4h} & -\frac{3}{20h} & \frac{1}{60h} & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{60h} & \frac{3}{20h} & -\frac{3}{4h} & 0 & \frac{3}{4h} & -\frac{3}{20h} & \frac{1}{60h} \\
0 & 0 & 0 & 0 & \frac{1}{60h} & -\frac{2}{15h} & \frac{1}{2h} & -\frac{4}{3h} & \frac{7}{12h} & \frac{2}{5h} & -\frac{1}{30h} \\
0 & 0 & 0 & 0 & -\frac{1}{30h} & \frac{1}{4h} & -\frac{5}{6h} & \frac{5}{3h} & -\frac{5}{2h} & \frac{77}{60h} & \frac{1}{6h} \\
0 & 0 & 0 & 0 & \frac{1}{6h} & -\frac{6}{5h} & \frac{15}{4h} & -\frac{20}{3h} & \frac{15}{2h} & -\frac{6}{h} & \frac{49}{20h}
\end{pmatrix}, \tag{2.21}$$

$$\mathfrak{D} \simeq \begin{pmatrix}
\frac{469}{90h^2} & -\frac{223}{10h^2} & \frac{879}{20h^2} & -\frac{949}{18h^2} & \frac{41}{h^2} & -\frac{201}{10h^2} & \frac{1019}{180h^2} & -\frac{7}{10h^2} & 0 & 0 & 0 \\
\frac{7}{10h^2} & -\frac{7}{18h^2} & -\frac{27}{10h^2} & \frac{19}{4h^2} & -\frac{67}{18h^2} & \frac{9}{5h^2} & -\frac{1}{2h^2} & \frac{11}{180h^2} & 0 & 0 & 0 \\
-\frac{11}{180h^2} & \frac{107}{90h^2} & -\frac{21}{10h^2} & \frac{13}{18h^2} & \frac{17}{36h^2} & -\frac{3}{10h^2} & \frac{4}{45h^2} & -\frac{1}{90h^2} & 0 & 0 & 0 \\
\frac{1}{90h^2} & -\frac{3}{20h^2} & \frac{3}{2h^2} & -\frac{49}{18h^2} & \frac{3}{2h^2} & -\frac{3}{20h^2} & \frac{1}{90h^2} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{90h^2} & -\frac{3}{20h^2} & \frac{3}{2h^2} & -\frac{49}{18h^2} & \frac{3}{2h^2} & -\frac{3}{20h^2} & \frac{1}{90h^2} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{90h^2} & -\frac{3}{20h^2} & \frac{3}{2h^2} & -\frac{49}{18h^2} & \frac{3}{2h^2} & -\frac{3}{20h^2} & \frac{1}{90h^2} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{90h^2} & -\frac{3}{20h^2} & \frac{3}{2h^2} & -\frac{49}{18h^2} & \frac{3}{2h^2} & -\frac{3}{20h^2} & \frac{1}{90h^2} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{90h^2} & -\frac{3}{20h^2} & \frac{3}{2h^2} & -\frac{49}{18h^2} & \frac{3}{2h^2} & -\frac{3}{20h^2} & \frac{1}{90h^2} \\
0 & 0 & 0 & -\frac{1}{90h^2} & \frac{4}{45h^2} & -\frac{3}{10h^2} & \frac{17}{36h^2} & \frac{13}{18h^2} & -\frac{21}{10h^2} & \frac{107}{90h^2} & -\frac{11}{180h^2} \\
0 & 0 & 0 & \frac{11}{180h^2} & -\frac{1}{2h^2} & \frac{9}{5h^2} & -\frac{67}{18h^2} & \frac{19}{4h^2} & -\frac{27}{10h^2} & -\frac{7}{18h^2} & \frac{7}{10h^2} \\
0 & 0 & 0 & -\frac{7}{10h^2} & \frac{1019}{180h^2} & -\frac{201}{10h^2} & \frac{41}{h^2} & -\frac{949}{18h^2} & \frac{879}{20h^2} & -\frac{223}{10h^2} & \frac{469}{90h^2}
\end{pmatrix}. \tag{2.22}$$

To clearly show the sparse pattern of the differentiation matrices, their matrix plots for the case $n = 100$ are provided in Figure 1.

Figure 1. The matrix plot of the sixth-order differentiation matrices for the first derivative (left) and the second derivative (right).



3. Method of solution

In order to consider all the boundary conditions, the technique of solving boundary value problems using differentiation matrices is based upon removing the first and last rows of the differentiation matrix [18]. As a matter of fact, such removed rows are then incorporated by the boundary conditions to fulfill the conditions of the under consideration PDE.

Following this, let us take into consideration that $\overline{\mathcal{D}}$ and $\overline{\mathfrak{D}}$ to stand for the differentiation matrices corresponding to the first and second derivatives (respectively) on a uniform spatial discretization (as discussed in Section 2), when the first and last rows had been removed. It should also be noticed that the differentiation matrices can be derived and saved one time for the grid points. In fact, differentiation matrices are not dependent on the problem itself but dependent on the grid nodes.

Remark 3.1. The advantage of implementing based on differentiation matrices allows us to re-run our implementations as easily as possible for pricing options when the parameters such as r, σ etc. are varying.

Thus, if $0 = x_1 < x_2 < \cdots < x_n = L$ be n equidistant collocation points in the space direction, and $\mathbf{y} \simeq V(x,t)$, $X = \text{DiagonalMatrix}(x)$, where $x = \{x_1, x_2, \cdots, x_n\}$ then the Black-Scholes PDE (2.1) can be written in the following semi-discretized notation

$$\dot{\mathbf{y}}_i = I_{i,:}\underbrace{\left(rI - rX\overline{\mathcal{D}} - \frac{1}{2}\sigma^2 X^2 \overline{\mathfrak{D}}\right)}_{\mathbf{M}}\mathbf{y}, \tag{3.1}$$

where $i = 2, \ldots, n-1$. Introducing $\mathbf{M} = (rI - rX\overline{\mathcal{D}} - \frac{1}{2}\sigma^2 X^2 \overline{\mathfrak{D}})$.

Substituting the following boundary conditions

$$\mathbf{y} = I_{:,i}\mathbf{y}_i + E\exp\{-r(T-t)\} \times I_{:,1}, \tag{3.2}$$

into (3.1) supplies the following linear constant-coefficient system of ODEs

$$\dot{\mathbf{y}}_i = \mathbf{A}\mathbf{y}_i + \mathbf{b}, \tag{3.3}$$

wherein

$$\mathbf{A} = \mathbf{M}_{i,i}, \qquad \mathbf{b} = E\exp\{-r(T-t)\} \times \mathbf{M}_{i,1}. \tag{3.4}$$

It is requisite to bear in mind that a new FD strategy using differentiation matrices is necessary to be constructed, because in case of a successful numerical behavior, then the numerical scheme can be extended for many other types of options. In particular, American options which are not solvable in an analytic sense. If the numerical method works for European style option, then this is the basis to get the solution for an American option as will be discussed in the next section.

The above-mentioned technique would result in a system of linear ODEs, as a usual consequence of the method of lines. Although this system could be solved by many different solvers, herein the second-order time-stepping method of Crank-Nicolson is considered in what follows [11]:

$$\left(I - \frac{1}{2}k\mathbf{A}\right)\mathbf{y}_i(t_j + k) = \mathbf{y}_i(t_j) + \frac{1}{2}k\left(\mathbf{A}\mathbf{y}_i(t_j) + \mathbf{b}(t_j) + \mathbf{b}(t_kj + k)\right), \tag{3.5}$$

wherein k is the negative time increment. Therefore, the total truncation error is $\mathcal{O}(h^6) + \mathcal{O}(k^2)$.

4. American put option

The model of American options is a linear complementary problem (LCP) [24] on an unbounded domain, viz, the model is defined as a PDE with free boundaries, which has no closed-form solution. Thus the major research on this problem is focused on presenting efficient and accurate numerical algorithms for American options.

In this section, the Black-Scholes model is taken into account for the American put options with a nonlinear penalty source term [24, 9]. In the penalty approach, the free boundary is removed by adding a small continuous penalty term to the Black-Scholes equation. The problem can then be solved on a fixed domain. Moreover, the technique can be used for any type of discretization, in any dimension, and on structured as well as unstructured meshes. It is also possible to use the penalty approach to handle American options together with other nonlinearities.

In this case, the American option pricing problem can be modeled by the following nonlinear PDE on a fixed domain as follows:

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV + F(V), \tag{4.1}$$

wherein $\tau = T - t$, $F(V)$ stands for the penalty term. Let $0 < \epsilon << 1$ be a small regularization parameter and $C > 0$ be a fixed constant [24], then we consider the following penalty function

$$F(V) = \frac{\epsilon C}{V(S,\tau) + \epsilon - q(S)}, \tag{4.2}$$

where $q(S) = E - S$. Here the initial and boundary conditions would be the same as in (2.1).

The construction and extension of the proposed semi-discretization scheme for the PDE (4.1) is straightforward, i.e., after applying the semi-discretization technique (as in (2.19)-(2.20)) by the sixth-order spatial discretization formulas we would obtain a system of semi-discretized ODEs which is *nonlinear* in this case.

Now, a time-stepping solver for the nonlinear system should be used to march along the temporal variable and value the put option.

Generally speaking, handling American option is resulted to a nonlinear PDE and after discretion to a nonlinear system of ODEs, so it is costlier than the European-style option in terms of computational burden. Note that the positivity constraint for American put value is

$$V(S,\tau) \geq \max\{E - S, 0\}, \tag{4.3}$$

for any $S \geq 0$.

5. Option pricing results

Numerical experiments are now performed for the new scheme (FD6) using Mathematica 10 [23] on a computer whose characteristics are Windows 7 Ultimate, Intel(R) Core(TM) i5-4400 CPU @ 3.10 GHz with 8GB of RAM and 64-bit Operating System.

Computed option prices are given for different values of r and σ and the merit of our scheme is illustrated in its ability to accurately compute the price in different cases. Note that both r and σ can be estimated from the past stock process data.

Comparisons with the exact solutions for the European case [8] are also provided

$$P(S,t) = E\exp\left(-r(T-t)\right)N(-d_2) - S\exp\left(-\delta(T-t)\right)N(-d_1), \tag{5.1}$$

where

$$d_1 = \frac{\ln S - \ln E + (r - \delta + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}, \tag{5.2}$$

and δ is the continuous dividend yield and

$$N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} \exp\left(-\frac{1}{2}x^2\right)dx. \tag{5.3}$$

The parameter $N(d_2)$ denotes the probability that the asset price will be above the exercise price. We also recall that put and call option prices are related via the Put-Call parity relation, which states that

$$C(S,t) + E\exp\left(-r(T-t)\right) = P(S,t) + S\exp\left(-\delta(T-t)\right). \tag{5.4}$$

Furthermore, the solution of a put of tends to the strike price while the solution of a call of tends to S, based on the boundary conditions.

For the sake of comparison, the second-order semi-discretization technique in [4] (without transaction costs) and the fourth-order semi-discretization technique of [22] are also taken into account, while denoted by FD2 and FD4, respectively. Here n and n_t stand for the number of grid points for spatial and temporal variables respectively. Furthermore, for the nonlinear time-stepping for the American option pricing a Newton's method might be used, see e.g. [3, 14] and the references cited therein.

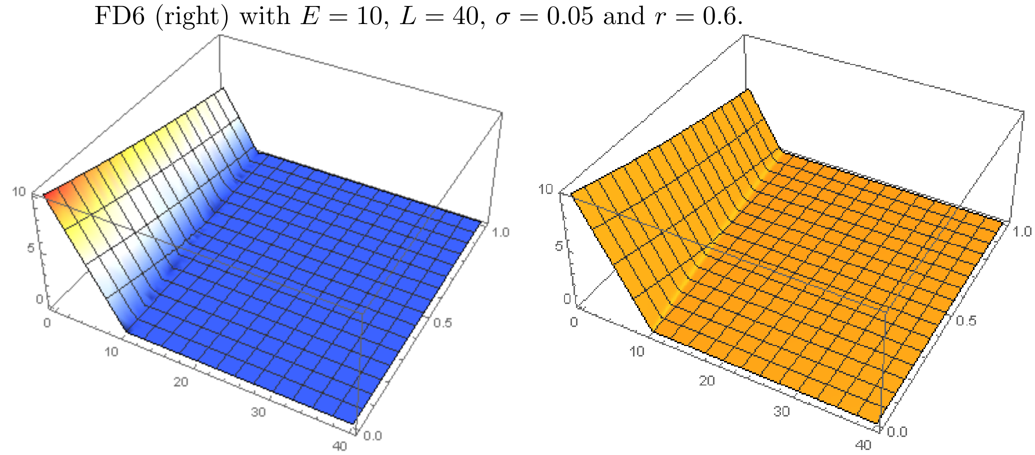
Our idea is to obtain an acceptable accuracy for the numerical methods with as small as possible of the number of nodes while the formulation and implementation is straightforward. We also state that the Crank-Nicolson produce spurious oscillation near the expiry time for fine state variable discretization and coarse time discretization [6].

Experiment 5.1. Consider pricing of a European put option with the following data

$$T = 1 \text{ year}, \quad n = 300, \quad n_t = 200, \quad \Delta x = h = \frac{L}{n-1}, \quad k = \frac{-T}{n_t}.$$

Figures 2-3 illustrate the exact and numerical solution of the European put options in two different cases of the interest rates and the stock volatilities. Results manifest that the new sixth-order semi-discretization technique produce results of acceptable accuracy in the whole of the domain.

FIGURE 2. The exact solution (left) and the numerical solution by FD6 (right) with $E = 10$, $L = 40$, $\sigma = 0.05$ and $r = 0.6$.



To compare the numerical methods as efficiently as possible, here we report the results of comparisons with some other numerical schemes of different orders in the following experiment.

Experiment 5.2. Consider the option pricing of a European put option with the following data

$$T = 1 \text{ year}, \quad h = \frac{L}{n-1}, \quad k = \frac{-T}{n_t}.$$

FIGURE 3. The exact solution (left) and the numerical solution by FD6 (right) with $E = 1$, $L = 4$, $\sigma = 0.1$ and $r = 0.1$.

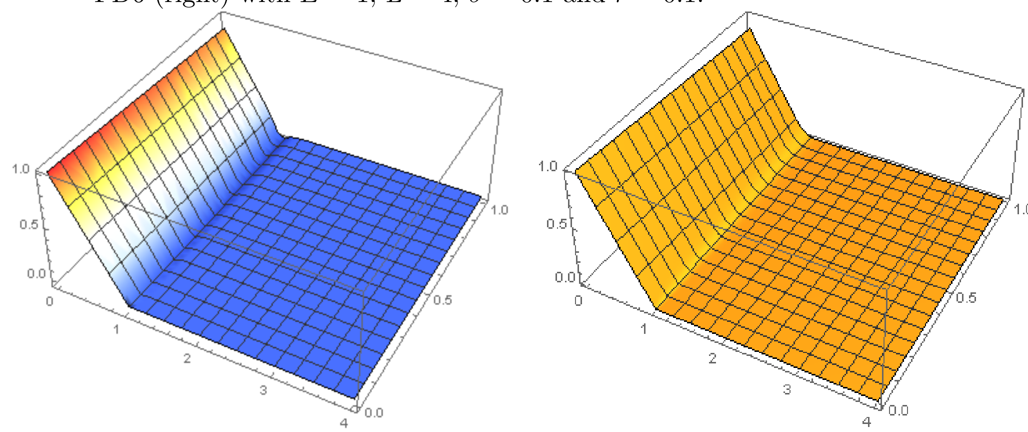


TABLE 1. Results of comparisons for Experiment 5.2 using $E = 10$, $L = 20$, $\sigma = 0.3$, $r = 0.3$.

Methods	FD2	FD4	FD6
$n_t = 105$, $n = 50$	1.0098×10^{-5}	9.84539×10^{-6}	9.74211×10^{-6}

TABLE 2. Results of comparisons for Experiment 5.2 using $E = 10$, $L = 20$, $\sigma = 0.05$, $r = 0.7$.

Methods	FD2	FD4	FD6
$n_t = 20$, $n = 25$	1.27647×10^{-3}	1.20472×10^{-3}	9.59918×10^{-4}

The comparisons are furnished in Tables 1-2. We have considered double precision arithmetic so as to minimize the round-off errors as much as possible. It is observed from the numerical results that proposed method is competitive in comparison with the existing ones.

The aim of this test is to show the high accuracy of the new semidiscretization FD approach as well as to show the rate of convergence (ROC) for numerical results which is calculated using the following relation. We can estimate the exponent p, also known as the ROC via the relation

$$p \approx \left| \log_2 \frac{V_{\text{approx}}(h) - V_{\text{approx}}(h/2)}{V_{\text{approx}}(h/2) - V_{\text{approx}}(h/4)} \right|. \tag{5.5}$$

This is used in the following test to manifest the rate of convergence.

Table 3. Results of comparisons for Experiment 5.3.

	FD2			FD4			FD6		
n	AE	CPU time	ROC	AE	CPU time	ROC	AE	CPU time	ROC
16	9.78×10^{-1}	0.011	-	9.23×10^{-1}	0.025	-	9.95×10^{-1}	0.063	-
32	1.27×10^{-1}	0.027	-	6.55×10^{-2}	0.065	-	5.10×10^{-2}	0.105	-
64	8.42×10^{-2}	0.146	2.21	2.65×10^{-2}	0.185	3.45	1.15×10^{-3}	2.650	4.24
128	2.25×10^{-2}	1.025	2.10	8.67×10^{-3}	1.562	3.85	2.21×10^{-5}	2.358	5.46
256	9.69×10^{-3}	1.595	2.06	1.29×10^{-3}	1.958	4.01	3.90×10^{-6}	2.150	5.95
512	1.72×10^{-3}	2.552	2.04	3.58×10^{-4}	3.250	4.05	3.60×10^{-6}	3.852	5.92

Table 4. Checking the stability along time for Experiment 5.3.

n_t	Price	ROC
100	F.	-
200	0.58094732322121334089	-
400	0.5809468712887259079	F.
800	0.5809467583193601570	2.00017566408
1600	0.5809467300778863308	2.0000443206
3200	0.5809467230175803243	2.0000127609

Experiment 5.3. In this test, we price an American put option with the following data

$$T = 1 \text{ year}, \quad r = 0.05, \quad \sigma = 0.2, \quad E = 10.$$

First of all, we consider the computational domain to be $[0, 3E]$ with the spatial uniform grid and various step sizes h and a fixed temporal step size $k = 0.05$ using a smoothed payoff. Note that by considering $C \neq 0$, e.g., $C = 100$ in (4.2), we are able to value an American option. Since the exact solution of American option pricing problem is not known, the reference value based on binomial tree for $N = 10000$ is used, $V(S^*, 1) = 0.583613582$. In Table 3, the absolute errors (AE) are reported. The computational times are also reported in seconds. In order to check the numerical rate of convergence along the temporal variable and to observe the numerical stability of the proposed sixth-order variant, we furnish Table 4 at which the number of spatial nodes are fixed at 64 but the number of temporal nodes are doubled each time. At the beginning, i.e., when the number of temporal nodes are a few, ($n_t = 100$) the scheme would diverge due to un-stability. But, for other cases, it converges and shows a uniform second order f accuracy along time as mentioned in Section 3.

From the results in Table 3, one may see the higher accuracy of the proposed method in contrast to the existing ones for valuing option prices. Similar numerical experiments have been carried out on variety of problems which confirm the above conclusions to a great extent.

6. Concluding remarks

In financial mathematics, put option is a contract at which the holder has the right (with no obligation) to sell some underlying asset at an agreed-upon price on or before the expiration date of the contract. One purchases a put option if he believes that the price for the asset will fall by the end of the contract.

In between, constructing higher order numerical techniques could lead to higher accuracy. As such, a new strategy based on the use of differentiation matrices was constructed and suggested for the numerical solution of European and American put options. The technique used a sixth-order FD approximations for the first and second derivatives along the space in the semi-discretiztion method. Then, the set of obtained (linear or nonlinear) ODEs had been solved by the numerical scheme of Crank-Nicolson.

We have shown the numerical results obtained from pricing the European/American put options. Numerical results showed a stable and efficient way for valuing put options. Moreover, the results led us to the extension of the presented idea to several other financial models such as pricing a zero-coupon bond using the Cox-Ingersoll-Ross (CIR) model [5] of the term structure of the interest rates in the forthcoming works.

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