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Invariant functions for solving multiplicative discrete and continuous ordinary differential equations

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Abstract In this paper, at first the elemantary and basic concepts of multiplicative discrete and continuous differentian and integration are introduced. Then for these kinds of differentiation invariant functions, the general solution of discrete and continuous multiplicative differential equations will be given. Finally a vast class of difference equations with variable coefficients and nonlinear difference and differential equations are investigated and solved by making use multiplicative difference and differential equations.

Keywords. Multiplicative Continuous calculus, Invariant Functions , Multiplicative Discrete calculus.

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1. Introduction

The classical calculus (or Newtonian calculus) was introduced in the 17th century by Isaac Newton and Gottfried Leibniz. This calculus some times called differential and integral calculus. This calculus and its beautiful result (differential equations) could solve many problems in physics and engineering. Therefore the 18th century was called as utilization century for Newtonian calculus [7, 13, 15]. As regarding that the classical calculus provides very useful and important tools for modeling and solving many physical and engineering problems, but there are several problems in physics and natural phenomena needed to different kind of calculus for modeling and solving these problems. Discrete additive and multiplicative calculus were introduced by many mathematicians. In order to more details, see [4, 13, 15]. In 1978, Jane Grossman, Michael Grossman and Robertz Katz introduced a new calculus which called Non-Newtonian calculus or geometric and bigeometric calculus, see [9, 10, 11, 12]. Afterward, this calculus was named as multiplicative continuous calculus. In recent years, many mathematicians have used this calculus for introducing new kind of derivative and integration operator. Consequently, new kind of differential equations was

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introduced as multiplicative continuous differential equations [4, 5, 6]. In this paper, at first we recall the concepts and definitions of discrete and continuous multiplicative derivative, and then we introduce their invariant functions. Next we will use invariant functions for solving discrete and continuous multiplicative differential equations [1, 3].

2. Multiplicative Discrete Differential Equations

To solve this kind of equations, we first remark discrete multiplicative derivative concept. We have discrete multiplicative derivative of discrete function of $f : \mathbb{Z} \to \mathbb{R}$ in form of

$$f^{[1]}(x) = \frac{f(x+1)}{f(x)}, \quad x \in \mathbb{Z}.$$

In [1] the authors introduced the invariant function for multiplicative discrete derivative as in the from

$$y(x) = C^{(\lambda+1)^x}, \tag{2.1}$$

where the basis C is a constant and λ is a parameter.

Remark 2.1. Some of properties of this kind of derivative have been given in [3] by the authors. Also, for the elementary functions their discrete multiplicative derivatives have been given. They used this function for finding general solution of multiplicative discrete differential equations with initial and boundary conditions.

For example we consider the following initial value problems

$$y^{[1]}\ (x) = y^a(x); \quad x > x_0, \quad y(x_0) = y_0,$$

where bracket is discrete multiplicative derivative and a is a constant. The solution of this problem is given by

$$y(x) = y_0^{(a+1)^{(x-x_0)}},$$

which the power expression in this relation is

$$y(x) = y_0^{(a+1)^{(x-x_0)}}.$$

Also, for the second order of this kind of differential equation we can consider the following problem

$$y^{[11]}\ (x) = y^{[1]}\ (x)^a.\, y(x)^b; \quad x > x_0; \quad x,\, x_0 \in \mathbb{Z},$$

where a, b are real constants.
The initial conditions are given as follows

$$y(x_0) = y_0, \quad y^{[1]}\ (x_0) = y_1.$$

According to the invariant function (2.1) the solution can be calculated by following process:
by replacing the first and second derivative of this function in differential equation (2.2), we have

$$C^{(\lambda+1)^x \lambda^2} = \left(C^{(\lambda+1)^x . \lambda}\right)^a \left(C^{(\lambda+1)^x}\right)^b,$$

or

$$C^{(\lambda+1)^x . \lambda^2} = C^{(\lambda+1)^x . (\lambda a+b)}.$$

Then the characteristic equation is obtained as

$$\lambda^2 - \lambda a - b = 0 \Rightarrow \lambda_1,\, \lambda_2 = \frac{a \pm \sqrt{a^2+4b}}{2}.$$

Therefore the general solution of the equation (2.2) is

$$y(x) = C_1^{(\lambda_1+1)^x} . \, C_2^{(\lambda_2+1)^x}.$$

By imposing given initial conditions we have

$$C_1^{(\lambda_1+1)^{x_0}\lambda_1} \, C_2^{(\lambda_2+1)^{x_0}\lambda_2} = y_1, \quad C_1^{(\lambda_1+1)^{x_0}} \, C_2^{(\lambda_2+1)^{x_0}} = y_0.$$

Finally by determining the unknown coefficients C_1, C_2, the solution of this problem is

$$y(x) = \left(y_0^{\lambda_2} y_1^{-1}\right)^{\frac{(\lambda_1+1)^{x-x_0}}{\lambda_2-\lambda_1}} . \left(y_0^{-\lambda_1} y_1\right)^{\frac{(\lambda_2+1)^{x-x_0}}{\lambda_2-\lambda_1}}, \quad x \in \mathbb{Z}. \tag{2.2}$$

At the end of this section, we consider the following boundary value problem for this kind of differential equation

$$y^{[1]}\,(x) = y^a(x); \quad x \in (x_0, x_1), \quad x,\, x_0,\, x_1 \in \mathbb{Z}, \tag{2.3}$$

with boundary condition

$$y(x_0) = A\; y^{\alpha}(x_1), \tag{2.4}$$

where A, a, α are real constants, α is a degree. By making use of invariant function (2.1) and using boundary condition (2.5) we get

$$C^{(a+1)^{x_0}} = A\left(C^{(a+1)^{x_1}}\right)^{\alpha} \quad \Rightarrow C = A^{\frac{1}{(a+1)^{x_0} - (a+1)^{x_1}\alpha}}.$$

Therefore the exact solution of problem (2.4)-(2.5) is

$$y(x) = A^{\frac{(a+1)^x}{(a+1)^{x_0} - (a+1)^{x_1}\alpha}} = \left(A^{\frac{1}{(a+1)^{x_0} - (a+1)^{x_1}\alpha}}\right)^{(a+1)^x}.$$

Now by considering the first and second discrete derivative of $y(x)$

$$y^{[1]}(x) = \frac{y(x+1)}{y(x)}, \quad y^{[11]}(x) = \frac{\frac{y(x+2)}{y(x+1)}}{\frac{y(x+1)}{y(x)}} = \frac{y(x)y(x+2)}{y^2(x+1)},$$

we can study second order multiplicative discrete the differential equation

$$y^{[11]}(x) = \left[y^{[1]}(x)\right]^a \left[y(x)\right]^b,$$

this equation can be written in the following from

$$y(x+2) = y^{a+2}(x+1)\; y(x)^{b-a-1},$$

also it can be written as the following difference equation

$$y_{n+2} = y_{n+1}^{a+2} . \, y_n^{b-a-1}.$$

Example 1. In the above obtained difference equation let $a = b = 1$ and

$$y(0) = 1, \quad y^{[1]}(0) = 2,$$

then, we will have an initial value problem including a nonlinear difference equation in the following from

$$y_{n+2} = y_{n+1}^3 \cdot \tfrac{1}{y(n)}, \quad y_0 = 1, \quad y_1 = 2.$$

By using the analytic solution (2.3) of equation (2.2) the exact and analytic solution of this problem will be as follows

$$y(x) = 2^{\frac{(\lambda_2+1)^x - (\lambda_1+1)^x}{\lambda_2 - \lambda_1}},$$

where

$$\lambda^2 - \lambda - 1 = 0 \Rightarrow \lambda_1 = \tfrac{1+\sqrt{5}}{2}, \quad \lambda_2 = \tfrac{1-\sqrt{5}}{2}.$$

3. Multiplicative Continuous Differential Equations

As mentioned in introduction, multiplicative continuous derivative is defined by the formula for the function $f : \mathbb{R} \to \mathbb{R}$

$$f^*(x) = \lim_{h\to 0} \Big(\frac{f(x+h)}{f(x)}\Big)^{\frac{1}{h}}.$$

We show that this formula can be written in the following form

$$f^*(x) = e^{\frac{f'(x)}{f(x)}}$$

$$\begin{aligned} f^*(x) &= \lim_{h\to 0} \Big(\frac{f(x+h)}{f(x)}\Big)^{\frac{1}{h}} = \lim_{h\to 0} \Big(\frac{f(x+h)}{f(x)} - \frac{f(x)}{f(x)} + 1\Big)^{\frac{1}{h}} \\ &= \lim_{h\to 0} \Big(1 + \frac{f(x+h) - f(x)}{f(x)}\Big)^{\frac{1}{h}} \\ &= \lim_{h\to 0} \Big[\Big(1 + \frac{f(x+h) - f(x)}{f(x)}\Big)^{\frac{f(x)}{f(x+h)-f(x)}}\Big]^{\frac{f(x+h)-f(x)}{h} \cdot \frac{1}{f(x)}} = e^{\frac{f'(x)}{f(x)}}. \end{aligned}$$

Remark 3.1. Some of properties of this kind of derivative were given in [11, 12].

For elementary functions, their multiplicative continuous derivatives are determined in the following

Example 2. a) $f(x) = x^n \Rightarrow f^*(x) = e^{\frac{n}{x}}$, $\qquad b) g(x) = \ln x \Rightarrow g^*(x) = e^{\frac{1}{x \ln x}}$.

Similarly, the second order multiplicative continuous derivative is defined as follows

$$f^{**}(x) = e^{(\ln f^*)'}(x) = e^{(\ln f)''}(x).$$

Similar to discrete multiplicative derivative, we are going to determinate the invariant function for the continuous case. For this, at first we consider the multiplicative continuous differentiation of the function $y_0(x) = e^{e^x}$.

$$\begin{aligned} y_0^*(x) &= \lim_{h\to 0} \Big(\frac{e^{e^{x+h}}}{e^{e^x}}\Big)^{\frac{1}{h}} = \lim_{h\to 0} \Big[\Big(e^{e^{x+h} - e^x}\Big)\Big]^{\frac{1}{h}} \\ &= e^{\lim_{h\to 0} \frac{e^{x+h} - e^x}{h}} = e^{e^x} = y_0(x). \end{aligned}$$

Similarly we can write: $y_0(x,\lambda) = e^{\left(e^{\lambda x}\right)}$,

$$y_0^*(x,\lambda) = \lim_{h\to 0}\left(\frac{y_0(x+h,\lambda)}{y_0(x)}\right)^{\frac{1}{h}} = \lim_{h\to 0}\left(\frac{e^{e^{\lambda(x+h)}}}{e^{e^{\lambda x}}}\right)^{\frac{1}{h}}$$

$$= \lim_{h\to 0} e^{\left(e^{\lambda(x+h)}-e^{\lambda x}\right)^{\frac{1}{h}}} = e^{\lim_{h\to 0} e^{\left(e^{\lambda(x+h)}-e^{\lambda x}\right)^{\frac{1}{h}}}}$$

$$= e^{\lim_{h\to 0}\frac{\lambda\left(e^{(x+h)}-e^{x}\right)h}{h}} = e^{\lambda e^{\lambda x}} = (e^{e^{\lambda x}})^{\lambda} = y_0^{\lambda}(x,\lambda).$$

Therefore we have

$$y_0^*(x,\lambda) = y_0^{\lambda}(x,\lambda). \tag{3.1}$$

Let us calculate its second derivative

$$y_0^{**}(x,\lambda) = \lim_{h\to 0}\left(\frac{y_0^*(x+h,\lambda)}{y_0^*(x,\lambda)}\right)^{\frac{1}{h}} = \lim_{h\to 0}\left(\frac{e^{\lambda e^{\lambda(x+h)}}}{e^{\lambda e^{\lambda x}}}\right)^{\frac{1}{h}}$$

$$= \lim_{h\to 0}\left(e^{(\lambda e^{\lambda(x+h)}-\lambda e^{\lambda x})}\right)^{\frac{1}{h}} = e^{\lambda \lim_{h\to 0}\frac{e^{\lambda(x+h)}-e^{\lambda x}}{h}} = e^{\lambda^2 . e^{\lambda x}}$$

$$= (e^{e^{\lambda x}})^{\lambda^2} = y_0^{\lambda^2}(x,\lambda).$$

By mathematical induction we can obtain for the arbitrary order multiplicative derivative

$$y^{\overbrace{*\cdots*}^{n}}(x) = y^{[n]}(x) = y_0^{\lambda^n}(x,\lambda).$$

By the following existence and uniqueness theorem, we are going to determine the analytic solutions of continous multiplicative differential equations.

Theorem 3.2. *Let f be continuous function on the open region G in $\mathbb{R}\times\mathbb{R}^+$ to (a,b), where $0 < a < b < \infty$. Assume that f satisfies the multiplicative analog of the Lipscitz condition. Take $(x_0, y_0) \in G$. Then there exists $\varepsilon > 0$ such that equation $y^*(x) = f(x, y(x))$ has a unique solution $y : (x_0-\epsilon, x_0+\epsilon) \to \mathbb{R}^+$ satisfying the condition $y(x_0) = y_0$ [4].*

Now, we can investigate multiplicative continuous differential equations with boundary and initial conditions. For this, we consider the general form of the second order multiplicative continuous differential equation

$$y^{**}(x)\ \left(y^*(x)\right)^a .\ \left(y^*(x)\right)^b = 1, \tag{3.2}$$

with following boundary conditions

$$y^*(0) = A, \quad y^*(1) = B. \tag{3.3}$$

By using the invariant function (3.1) and substituting the first and second order derivatives in equation (3.2)

$$e^{\lambda^2 e^{\lambda x}} .\ e^{\lambda a e^{\lambda x}}\ e^{e^{\lambda x} b} = 1 \Rightarrow e^{(\lambda^2+\lambda a+b)e^{\lambda x}} = 1,$$

therefore, we obtain the characteristic equation

$$\lambda^2 + \lambda a + b = 0,$$

assume its roots are distinct and real numbers, that is

$$\lambda_1,\ \lambda_2 = \frac{-a \pm \sqrt{a^2 - 4b}}{2},$$

then the general solution of equation (3.2) will be

$$y(x) = (e^{e^{\lambda_1 x}})^{c_1}.\ (e^{e^{\lambda_2 x}})^{c_2} = e^{c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}}. \tag{3.4}$$

Now, by imposing the boundary conditions (3.3), we have

$$A = e^{\lambda_1 c_1 . \lambda_2 c_2}, \quad B = e^{\lambda_1 c_1 e^{\lambda_2 . 1}}\ e^{\lambda_2 c_2 e^{\lambda_2 . 1}}, \tag{3.5}$$

applying logarithm operation on both sides of relations (3.5) yields the following algebraic system

$$\begin{cases} \lambda_1 c_1 + \lambda_2 c_2 = \ln A, \\ \lambda_1 c_1 e^{\lambda_2} + \lambda_2 c_2 e^{\lambda_2} = \ln B. \end{cases}$$

Assume the determinant of this system is not vanished, that is

$$\begin{vmatrix} \lambda_1 & \lambda_2 \\ \lambda_1 e^{\lambda_1} & \lambda_2 e^{\lambda_2} \end{vmatrix} = \lambda_1 \lambda_2 (e^{\lambda_2} - e^{\lambda_1}) \neq 0,$$

we have $c_1 = \frac{e^{\lambda_2} lnA + lnB}{\lambda_1 e^{\lambda_1 + \lambda_2}}$ and $c_2 = -\frac{e^{\lambda_1} lnA + lnB}{\lambda_2 e^{\lambda_1 + \lambda_2}}$. By replacing c_1, c_2 in general solution (3.4), we get the solution of BVP (3.2)-(3.3).
Now let us define an analog of Riemann integral in multiplicative calculus. Let f be positive bounded function on $[a, b]$, where $-\infty < a < b < \infty$. Consider the partition $\mathcal{P} = \{x_0, x_1, ..., x_n\}$ of $[a, b]$. Take the numbers $\xi_1, \xi_2, ..., \xi_n$ associated with the partition $\mathcal{P}$. The first step in the definition of proper Riemann integral of f on $[a, b]$ is the formation of the integral sum

$$S(f, \mathcal{P}) = \sum_{i=1}^{n} f(\xi_i)(\xi_i - \xi_{i-1}).$$

To define the multiplicative integral of f on $[a, b]$ we will replace the sum by product and the product by raising to power

$$P(f, \mathcal{P}) = \prod_{i=1}^{n} f(\xi_i)^{\xi_i - \xi_{i-1}}. \tag{3.6}$$

The function f is said to be integrable in the multiplicative sense or * integrable if there exists a number P having the property: for every $\varepsilon > 0$ there exists a partition $\mathcal{P}_\varepsilon$ of $[a, b]$ such that $|P(f, \mathcal{P}) - P| < \varepsilon$ for every refinement $\mathcal{P}$ of $\mathcal{P}_\varepsilon$ independently on selection of the numbers associated with the partition $\mathcal{P}$. The symbol

$$\int_a^b f(x)^{dx},$$

reflecting the feature of the product in (3.6), is used for the number P and it is called the multiplicative integral f or * integral of on $[a, b]$. It is reasonable to let

$$\int_a^b f(x)^{dx} = \left(\int_b^a f(x)^{dx}\right)^{-1}.$$

It is easily seen that if f is positive and Riemann integrable on $[a, b]$, then it is * integrable on $[a, b]$ and

$$\int_a^b f(x)^{dx} = e^{\int_a^b (\ln \circ f(x))dx}.$$

Indeed, since the Riemann integral of $\ln \circ f$ on $[a, b]$ exists, the continuity of the exponential function and

$$P(f, \mathcal{P}) = e^{\sum_{i=1}^{n}(x_i - x_{i-1})(\ln \circ f)(\xi_i)} = e^{S(\ln \circ f, \mathcal{P})},$$

imply the above statement [4].

Example 3. Consider the nonlinear ordinary differential equation:

$$y''y - y'^2 = g(x)\, y^2. \tag{3.7}$$

At first, by using the multiplicative continuous derivative and some algebraic operation, we reduce the nonlinear equation (3.7) to a continuous multiplicative differential equation as follows

$$y''y - (y'^2) = y^2\, g(x) \Rightarrow \frac{y''y - (y'^2)}{y^2} = g(x) \Rightarrow e^{\frac{y''y-(y'^2)}{y^2}} = e^{g(x)} = f(x),$$
$$e^{\left(\frac{y'}{y}\right)'} = f(x) \Rightarrow y^{**}(x) = f(x).$$

For solving this multiplicative differential equation, it is enough to take multiplicative continuous integerate from $f(x)$. This process will be presented in the following examples.

Example 4. Consider the equation

$$\frac{y''y - (y'^2)}{y^2} = \cos x. \tag{3.8}$$

By applying the above process we get

$$\mathrm{e}^{\frac{y''y-(y'^2)}{y^2}} = e^{\cos x} \Rightarrow e^{(\ln y)''(x)} = e^{\cos x} \Rightarrow y^{**}(x) = e^{\cos x}.$$

This is a multiplicative O.D.E, for solving it, we apply two time multiplicative continuous integration for this equation

$$y^*(x) = c_1 \int_0^x (e^{\cos t})^{dt} = c_1\, e^{\int_0^x \ln e^{\cos t} dt} = c_1\, e^{\sin x},$$
$$y(x) = c_2\, e^{\int_0^x \left(\ln c_1 + \ln e^{\sin x}\right) dx} = c_2\, e^{x \ln c_1}.e^{-\cos x}.$$

Thus, the general solution of equation (3.7) is

$$y(x) = c_2\, e^{x \ln c_1} . e^{-\cos x}.$$

Example 5. Consider the homogeneous second order multiplicative continuous differential equation

$$y^{**}(x) = 1.$$

The general solution is

$$y(x) = c_1^x .\, c_2.$$

It is interested to know that this equation is similar to ordinary case $y^{''}(x) = 0$ and its general solution is $y(x) = c_1 x + c_2$.

4. Conclusion

In this paper we recalled the preliminary definitions of discrete and continuous multiplicative differentiations. Then their related invariant functions were introduced. By means of these functions we could solve the related multiplicative differential equations with suitable initial and boundary conditions. We can extend the multiplicative concepts and methods for investigation and solving advanced nonlinear difference and differential equations. Note that this process was done by converting the given nonlinear difference and differential equation to a multiplicative continuous differential equation as shown in examples 3 and 4.

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