Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 5, No. 4, 2017, pp. 280-300

A numerical study of electrohydrodynamic flow analysis in a circular cylindrical conduit using orthonormal Bernstein polynomials

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Abstract In this work, the nonlinear boundary value problem in electrohydrodynamics flow of a fluid in an ion-drag configuration in a circular cylindrical conduit is studied numerically. An effective collocation method, which is based on orthonormal Bernstein polynomials is employed to simulate the solution of this model. Some properties of orthonormal Bernstein polynomials are introduced and utilized to narrow down the computation of nonlinear boundary value problem to the solution of algebraic equations. Also, by using the residual correction process, an efficient error estimation is introduced. Graphical and tabular results are presented to investigate the influence of the strength of nonlinearity α and Hartmann electric number Ha^2 on velocity profiles. The significant merit of this method is that it can yield an appropriate level of accuracy even with large values of α and Ha^2. Compared with recent works, the numerical experiments in this study show a good agreement with the results obtained by using MATLAB solver bvp5c and its competitive ability.

Keywords. Electrohydrodynamics flow, Circular cylindrical conduit, Hartmann electric number, Orthonormal Bernstein polynomials.

2010 Mathematics Subject Classification. 65L10, 65L60, 41A10.

Received: 1 December 2016 ; Accepted: 24 September 2017.

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1. Introduction

Electro fluid systems have received much attention in several industrial and engineering processes. In recent years, the effects of an electric field on fluids have been studied from theoretical viewpoint to develop new processes. The interaction of electrohydrodynamics and hydrodynamics is often referred to as electrohydrodynamics(EHD) and appears whenever a dielectric fluid moves in the electric field [50, 47]. Using electrical fields frequently, Electrohydrodynamics (EHD) control the transport phenomena in the flowing fluids.

EHD flows have practical applications in many areas of chemical and industrial engineering, such as pumps and inkjet devices [16], electrostatic precipitators [56], polymer nanotechnology [42], colloidal particle orientation with an alternating or direct current electric field applied normally to an interface [51], MEMS devices [49], dielectric pump designs [15], micro-electronic thermal management using electro gasdynamic pumps [10], co-planar microelectrode designs [23], electrostatic precipitators [28], regulation of nickel particle granular hopper flows in medical powder processing [26], ion flow in divertor tokamak reactors [11], transverse or radial electrical fields including combustion controls [55], microfluidic particle trapping without any mechanically moving parts [3], small-scale naval vessel propulsion [35], smart electrorheological fluids [8], electro-fluid control of aggregative and deaggregative behavior of colloidal particles near electrodes [48], electro-spray liquid atomization using Taylor cone electrohydrodynamic jets [40], EHD control and manipulation of aerodynamic boundary layer separation using plasma actuators for drag reduction [17], EHD printing technology [12], protein biomolecule separation with pulsed electrofluid dynamics [31] boundary layer control in ionized hypersonic flows [46], etc.

The EHD flow of a fluid in an ion drag configuration in a circular cylindrical conduit is governed by a singular nonlinear boundary value problem. It was first dealt with by McKee in 1997 [37]. McKee et al. in [37] obtained the perturbation solutions of fluid velocities for both small and large values of the parameter governing the nonlinearity of the problem, α. They used a combination of Gauss-Newton finite-difference method and Runge-Kutta shooting method to provide a numerical solution for the fluid velocity over a large range of values of α. Furthermore, they proposed that for $\alpha \ll 1$ and $\alpha \gg 1$, the perturbation solutions can be expanded in a series with $O(\alpha^n)$ and $O(1)$ leading order terms, respectively. The existence and uniqueness of a solution to the EHD flow in a circular cylindrical conduit are proved by Paullet [43]. For large values of α, the obtained results in [43] are qualitatively different from those calculated in [37]. This stems from the fact that, for $\alpha \gg 1$ the solutions are $O(1/\alpha)$, not $O(1)$ as proposed in the perturbation expansion used by McKee and his coworkers [43]. Mastroberardino [36] proposed analytical solutions based on the homotopy analysis method (HAM) for various values of the Hartmann electrical number Ha^2 and the non-linearity parameter α, and the convergence of these solutions is discussed. Pandey et al. [41] employed two semi-analytic algorithms based on the optimal homotopy asymptotic method (OHAM) and the optimal homotopy analysis method to solve the EHD flow equation for the range of values of the relevant parameters. Authors of [27] developed a numerical scheme to solve the EHD flow equation using a new homotopy perturbation method (NHPM). This method is an improvement of the classical homotopy perturbation method and it effectively depends only on two components of the homotopy series. Besides, in contrast to HAM and HPM, this method is not required to solve the functional equations in each iteration. Moghtadaei et al. [38] provided a combination of the hybrid spectral collocation technique and the spectral homotopy analysis method (SHAM) to solve the nonlinear boundary value problem for the EHD flow of a fluid in an ion drag configuration in a circular cylindrical conduit. Mosayebdorcheh [39] obtained an approximate solution of the EHD flow equation using the differential transform method (DTM) for $0 < \alpha < 1$. This method overcame the nonlinearity and singularity of the problem without any need to restrict assumptions or linearization. Rostamy et al. [45] presented an algorithm by using a pseudo-spectral collocation method to obtain the approximate solutions of the EHD flow of a fluid in a circular. Anwar Bég et al. [1] used the spectral collocation method based on Chebyshev polynomials and DTM-Padé approximation [2] for the numerical simulation of this differential equation. Also, Ghasemi et al. [22] presented a least square method (LSM), and Hasankhani et al. [21] introduced two analytical techniques by using Galerkin Method, and Collocation Method to obtain an approximate solution of this equation for various values of the relevant parameters.

Bernstein polynomials were first used by Sergei Natanovich Bernstein in a constructive proof for the Stone-Weierstrass approximation theorem. It plays an important role in computer graphics [29, 19]. These polynomials have many useful properties, such as, continuity, positivity, unity partition of the basis set over the interval $[a, b]$ and greater flexibility in imposing boundary conditions at the end points of the interval. Moreover, Farouki et al. [20] showed that the computation on Bernstein bases are more stable than computations on other non-negative polynomial basis. Recently, these polynomials have been applied to solve various kinds of ordinary and partial differential equations, integral equations and integro-differential equations including KdV equation [5], some classes of integral equations [34], Volterra integral equations [6], Fredholm integral equations [9], ordinary differential equations [7, 52], parabolic equation subject to specification of the mass [53], high even-order differential equations [13, 14], nonlinear age-structured population models [54], nonlinear Volterra-Fredholm-Hammerstein integral equations [32, 33], fractional heat- and wave-like equations [44], etc.

Heydari et. al [25] applied the Gram-Schmidt orthogonalization process to find orthogonal Bernstein polynomials for the solution of heat transfer of a micropolar fluid through a porous medium with radiation. Bellucci [4] introduced an explicit representation of orthogonal Bernstein polynomials and demonstrated that they can be generated from a linear combination of non-orthogonal Bernstein polynomials. Bellucci showed that highly accurate approximations to curves and surfaces can be obtained by using small sized basis sets, when the orthogonal Bernstein polynomials are used.

The aim of the present study is to propose a numerical method based on orthonormal Bernstein polynomials to solve the nonlinear boundary value problem in the EHD flow of a fluid in an ion-drag configuration in a circular cylindrical conduit for all the values of the relevant parameters.

Nomenclature

$B_{i,n}(x)$	Bernstein polynomial of degree n
$\mathfrak{B}_{i,n}(x)$	Orthogonal Bernstein polynomial of order n
$\bar{\mathfrak{B}}_{i,n}(x)$	Orthonormal Bernstein polynomial of order n
$\mathbb{B}_n f(t)$	Bernstein function of order n
r	Dimensionless distance from the center of cylindrical conduit
$w(r)$	Dimensionless fluid velocity
Ha	Hartmann Electric number
z	Axial coordinate
a	Insulating wall supporting screens
V	External voltage
$\vec{E}_0$	Electrical field
$\vec{j}$	Current density
K	Ion mobility
$\vec{v}$	Generalized vectors for velocity
p	Pressure
t_c	Charge relaxation time for the ions
t_f	Fluid transport time
U	Typical velocity scale
L	Length of circular cylindrical conduit

Greek symbols

α	Measure of the strength of non-linearity
ρ_f	Free charge density of the ion/fluid medium
$\frac{\partial p}{\partial z}$	Axial pressure gradient
μ	Dynamic viscosity of the dielectric fluid
ε_0	Constant permittivity of free space
ρ_0	Charge density at the inlet screen
λ	Eigenvalues of of Sturm-Liouville problem

The rest of this paper is organized as follows. In Section 2, the mathematical formulation of the Electrohydrodynamic ion drag flow in a circular cylindrical conduit is presented. Bernstein and orthonormal Bernstein polynomials and their properties are given in Section 3 and Section 4, respectively. After introducing a function approximation with orthonormal Bernstein polynomials in Section 5, the collocation method based on this approximation is dealt with for the EHD flow in Section 6. In Section 7, we will give an error analysis of the present method by comparing the estimated of the error function and the actual absolute error. The results and discussion for the all values of the relevant parameters are presented in Section 8. Finally, the conclusions of the research is provided in the last Section.

2. Mathematical modeling

In this Section, we present a brief review of governing equations which is taken from [18, 37]. Let us choose the axisymmetric coordinate system (r, z) where r is the radial coordinate and z is the axial coordinate. Consider that a circular cylindrical conduit of radius a has an insulating wall supporting screens at $z = 0$ and $z = L$ [18] in a fully developed flow in the ion-drag configuration as shown in Figure 1. Voltage V produces an electric field $\vec{E}_0$ across the length of the cylinder. The current density $\vec{j}$, by assuming only a single type of ionized particle in the dielectric fluid medium, can be obtained as:

$$\vec{j} = \rho_f[K\vec{E}_0 + \vec{v}], \tag{2.1}$$

where ρ_f is the free charge density of the ion/fluid combination and K is the ion mobility. For a fully developed solution, McKee et al. [37] reduced the current density, as well as the free charge density and generalized vectors for velocity in the following form:

$$\vec{j} = (0, 0, j(r)),\ \vec{v} = (0, 0, w(r)),\ \rho_f = \rho(r), \tag{2.2}$$

while the pressure gradient, $\frac{\partial p}{\partial z}$, is constant.

Using (2.2), the Navier-Stokes equation reduces to the following ordinary differential equation [37]:

$$\frac{\partial p}{\partial z} = \rho_f E_0 + \frac{\mu}{r}\frac{d}{dr}\left[r\frac{dw}{dr}\right], \tag{2.3}$$

where μ is the dynamic viscosity of the dielectric fluid. There are two different time scales for a typical velocity scale U and length L, which determine the coupling between the fluid and the ions, as follows:

$$t_c = \frac{\varepsilon_0}{K\rho_0},\ t_f = \frac{L}{U}, \tag{2.4}$$

where ε_0 is the constant permittivity of free space, ρ_0 is the charge density at the inlet screen, t_c is the charge relaxation time for the ions and t_f is the fluid transport time.

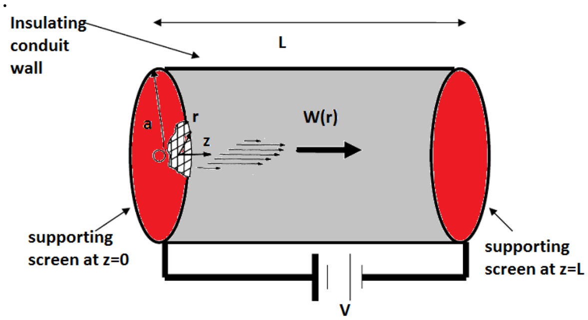
The ratio of charge relaxation time (t_c) to fluid transport time (t_f) is the electrical Reynolds number. For large values of this ratio, fluid convection determines ρ_f. Assuming that $j(z = 0) = j_0$, from equation (2.1), the charge density can be evaluated as:

$$\rho_f(r) = \frac{j_0}{KE_0 + w(r)}. \tag{2.5}$$

To determine the fully developed fluid velocity $w(r)$, by integrating (2.5) to (2.3), we obtain the following nonlinear differential equation, as given by McKee et al. [37]:

$$\frac{\partial p}{\partial z} = \frac{j_0 E_0}{KE_0 + w} + \frac{\mu}{r}\frac{d}{dr}\left[r\frac{dw}{dr}\right]. \tag{2.6}$$

FIGURE 1. Electrohydrodynamic flow in a circular cylindrical conduit [18].



The boundary conditions are specified by taking the velocity and the velocity gradient as vanishing at the conduit center and wall as follows:

$$\frac{dw}{dr} = 0 \ at \ r = 0; \ w = 0 \ at \ r = a, \tag{2.7}$$

It should be pointed out that the fluid velocity $w(r)$ is symmetric and bounded about $r = 0$. By using the transformed variables $r^* = r/a$ and $w^* = -(w/KE_0\alpha)$, we can get:

$$\frac{d^2w^*}{dr^{*2}} + \frac{1}{r^*}\frac{dw^*}{dr^*} + Ha^2\left[1 - \frac{w^*}{1-\alpha w^*}\right] = 0, \tag{2.8}$$

where $\alpha = \frac{K}{j_0}\frac{\partial p}{\partial z} - 1$ is measures the non-linearity and $Ha = \sqrt{\frac{j_0 a^2}{\mu K^2 E_0}}$ is the electric Hartmann number. Also, boundary conditions (2.7) become:

$$\frac{dw^*}{dr^*} = 0 \ at \ r^* = 0; \ w^* = 0 \ at \ r^* = 1. \tag{2.9}$$

Theorem 2.1. [43]*: For any $\alpha > 0$ and any $Ha^2 \neq 0$, there exists a solution to the differential equation (2.8) with boundary conditions (2.9). Furthermore, this solution is monotonically decreasing and satisfies $0 < w^*(r^*) < 1/(1+\alpha)$ for all $r^* \in (0,1)$.*

3. Bernstein polynomials and their properties

Bernstein polynomials have important applications in computer graphics and have been applied for approximations of functions in many areas of mathematics and other fields such as smoothing in statistics and constructing Bézier curves [19, 24, 29]. Bernstein polynomials of the degree n are defined on the interval $[a,b]$ as [7]

$$B_{i,n}(x) = \binom{n}{i}\frac{(x-a)^i(b-x)^{n-i}}{(b-a)^n}, \ 0 \le i \le n, \tag{3.1}$$

where the binomial coefficients are calculated by $\binom{n}{i} = \frac{n!}{i!(n-i)!}$. The properties of Bernstein polynomials have been nvestigated by many authors some of which to be mentioned briefly here.

These Bernstein polynomials form a basis on $[a, b]$ and there are $n+1$, nth-degree polynomials. If $i < 0$ or $i > n$ we set $B_{i,n}(x) = 0$. Also for all $i = 0, 1, \cdots, n$ and all x in $[a, b]$, we have $B_{i,n}(x) \geq 0$. In addition, these polynomials can be generated by a recursive definition over the interval $[a, b]$ as follows:

$$B_{i,n}(x) = \frac{b-x}{b-a} B_{i,n-1}(x) + \frac{x-a}{b-a} B_{i-1,n-1}(x). \tag{3.2}$$

The binomial expansion of the right-hand side of the equality $(b-a)^n = ((x-a)+(b-x))^n$ shows that the sum of all Bernstein polynomials of the degree n is the constant 1, i.e,

$$\sum_{i=0}^{n} B_{i,n}(x) = 1.$$

One of the benefits of the Bernstein polynomial approximation of a continuous function f is that it approximates f on $[a, b]$ using only the values of f at $x_i = a+(b-a)i/n$, $i = 0, 1, \ldots, n$, that is,

$$f(x) \simeq \mathbb{B}_n f(x) = \sum_{i=0}^{n} f(x_i) B_{i,n}(x).$$

The above approximation is preferred when the evaluation of f is difficult, expensive and time consuming.

An explicit expression for the derivatives of Bernstein polynomials of any degree and any order in terms of Bernstein polynomials on $[0, 1]$, introduced by Doha et al. [13] is as follows:

$$\frac{d^k}{dx^k} B_{i,n}(x) = \frac{n!}{(n-k)!} \sum_{j=max\{0,i+k-n\}}^{min\{i,k\}} (-1)^{j+k} \binom{k}{j} B_{i-j,n-k}(x). \tag{3.3}$$

It can easily be shown that for Bernstein polynomials on $[a, b]$ [25]:

$$\frac{d^k}{dx^k} B_{i,n}(x) = \frac{1}{(b-a)^k} \frac{n!}{(n-k)!} \sum_{j=max\{0,i+k-n\}}^{min\{i,k\}} (-1)^{j+k} \binom{k}{j} B_{i-j,n-k}(x). \tag{3.4}$$

The product of two Bernstein polynomials is also a Bernstein polynomial which is given by:

$$B_{i,j}(x) B_{k,m}(x) = \frac{\binom{j}{i}\binom{m}{k}}{\binom{j+m}{i+k}} B_{i+k,j+m}(x). \tag{3.5}$$

All Bernstein polynomials of the same order have the same definite integral over the interval $[a, b]$, namely

$$\int_a^b B_{i,n}(x) dx = \frac{b-a}{n+1}. \tag{3.6}$$

The definite integrals of the products of Bernstein polynomials can be found by using (3.5) and (3.6), as follows:

$$I_{k,i}^n = \int_a^b B_{k,n}(x)B_{i,n}(x)dx = \frac{\binom{n}{k}\binom{n}{i}}{(2n+1)\binom{2n}{k+i}}(b-a). \tag{3.7}$$

4. Orthonormal Bernstein polynomials

Heydari at el. [25] derived the orthogonal Bernstein polynomials of Bernstein polynomials by using the Gram-Schmidt orthogonalization process and denoted them by $\{\mathfrak{B}_{i,n}(x)\}_{i=0}^n$. These polynomials are given by:

$$\mathfrak{B}_{i,n}(x) = \sum_{j=0}^{n} \tilde{\alpha}_{ij} B_{j,n}(x) = \sum_{j=0}^{i} \tilde{\alpha}_{ij} B_{j,n}(x),\ i = 0, 1, \dots, n, \tag{4.1}$$

where $\tilde{\alpha}_{ij} = A_{ij}^{-1},\ i, j = 0, 1, \dots, n$ are the elements of A^{-1} and

$$A = \begin{bmatrix} \alpha_{00} & 0 & 0 & \cdots & 0 \\ \alpha_{10} & \alpha_{11} & 0 & \cdots & 0 \\ \alpha_{20} & \alpha_{21} & \alpha_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_{n0} & \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix},$$

$$\alpha_{ij} = \begin{cases} \frac{\langle B_{i,n}(x), \mathfrak{B}_{j,n}(x)\rangle}{||\mathfrak{B}_{j,n}(x)||^2}, & 0 \le j \le i-1, \\ 1, & i = j, \\ 0, & i+1 \le j \le n. \end{cases} \tag{4.2}$$

Note that matrix A is a lower triangular matrix and $\det(A) = \prod_{i=0}^n \alpha_{ii} = 1$, so A is an invertible matrix. The orthogonal Bernstein polynomials in equation (4.1) satisfy the orthogonality relationship [25];

$$\int_a^b \mathfrak{B}_{i,n}(x)\mathfrak{B}_{j,n}(x)dx = I_i^n \delta_{ij}, \tag{4.3}$$

where

$$\begin{aligned} I_i^n &= \int_a^b [\mathfrak{B}_{i,n}(x)]^2 dx = \int_a^b \left(\sum_{l=0}^{i} \tilde{\alpha}_{il} B_{l,n}(x)\right)\left(\sum_{k=0}^{i} \tilde{\alpha}_{ik} B_{k,n}(x)\right) dx \\ &= \sum_{l=0}^{i}\sum_{k=0}^{i} \tilde{\alpha}_{il}\tilde{\alpha}_{ik} I_{l,k}^n, \end{aligned} \tag{4.4}$$

and δ_{ij} is the Kronecker delta function.

Bellucci [4] obtained an explicit representation of orthonormal Bernstein polynomials by analyzing the resulting orthonormal polynomials after using the Gram-Schmidt process on Bernstein polynomials of varying degree n. These polynomials are a product of a factorable polynomial and a non-factorable polynomial. There exists a pattern of the form $\left(\sqrt{2(n-i)+1}\right)(1-x)^{n-i}$ for the factorable part of them, and a pattern in the form of

$$\sum_{j=0}^{i}(-1)^j\binom{2n+1-j}{i-j}\binom{i}{j}x^{i-j},\ i=0,1,\ldots,n,$$

for the non-factorable part, which can be determined by analyzing the binomial coefficients, is present in Pascal's triangle. By doing this, an explicit representation can be obtained for the orthonormal Bernstein polynomials on the interval $[0,1]$ as follows:

$$\bar{\mathfrak{B}}_{i,n}(x)=\left(\sqrt{2(n-i)+1}\right)(1-x)^{n-i}\sum_{j=0}^{i}(-1)^j\binom{2n+1-j}{i-j}\binom{i}{j}x^{i-j}. \tag{4.5}$$

Moreover, using (3.1) on the interval $[0,1]$, (4.5) can be written in a simpler form in terms of the non-orthonormal Bernstein basis functions as follows:

$$\bar{\mathfrak{B}}_{i,n}(x)=\left(\sqrt{2(n-i)+1}\right)\sum_{j=0}^{i}(-1)^j\frac{\binom{2n+1-j}{i-j}\binom{i}{j}}{\binom{n-j}{i-j}}B_{i-j,n-j}(x). \tag{4.6}$$

By changing the variable $x=(t-a)/(b-a)$, we will have the orthonormal Bernstein polynomials on the arbitrary interval [a, b] as:

$$\bar{\mathfrak{B}}_{i,n}(t)=\left(\sqrt{\frac{2(n-i)+1}{b-a}}\right)\sum_{j=0}^{i}(-1)^j\frac{\binom{2n+1-j}{i-j}\binom{i}{j}}{\binom{n-j}{i-j}}B_{i-j,n-j}\left(\frac{t-a}{b-a}\right). \tag{4.7}$$

The orthonormal Bernstein polynomial, $\bar{\mathfrak{B}}_{j,n}(x)$ on $[0,1]$ is the nth eigenfunction of the singular Sturm-Liouville problem [4]:

$$\begin{aligned}\frac{d}{dx}\left[x(1-x)^2\frac{d\mathfrak{B}\bar{(}x)}{dx}\right]&+n(n+2)(1-x)\mathfrak{B}\bar{(}x)\\&+(n-j+1)(j-n)\mathfrak{B}\bar{(}x)=0,\end{aligned} \tag{4.8}$$

with the orthogonality property:

$$\int_0^1\bar{\mathfrak{B}}_{i,n}(x)\bar{\mathfrak{B}}_{j,n}(x)dx=\delta_{ij}. \tag{4.9}$$

Also, using (4.7) and (3.7), the orthonormal polynomials necessarily satisfy the following relationships over the interval $[0,1]$:

$$\int_0^1\bar{\mathfrak{B}}_{i,n}(x)B_{j,n}(x)dx=\begin{cases}\sqrt{(2(n-i)+1)}\sum_{k=0}^{i}(-1)^k\frac{\binom{2n+1-k}{i-k}\binom{i}{k}\binom{n}{j}}{[2n+1-k]\binom{2n-k}{i+j-k}}, & j\geq i,\\ 0, & j<i.\end{cases} \tag{4.10}$$

In the end of this section, we will prove the following theorem, for the derivatives of $\bar{\mathfrak{B}}_{i,n}(x)$ at the end points of the interval $[a,b]$.

Theorem 4.1. *For $k = 0, 1, \cdots, n$, we have*

$$\frac{d^k}{dx^k}\bar{\mathfrak{B}}_{i,n}(a) = \sqrt{\frac{2(n-i)+1}{(b-a)^{2k+1}}}\sum_{j=0}^{i}\frac{(-1)^{i+k}\binom{2n+1-j}{i-j}\binom{i}{j}\binom{k}{i-j}(n-j)!}{\binom{n-j}{i-j}(n-j-k)!}\gamma_{i-j,k}, \tag{4.11}$$

and

$$\frac{d^k}{dx^k}\bar{\mathfrak{B}}_{i,n}(b) = \sqrt{\frac{2(n-i)+1}{(b-a)^{2k+1}}}\sum_{j=0}^{i}\frac{(-1)^{n-i+j}\binom{2n+1-j}{i-j}\binom{i}{j}\binom{k}{n-i}(n-j)!}{\binom{n-j}{i-j}(n-j-k)!}\gamma_{n-i,k}, \tag{4.12}$$

where

$$\gamma_{i,k} = \begin{cases} 1, & i \leq k, \\ 0, & i > k. \end{cases} \tag{4.13}$$

Proof. For a fixed value of k if $i = 0, 1, \cdots, k$, then $min\{i, k\} = i$. Moreover $B_{0,n}(a) = 1$ and $B_{i,n}(a) = 0,\ i = 1, 2, \cdots, n$. So, from (3.4), we can get $j \leq i$ and

$$\frac{d^k}{dx^k}B_{i,n}(a) = \frac{(-1)^{i+k}}{(b-a)^k}\frac{n!}{(n-k)!}\binom{k}{i}. \tag{4.14}$$

Also, if $i = k+1, k+2, \cdots, n$, then $min\{i, k\} = k$. So, from (3.4), we can get $j < i$ and $\frac{d^k}{dx^k}B_{i,n}(a) = 0$. Thus we have

$$\frac{d^k}{dx^k}B_{i,n}(a) = \frac{(-1)^{i+k}}{(b-a)^k}\frac{n!}{(n-k)!}\binom{k}{i}\gamma_{i,k}. \tag{4.15}$$

Similarly, it can be easily shown that for $x = b$:

$$\frac{d^k}{dx^k}B_{i,n}(b) = \frac{(-1)^{n-i}}{(b-a)^k}\frac{n!}{(n-k)!}\binom{k}{n-i}\gamma_{n-i,k}. \tag{4.16}$$

By (4.7), we can write

$$\frac{d^k}{dx^k}\bar{\mathfrak{B}}_{i,n}(x) = \left(\sqrt{\frac{2(n-i)+1}{b-a}}\right)\sum_{j=0}^{i}(-1)^j\frac{\binom{2n+1-j}{i-j}\binom{i}{j}}{\binom{n-j}{i-j}}\frac{d^k}{dx^k}B_{i-j,n-j}(x). \tag{4.17}$$

Therefore from (4.15)-(4.17) we have:

$$\frac{d^k}{dx^k}\bar{\mathfrak{B}}_{i,n}(a) = \left(\sqrt{\frac{2(n-i)+1}{b-a}}\right)\sum_{j=0}^{i}(-1)^j\frac{\binom{2n+1-j}{i-j}\binom{i}{j}}{\binom{n-j}{i-j}}\frac{(-1)^{i-j+k}(n-j)!\binom{k}{i-j}}{(b-a)^k(n-j-k)!}\gamma_{i-j,k}.$$

$$\frac{d^k}{dx^k}\bar{\mathfrak{B}}_{i,n}(b) = \left(\sqrt{\frac{2(n-i)+1}{b-a}}\right)\sum_{j=0}^{i}(-1)^j\frac{\binom{2n+1-j}{i-j}\binom{i}{j}}{\binom{n-j}{i-j}}\frac{(-1)^{n-i}(n-j)!\binom{k}{n-i}}{(b-a)^k(n-j-k)!}\gamma_{n-i,k}.$$

□

5. Function approximation

We define

$$L^2[a,b] = \{f : [a,b] \longrightarrow \mathbb{R} | f \ is\ measurable\ and\ \|f\| < \infty\}, \tag{5.1}$$

where

$$\|f\| = \left(\int_a^b |f(x)|^2 dx \right)^{\frac{1}{2}}, \tag{5.2}$$

is the norm induced by the scalar product

$$\langle f, g \rangle = \int_b^b f(x)g(x)dx. \tag{5.3}$$

Suppose that $H = L^2[a,b]$, and let $\{\bar{\mathfrak{B}}_{0,n}(x), \bar{\mathfrak{B}}(x)_{1,n}, \cdots, \bar{\mathfrak{B}}(x)_{n,n}\} \subset H$ be the set of orthonormal Bernstein functions of the order n, by suppose that

$$S_n = Span\{\bar{\mathfrak{B}}_{0,n}(x), \bar{\mathfrak{B}}(x)_{1,n}, \cdots, \bar{\mathfrak{B}}(x)_{n,n}\}. \tag{5.4}$$

Now, we define $P_n : L^2[a,b] \longrightarrow S_n$ by

$$P_n f(x) = \sum_{i=0}^{n} f_i \bar{\mathfrak{B}}_{i,n}(x). \tag{5.5}$$

Theorem 5.1. [30] *For every given w in a Hilbert space H and every given closed subspace Z of H there is a unique best approximation to w from Z.*

Since $H = L^2[a,b]$ is Hilbert space and S_n is a finite-dimensional subspace, S_n is a closed subspace of H, Thus S_n is a complete subspace of H. So, if w is an arbitrary element in H, by Theorem 5.1, f has the unique best approximation from S_n such as f^*, that is

$$\exists f^* \in S_n;\ \forall g \in S_n\ \|f - f^*\| \leq \|f - g\|. \tag{5.6}$$

Since $f^* \in S_n$, there exists the unique coefficients $f_0, f_1, \ldots, f_n$ such that

$$f(x) \simeq f^*(x) = \sum_{i=0}^{n} f_i \bar{\mathfrak{B}}_{i,n}(x), \tag{5.7}$$

where the coefficients f_i can be obtained by

$$f_i = \langle f(x), \bar{\mathfrak{B}}_{i,n}(x) \rangle,\ i = 0, 1, \ldots, n. \tag{5.8}$$

Theorem 5.2. *Suppose $f \in C^n[a,b]$, $f^{(n+1)}$ exists on $[a,b]$, and $x_0 = \frac{a+b}{2}$. If f^* is the best approximation of f out of S_n then*

$$\|f - f^*\| \leq \frac{\hat{A}(b-a)^{\frac{2n+3}{2}}}{2^{n+1}(n+1)!\sqrt{2n+3}}, \tag{5.9}$$

where $\hat{A} = max_{a \leq \xi_x \leq b} \mid f^{n+1}(\xi_x) \mid$.

Proof. We know that the set

$$\left\{1, \left(x-\frac{a+b}{2}\right), \left(x-\frac{a+b}{2}\right)^2, \dots, \left(x-\frac{a+b}{2}\right)^n\right\},$$

is a basis for a polynomials space of the degree n. Using Taylor series expansion, for every $x \in [a,b]$, there exists a number ξ_x between $x_0 = \frac{a+b}{2}$ and x with

$$f(x) = P_n(x) + R_n(x),$$

where

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{n!}(x-x_0)^k, \; R_n(x) = \frac{f^{n+1}(\xi_x)(x-x_0)^{n+1}}{(n+1)!}.$$

According to Theorem 5.1, since f^* is the best approximation f out of S_n and $P_n(x) \in S_n$ we have

$$\begin{aligned}
\|f-f^*\|^2 \le \|f-P_n(x)\|^2 &= \int_a^b |f(x)-P_n(x)|^2 dx \\
&= \int_a^b \left(\frac{f^{n+1}(\xi_x)(x-x_0)^{n+1}}{(n+1)!}\right)^2 dx \\
&\le \frac{\hat{A}^2}{[(n+1)!]^2} \int_a^b \left(x-\frac{a+b}{2}\right)^{2n+2} dx \\
&= \frac{\hat{A}^2}{[(n+1)!]^2(2n+3)} \left[\left(\frac{b-a}{2}\right)^{2n+3} - \left(\frac{a-b}{2}\right)^{2n+3}\right] \\
&= \frac{\hat{A}^2(b-a)^{2n+3}}{2^{2n+2}[(n+1)!]^2(2n+3)}, \qquad (5.10)
\end{aligned}$$

and by taking the square roots of the two sides (5.10) and (5.9) is obtained. □

6. Orthonormal Bernstein collocation method (OBCM)

In this section, we use the orthonormal Bernstein collocation method to solve boundary value problem (2.8) with boundary conditions (2.9). First, let's suppose that $w^* = w$ and rewrite equations (2.8) and (2.9) in the following form:

$$\begin{cases} (1-\alpha w(r))[rw''(r)+w'(r)] + rHa^2\left[1-(\alpha+1)w(r)\right] = 0, \; 0<r<1, \\ w'(0)=0, \; w(1)=0. \end{cases} \qquad (6.1)$$

To solve (6.1), we approximate the solution for $w(r)$ as:

$$w(r) \simeq P_n w(r) = \sum_{i=0}^{n} w_i \bar{\mathfrak{B}}_{i,n}(r). \qquad (6.2)$$

Now, we construct the residual function by substituting $w(r)$ by $P_n w(r)$ in (6.1) as:

$$Res(r) = (1 - \alpha P_n w(r))[r P_n'' w(r) + P_n' w(r)] + r Ha^2 \left[1 - (\alpha + 1) P_n w(r)\right]. \tag{6.3}$$

Using the collocation points $r_j = \frac{j}{n}$, $j = 0, 1, \ldots, n$, the equations for obtaining the coefficients w_is come through equalizing $Res(r)$ to zero at these points plus two boundary conditions as follows:

$$Res(r_j) = 0, \ j = 0, 1, \ldots, n-2, \tag{6.4}$$

$$P_n' w(r_0) = 0, \tag{6.5}$$

$$P_n w(r_n) = 0. \tag{6.6}$$

Using Theorem 4.1, we rewrite boundary conditions (6.5) and (6.6), respectively, as follows:

$$\sum_{i=0}^{n} (-1)^i \left(i^2 - 2\,in - i - n\right) \sqrt{2(n-i)+1}\, w_i = 0, \tag{6.7}$$

$$(n+1) w_n = 0. \tag{6.8}$$

Equations (6.4), (6.7) and (6.8) generate a set of $(n+1)$ nonlinear equations that can be solved by Newton's method for the unknown coefficients w_is.

7. Estimation of error function

In this section, an error estimation is presented for the orthonormal Bernstein approximate solution of equation (2.8). Let us call

$$E(r) = w(r) - P_n w(r), \tag{7.1}$$

as the error function of the orthonormal Bernstein approximation $P_n w(r)$ to $w(r)$, where $w(r)$ is the exact solution of (2.8)(we suppose that $w^* = w$). By substituting $P_n w(r)$ in (2.8) and (2.9), we can obtain the following equations:

$$P_n'' w(r) + \frac{1}{r} P_n' w(r) + Ha^2 \left[1 - \frac{P_n w(r)}{1 - \alpha P_n w(r)}\right] = R_n(r), \tag{7.2}$$

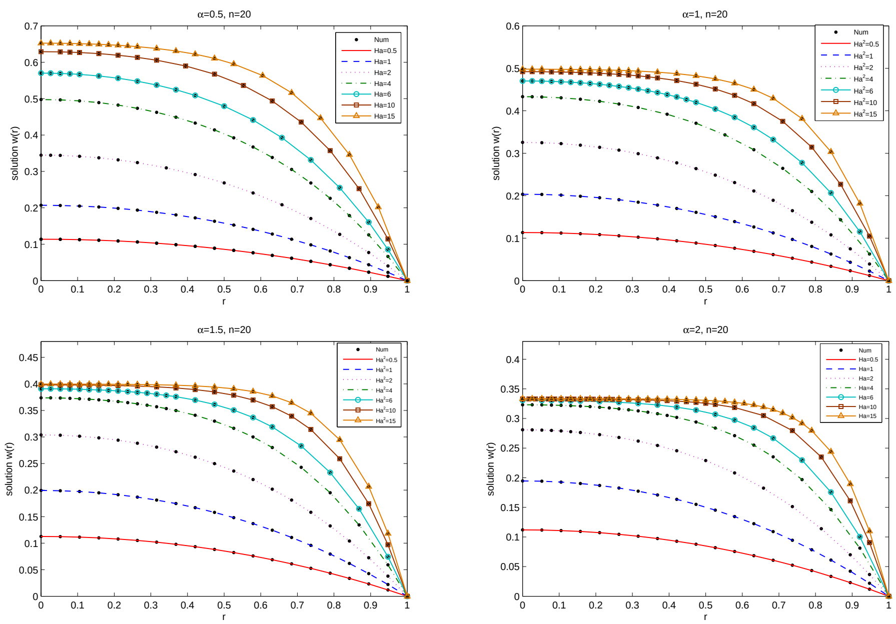
$$P_n' w(0) = 0, \ P_n w(1) = 0, \tag{7.3}$$

where $R_n(r)$ is the residual function associated with $P_n w(r)$. Subtracting (7.2) and (7.4) from (2.8) and (2.9), respectively, the error estimate function (EEF) $E(r)$ satisfies the equations

$$\begin{aligned} &E''(r) + \frac{1}{r} E'(r) - Ha^2 \left[\frac{E(r)}{(1 - \alpha P_n w(r))(1 - \alpha(E(r) + P_n w(r))}\right] = -R_n(r), \\ &E'(0) = 0, \ E(1) = 0. \end{aligned} \tag{7.4}$$

We can proceed to find an approximation of $P_n E(r)$ to the error function $E(r)$ in the same way as we did earlier for the solution to the problem in (2.8).

FIGURE 2. Plot of velocity function, with ($\alpha = 0.5, 1.0.1.5, 2$) and ($Ha^2 = 0.5, 1, 2, 4, 6, 10, 15$).



8. Results and discussion

In this section, we present the numerical results of the orthonormal Bernstein collocation method (OBCM) to solve the singular boundary value problem for the electrohydrodynamic flow of a fluid in an ion drag configuration in a circular cylindrical conduit. To demonstrate the applications of the proposed method, the results of OBCM have been compared with a numerical method based on "bvp5c" function using MATLAB software and the results reported by Ghasemi et al. [22]. It is to be noted that, all the OBCM results presented in this study are obtained using $n = 20$.

The axial velocity profile $w(r)$ for different values of the non-linearity parameter α and Hartmann electrical number Ha^2 are displayed in Figure 2- 4. We have reproduced two sets of profiles using the same set of parameters as employed by McKee [37] (small α that from 1 to 2 and large α that from 4 to 10) to confirm the efficiency of the new procedure. As it is suggested by Figure 2, the axial velocity increases with an increase in Hartmann electrical number Ha^2, all along the conduit for a fixed value of the non-linearity parameter α. Inspection of Figure 3 reveals that, for a fixed value of Hartmann electrical number Ha^2, when α increases, the fluid velocity $w(r)$ decreases. Furthermore, it is observed that this decrease is strong at the inlet $r = 0$.

The effects of large values of the non-linearity parameter α on the fluid velocity $w(r)$ are presented in Figure 4. Paullet [43] proved that solutions of (2.8) with boundary conditions (2.9) are bounded above by $1/(1 + \alpha)$ (See Theorem 2.1). So, in cases

Figure 3. Plot of velocity function, with (Ha^2 =1, 2, 6, 10, 20, 50, 75) and ($\alpha = 0.5, 1, 1.5, 2$).

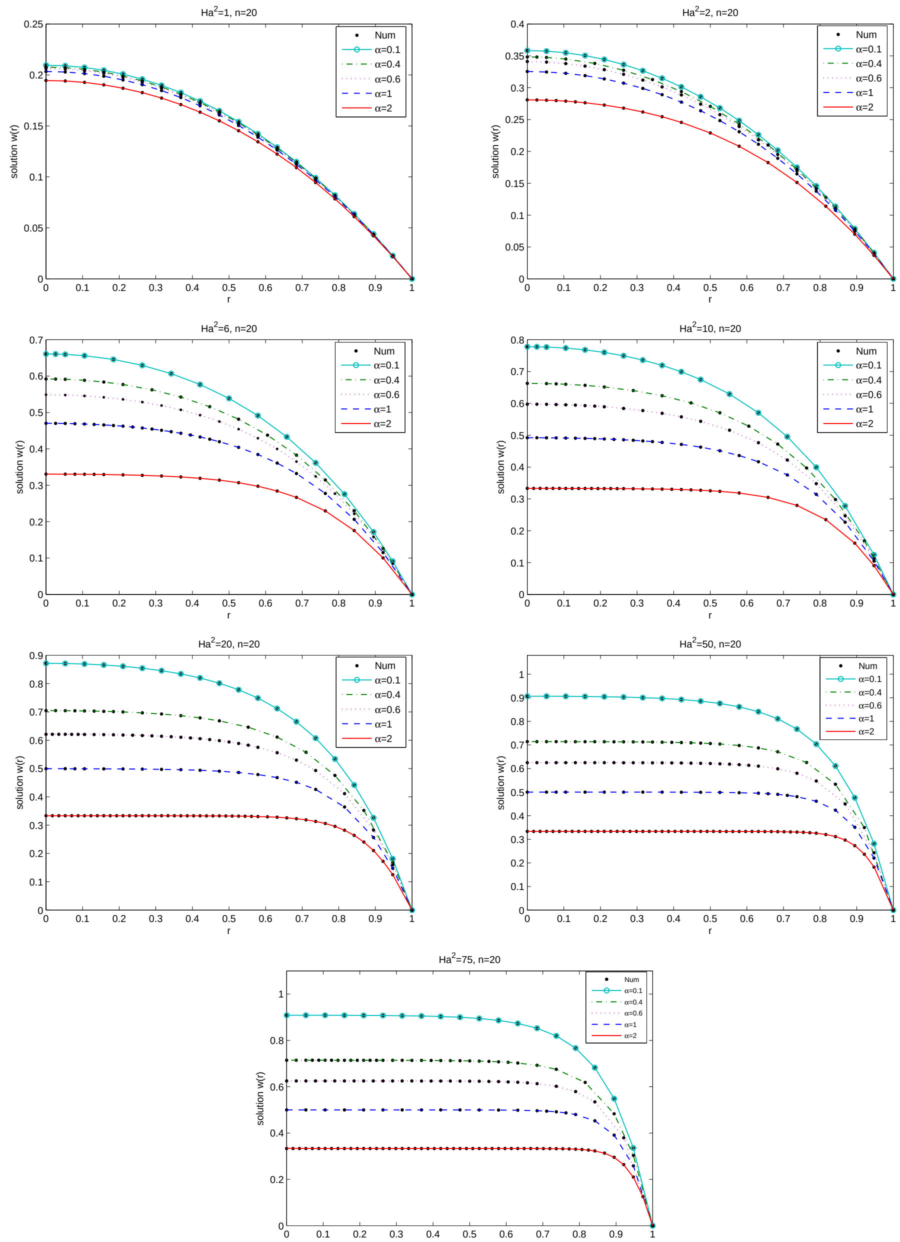


FIGURE 4. Plot of velocity function, with ($\alpha = 4, 8, 10$) and ($Ha^2 = 1, 10, 100$).

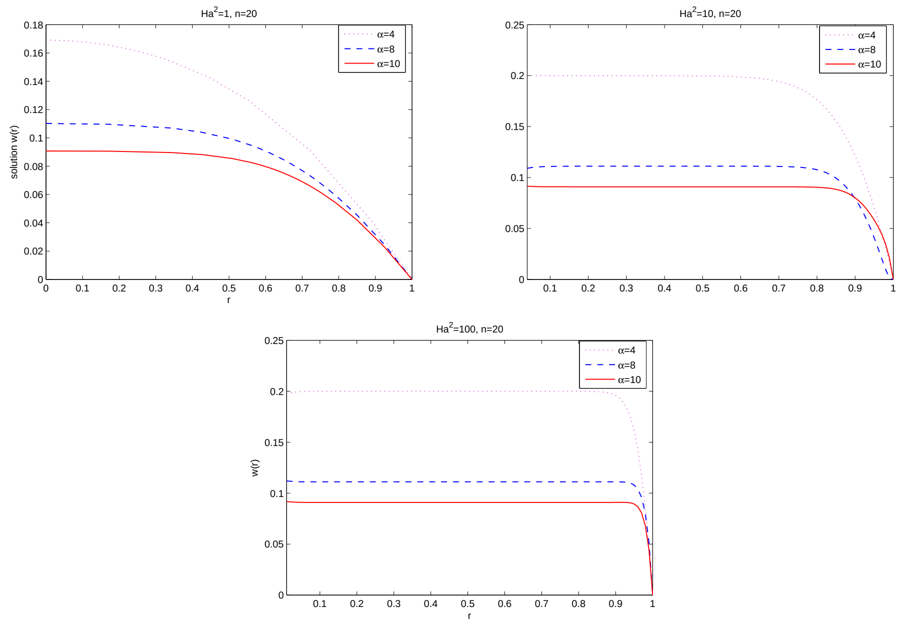
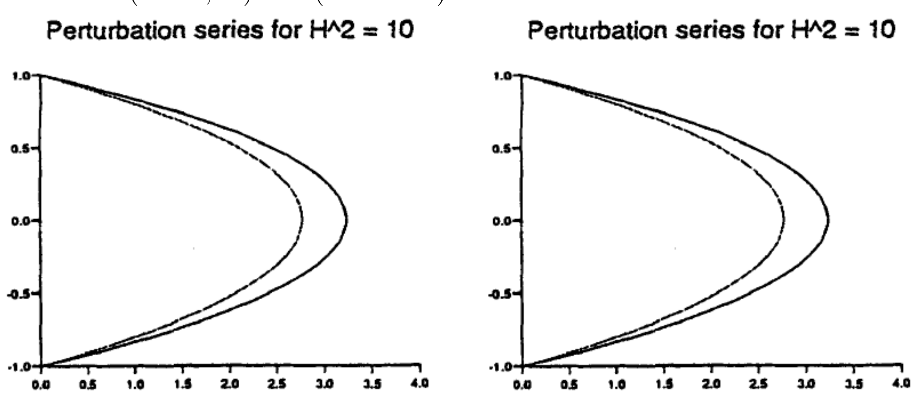
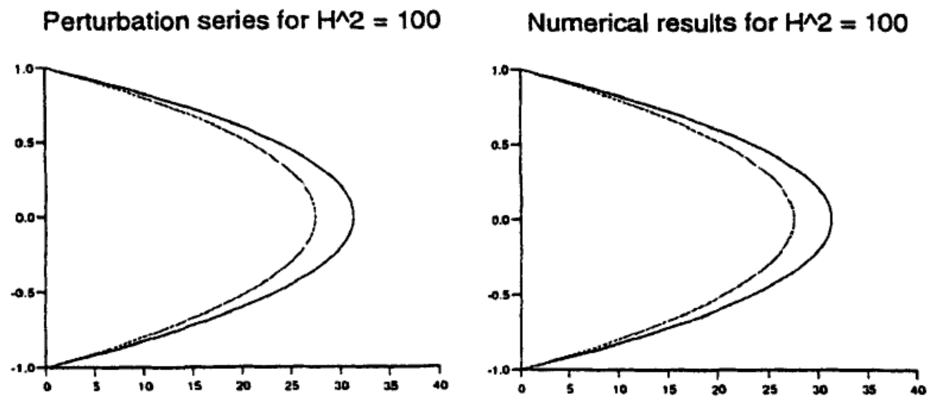


FIGURE 5. Plot of velocity function introduced by McKee in [37], with ($\alpha = 4, 10$) and ($Ha^2 = 10$).



$\alpha = 4, 8, 10$, the solutions are bounded above by $0.2, 0.\bar{1}, 0.\bar{0}\bar{9}$, respectively. Comparing the results obtained using OBCM in Figure 4 with the results achieved by the methods of McKee et al. [37] (See Figures 5 and 6) and Paullet [43], one can easily find that the present approximations support Paullet's numerical results for any $\alpha > 0$. As it

FIGURE 6. Plot of velocity function introduced by McKee in [37], with ($\alpha = 4, 10$) and ($Ha^2 = 100$).



is noted, because of the nonlinear term in the rational function form, the "bvp5c" Matlab code cannot solve differential equation (2.8) for the parameters in Figure 4.

Tables 1 and 2 display the numerical results for $w(0)$ at several values of Hartmann electrical number Ha^2 and the non-linearity parameter α. These tables also provide a comparison of the presented method, numerical method(NM) and the least square method (LSM) [22].

We compare the actual absolute error functions and the estimated absolute error functions for solution of (6.1) and some parameters of α, Ha^2 in Figure 7. In this Figure, $e(r) = |w(r) - \hat{w}(r)|$ shows the actual absolute error. Also, as it is suggested in Figure 7 the actual and the estimated absolute errors are almost the same. This comparison demonstrates the validity and applicability of the presented technique. Table 1 shows the comparison of LSM method in [22] with OBCM method in predicting $w(r)$. This table confirms that OBCM method has a lower absolute error than LSM method. Table 2 illustrates the $w(0)$'s values for the numerical method (NM) and the peresent method.

9. CONCLUSIONS

In this paper, we presented the orthonormal Bernstein collocation method (OBCM) to solve the singular nonlinear boundary value problem for the electrohydrodynamic flow of a fluid in an ion drag configuration in a circular cylindrical conduit. The OBCM results are compared with the results obtained by using the MATLAB solver bvp5c, and a very close accuracy is observed. The obtained results showed that the OBCM scheme overcame the singularity and non-linearity of the governing equation without any need to linearization tools. It should be noted that the proposed method in this work can be used for the solution of nonlinear differential equations in scientific and engineering fields.

FIGURE 7. Plot of error estimate functions (EEF) and absolute error for some parameters of α, Ha^2.

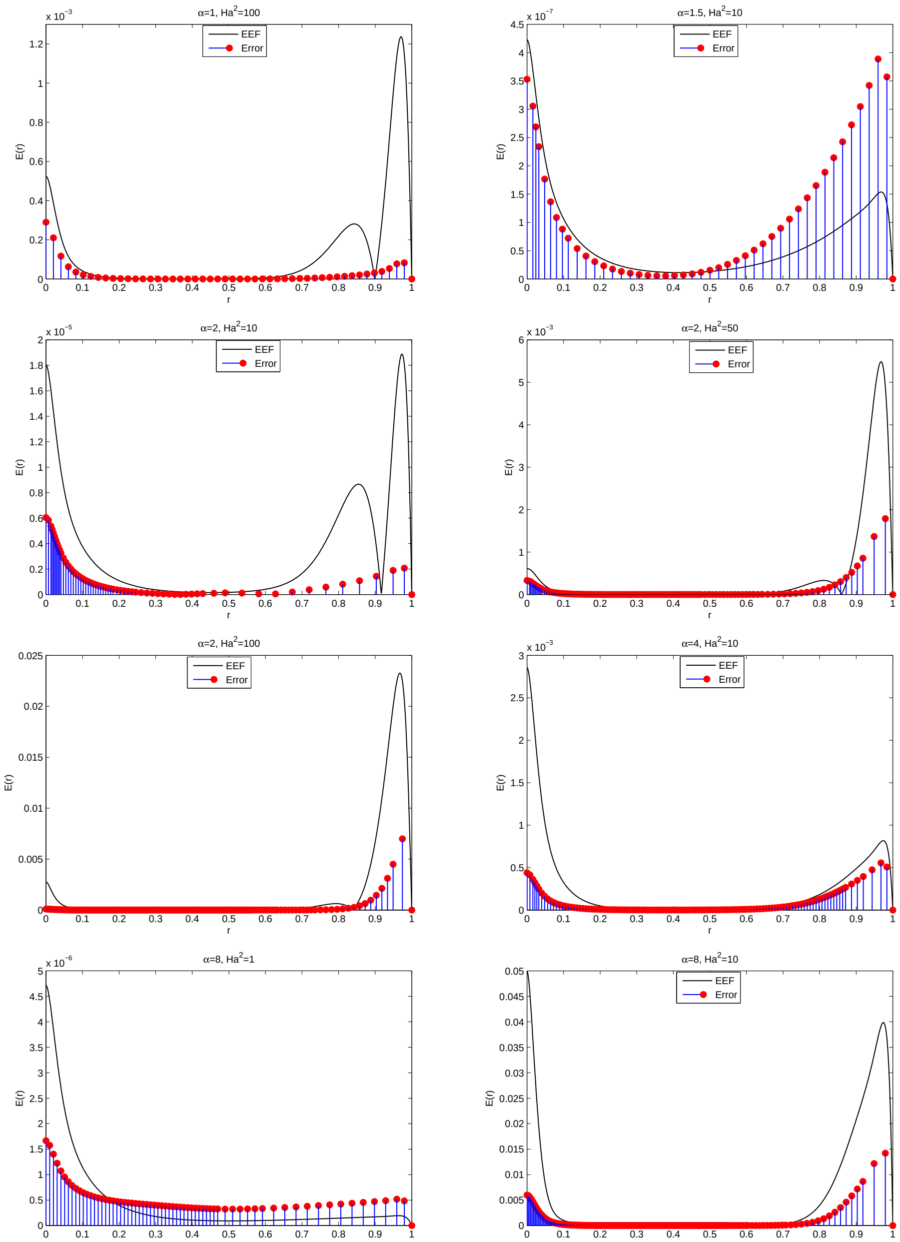


TABLE 1. Comparison of LSM [22] and present method in predicting $w(r)$ values on 20th order approximations

r	Error of LSM [22] $Ha^2 = 0.5$ $\alpha = 0.5$	$Ha^2 = 1$ $\alpha = 1$	Error of the OBCM $Ha^2 = 0.5$ $\alpha = 0.5$	$Ha^2 = 1$ $\alpha = 1$
0	$8.29922e-06$	$8.184222e-05$	$3.202287e-08$	$1.938379e-06$
0.1	$6.33578e-06$	$6.434110e-05$	$3.144691e-08$	$1.900423e-06$
0.2	$4.02863e-06$	$3.798456e-05$	$2.972748e-08$	$1.789965e-06$
0.3	$2.42051e-06$	$2.108585e-05$	$2.701783e-08$	$1.617795e-06$
0.4	$1.94360e-06$	$1.568848e-05$	$2.353848e-08$	$1.399656e-06$
0.5	$2.03781e-06$	$1.686197e-05$	$1.955166e-08$	$1.153631e-06$
0.6	$2.16222e-06$	$1.846308e-05$	$1.532665e-08$	$8.973791e-07$
0.7	$2.04545e-06$	$1.649501e-05$	$1.110608e-08$	$6.457852e-07$
0.8	$1.24095e-06$	$1.079317e-05$	$7.079086e-09$	$4.094229e-07$
0.9	$1.00024e-06$	$5.508480e-06$	$3.363817e-09$	$1.939062e-07$

TABLE 2. The value of $w(0)$ for various relevant parameters on 20th order approximations

Ha^2	$\alpha = 0.5$ NM	OBCM	$\alpha = 1$ NM	OBCM	$\alpha = 2$ NM	OBCM
1	0.20700807	0.20700815	0.20341385	0.20341579	0.19459639	0.19459641
2	0.34471303	0.34472730	0.32545491	0.32545427	0.28092892	0.28092892
6	0.57002550	0.57002550	0.47017920	0.47017854	0.33053379	0.33053209
10	0.62896071	0.62896071	0.49202969	0.49202968	0.33297071	0.33296462
20	0.66043310	0.66043311	0.49932528	0.499325101	0.33332506	0.33339417
50	0.66650870	0.66650608	0.49999536	0.499947628	0.33337147	0.33366547

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