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Exact solutions of a linear fractional partial differential equation via characteristics method

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Abstract In recent years, many methods have been studied for solving differential equations of fractional order, such as Lie group method, invariant subspace method and numerical methods, [1, 2, 3, 4]. Among this, the method of characteristics is an efficient and practical method for solving linear fractional differential equations (FDEs) of multi-order. In this paper we apply this method for solving a family of linear (2+1)-dimensional fractional differential equations (FDEs) of multi order α, β and γ.

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1. Introduction

The analysis and applications of FDEs are developed in the past few centuries.

In this paper, we are going to find explicit solutions for the following equation of α, β and γ orders (multi-order),

$$f(t,x,y)\frac{\partial^\alpha u}{\partial t^\alpha} + g(t,x,y)\frac{\partial^\beta u}{\partial x^\beta} + k(t,x,y)\frac{\partial^\gamma u}{\partial y^\gamma} = h(t,x,y). \tag{1.1}$$

The present paper is organized as follows: in section 2, a summary of difinitions and properties of fractional calculus are expressed, and the characteristics method for linear FDEs is explained too. In section 3, in order to clear the mentioned method, by substituting certain functions within functions f, g and k, some exact solutions have been achived for Eq. (1.1).

2. Preliminaries

In this section we recall some basic defintions and results about fractional calculus, and then we will talk about the method of characteristics .

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2.1. **Fractional calculus and some properties.** Here, we give Riemann- Liouville fractional integral and derivative definitions and some important notions and results about fractional calculus, [7].

- Integration with respect to $(dt)^\alpha$ due to Riemann-Liouville is

$$J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{(\alpha-1)} f(\tau) d\tau, \qquad \alpha \in \mathbb{R}^+,$$

  where

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha 1} e^{-x} dx,$$

  is the Euler-Gamma function.
- Suppose $u(t,x)$ is a smooth function of t, x. Then, the Riemann-Liouville derivative is defined by,

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \begin{cases} \dfrac{\partial^n u}{\partial t^n} & \alpha = n, \\ \dfrac{1}{\Gamma(n-\alpha)} \dfrac{\partial^n}{\partial t^n} \displaystyle\int_0^t (t-s)^{n-\alpha-1} u(s,x) ds & 0 \leq n-1 < \alpha < n. \end{cases} \tag{2.1}$$

These two definitions yield properties and notations that would be used in the paper if necessary.

2.2. **Characteristics method for FDEs.** Consider linear FDE (1.1), with the fractional Taylors series in three variables [5],

$$\begin{aligned} du = & \frac{\partial^\alpha u(t,x,y)}{\Gamma(1+\alpha)\partial t^\alpha} (dt)^\alpha + \frac{\partial^\beta u(t,x,y)}{\Gamma(1+\beta)\partial x^\beta} (dx)^\beta \\ & + \frac{\partial^\gamma u(t,x,y)}{\Gamma(1+\gamma)\partial y^\gamma} (dy)^\gamma, \quad 0 < \alpha, \beta, \gamma < 1. \end{aligned} \tag{2.2}$$

We derive the total derivative in more generalized characteristics curves

$$\begin{cases} \dfrac{du}{ds} = h(t,x,y), \\ \dfrac{(dt)^\alpha}{\Gamma(1+\alpha)ds} = f(t,x,y), \\ \dfrac{(dx)^\beta}{\Gamma(1+\beta)ds} = g(t,x,y), \\ \dfrac{(dy)^\gamma}{\Gamma(1+\gamma)ds} = k(t,x,y). \end{cases}$$

If consider $\alpha = \beta = \gamma = 1$ then the auxiliary system of partial differential equations associated with the FPDE (1.1) is Jumarie's Lagrange method of characteristics [6],

$$\frac{(dt)^\alpha}{f(t,x,y)} = \frac{(dx)^\alpha}{g(t,x,y)} = \frac{(dy)^\alpha}{k(t,x,y)} = \frac{\Gamma(1+\alpha)du}{h(t,x,y)} = \frac{d^\alpha u}{h(t,x,y)}. \tag{2.3}$$

3. Examples of Eq. (1.1)

- Case 1: If consider $f = g = k = 1$, and $h = 0$, Eq. (1.1) changes to the following equation,

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial^\beta u}{\partial x^\beta} + \frac{\partial^\gamma u}{\partial y^\gamma} = 0. \tag{3.1}$$

Thus, the generalized characteristics curves satisfy:

$$\begin{cases} \dfrac{du}{ds} = 0, \\ \dfrac{(dt)^\alpha}{\Gamma(1+\alpha)ds} = 1, \\ \dfrac{(dx)^\beta}{\Gamma(1+\beta)ds} = 1, \\ \dfrac{(dy)^\gamma}{\Gamma(1+\gamma)ds} = 1. \end{cases} \tag{3.2}$$

Integration of the two sides of the system (3.2) yields,

$$\begin{cases} u = c_1, \\ \dfrac{t^\alpha}{\Gamma(1+\alpha)} = s + c_2, \\ \dfrac{x^\beta}{\Gamma(1+\beta)} = s + c_3, \\ \dfrac{y^\gamma}{\Gamma(1+\gamma)} = s + c_4, \end{cases} \tag{3.3}$$

where c_1, c_2, c_3 and c_4 are integral constants. By eliminating the parameter s in the system (3.3), we can reach an exact solution of equation (3.1), in the following form:

$$u(t,x,y) = f\left(\frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{x^\beta}{\Gamma(1+\beta)}, \frac{x^\beta}{\Gamma(1+\beta)} - \frac{y^\gamma}{\Gamma(1+\gamma)}\right). \tag{3.4}$$

On the other hands

$$\begin{aligned} u &= \frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{x^\beta}{\Gamma(1+\beta)}, \\ u &= \frac{x^\beta}{\Gamma(1+\beta)} - \frac{y^\gamma}{\Gamma(1+\gamma)}, \\ u &= \frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{y^\gamma}{\Gamma(1+\gamma)}, \end{aligned}$$

are solutions of equation (3.1).

- Case 2: Under consideration $f(t,x,y) = e^{-at}$, $g(t,x,y) = e^{-bx}$, $k(t,x,y) = e^{-cy}$ and $h(t,x,y) = 0$, Eq. (1.1), becomes to the following form:

$$e^{-at}\frac{\partial^\alpha u}{\partial t^\alpha} + e^{-bx}\frac{\partial^\beta u}{\partial x^\beta} + e^{-cy}\frac{\partial^\gamma u}{\partial y^\gamma} = 0. \tag{3.5}$$

So we have

$$\begin{cases} \dfrac{du}{ds} = 0, \\ \dfrac{(dt)^{\alpha}}{\Gamma(1+\alpha)ds} = e^{-at}, \\ \dfrac{(dx)^{\beta}}{\Gamma(1+\beta)ds} = e^{-bx}, \\ \dfrac{(dy)^{\gamma}}{\Gamma(1+\gamma)ds} = e^{-cy}, \end{cases} \tag{3.6}$$

as a system of characteristics curves. Integrating the system (3.6) gives the following solution:

$$\begin{cases} \dfrac{du}{ds} = 0, \\ E_t(\alpha, a) = s + c_2, \\ E_x(\beta, b) = s + c_3, \\ E_y(\gamma, c) = s + c_4, \end{cases} \tag{3.7}$$

where $E_z(\alpha)$ is a Mittage-Leffler function. If consider $\alpha = \beta = \gamma = \frac{1}{2}$, then by omitting the parameter s we have

$$u(t,x,y) = a^{-\frac{1}{2}}\mathrm{Erf}(at)^{\frac{1}{2}} - b^{-\frac{1}{2}}e^{bx}\mathrm{Erf}(ax)^{\frac{1}{2}},$$
$$u(t,x,y) = b^{-\frac{1}{2}}\mathrm{Erf}(bx)^{\frac{1}{2}} - c^{-\frac{1}{2}}e^{cy}\mathrm{Erf}(cy)^{\frac{1}{2}},$$
$$u(t,x,y) = a^{-\frac{1}{2}}\mathrm{Erf}(at)^{\frac{1}{2}} - c^{-\frac{1}{2}}e^{cy}\mathrm{Erf}(cy)^{\frac{1}{2}},$$

as exact solutions of equation (3.5). Here Erf is the Error function.

- Case 3: Substuiting $f(t,x,y) = \ln t$, $g(t,x,y) = \ln x$, $k(t,x,y) = \ln y$ and $h(t,x,y) = 0$ turns FDE (1.1), to

$$\ln t\frac{\partial^{\alpha}u}{\partial t^{\alpha}} + \ln x\frac{\partial^{\beta}u}{\partial x^{\beta}} + \ln y\frac{\partial^{\gamma}u}{\partial y^{\gamma}} = 0. \tag{3.8}$$

Consequently, the generalized curve satisfies the following system:

$$\begin{cases} \dfrac{du}{ds} = 0, \\ \dfrac{(dt)^{\alpha}}{\Gamma(1+\alpha)ds} = \ln t, \\ \dfrac{(dx)^{\beta}}{\Gamma(1+\beta)ds} = \ln x, \\ \dfrac{(dy)^{\gamma}}{\Gamma(1+\gamma)ds} = \ln y. \end{cases} \tag{3.9}$$

Integrating the system (3.9) gives the solution:

$$\begin{cases} \dfrac{du}{ds} = 0, \\ -J_t^{\alpha} \ln t = ds, \\ -J_x^{\beta} \ln x = ds, \\ -J_y^{\gamma} \ln y = ds, \end{cases} \tag{3.10}$$

The above fractional integrals, give the following results as exact solutions of Eq. (3.8)

$$\begin{aligned} u(t,x,y) = & \frac{x^{\beta}}{\Gamma(1+\beta)} \left[\ln x - \lambda + \psi(1+\beta)\right] \\ & - \frac{t^{\alpha}}{\Gamma(1+\alpha)} \left[\ln t - \lambda + \psi(1+\alpha)\right], \\ u(t,x,y) = & \frac{y^{\gamma}}{\Gamma(1+\gamma)} \left[\ln y - \lambda + \psi(1+\gamma)\right] \\ & - \frac{x^{\beta}}{\Gamma(1+\beta)} \left[\ln x - \lambda + \psi(1+\beta)\right], \end{aligned}$$

where λ is Euler constant and ψ is diagmma function [7, 8].

- Case 4: According to the substitution $f(t,x,y) = \csc(at)$, $g(t,x,y) = \csc(bx)$, $h(t,x,y) = \csc(cy)$ and $h(t,x,y) = 0$, we derive

$$\csc(at)\frac{\partial^{\alpha} u}{\partial t^{\alpha}} + \csc(bx)\frac{\partial^{\beta} u}{\partial x^{\beta}} + \csc(cy)\frac{\partial^{\gamma} u}{\partial y^{\gamma}} = 0. \tag{3.11}$$

The corresponding generalized curves are:

$$\begin{cases} \dfrac{du}{ds} = 0, \\ \dfrac{(dt)^{\alpha}}{\Gamma(1+\alpha)ds} = \csc(at), \\ \dfrac{(dt)^{\alpha}}{\Gamma(1+\alpha)ds} = \csc(bx), \\ \dfrac{(dt)^{\alpha}}{\Gamma(1+\alpha)ds} = \csc(cy). \end{cases} \tag{3.12}$$

A straightforward calculation shows that the system (3.12) has the following solution:

$$u(t,x,y) = f(S_t(\alpha,a) - S_x(\beta,b), S_x(\beta,a) - S_y(\gamma,c)),$$

where $S_z(\alpha,a) = \frac{1}{2i}[E_z(\alpha,i\alpha) - E_z(\alpha,-ia)], E_z(\alpha,a) = z^a e^{az}\nu^*(\alpha,az)$ and $\nu^*(\alpha,z) = e^z \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha+k+1)}$.

If consider $\alpha=\beta=\gamma=\frac{1}{2}$, then we have:

$$
\begin{aligned}
u=&\frac{\sqrt{2}}{a}\left[\sin(at)C(p)-\cos(at)S(p)\right]\\
&-\frac{\sqrt{2}}{b}\left[\sin(bx)C(q)-\cos(bz)S(q)\right],\\
u=&\frac{\sqrt{2}}{b}\left[\sin(bx)C(q)-\cos(bx)S(q)\right]\\
&-\frac{\sqrt{2}}{c}\left[\sin(cy)C(z)-\cos(cy)S(z)\right],
\end{aligned}
$$

are exact solutions for Eq. (3.11), where $C(z)$, $S(z)$ are Fresenal integrals and $p=\sqrt{\frac{2at}{\pi}}$,$q=\sqrt{\frac{2bx}{\pi}}$, $z=\sqrt{\frac{2cy}{\pi}}$, [7].

- Case 5: With consideration polynomial expersion for f,g,k, we have

$$
t^a\frac{\partial^\alpha u}{\partial t^\alpha}+x^b\frac{\partial^\beta u}{\partial x^\beta}+y^c\frac{\partial^\gamma u}{\partial y^\gamma}=0. \tag{3.13}
$$

After the integration of generalized characteristics curves,

$$
\begin{cases}
\dfrac{du}{ds}=0,\\
\dfrac{(dt)^\alpha}{\Gamma(1+\alpha)ds}=t^a,\\
\dfrac{(dt)^\alpha}{\Gamma(1+\alpha)ds}=x^b,\\
\dfrac{(dt)^\alpha}{\Gamma(1+\alpha)ds}=y^c,
\end{cases} \tag{3.14}
$$

we obtain the following exact solution to the Eq. (3.13):

$$
\begin{aligned}
u=f\Bigg(&\frac{\Gamma(1-a)}{\Gamma(1-a+\alpha)}t^{\alpha-a}-\frac{\Gamma(1-b)}{\Gamma(1-b+\beta)}x^{\beta-b},\frac{\Gamma(1-b)}{\Gamma(1-b+\beta)}x^{\beta-b}\\
&-\frac{\Gamma(1-c)}{\Gamma(1-c+\gamma)}y^{\gamma-c}\Bigg).
\end{aligned}
$$

For example, if consider $a=b=c=1$ and $\alpha=\beta=\gamma=\frac{1}{2}$ then

$$
\begin{aligned}
u&=\frac{1}{\sqrt{\pi}}\left(\frac{1}{\sqrt{t}}-\frac{1}{\sqrt{x}}\right),\\
u&=\frac{1}{\sqrt{\pi}}\left(\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{y}}\right),\\
u&=\frac{1}{\sqrt{\pi}}\left(\frac{1}{\sqrt{t}}-\frac{1}{\sqrt{y}}\right),
\end{aligned}
$$

satisfied in equation.

4. Conclusion

The method of characteristics is a useful and classical method which plays an important role in mathematical physics. Until now, several methods have been employed for solving FDEs. But this methods is dedicated to solve FDE with one fractional differential such as α, so we show how could we solve FDEs with several fractional differential such as α, β and γ via the characteristics method. In the present paper we applied this method for solving a linear FDE with multiple fractional orders including several examples.

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