Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 5, No. 3, 2017, pp. 232-245

The existence result of a fuzzy implicit integro-differential equation in semilinear Banach space

Masoumeh Zeinali
Faculty of mathematical sciences, University of Tabriz, Tabriz, Iran.
E-mail: zeinali@tabrizu.ac.ir

Abstract In this paper, the existence and uniqueness of the solution of a nonlinear fully fuzzy implicit integro-differential equation arising in the field of fluid mechanics is investigated. First, an equivalency lemma is presented by which the problem understudy is converted to the two different forms of integral equation depending on the kind of differentiability of the solution. Then, the conditions required to guarantee the existence of a solution for the equivalent integral equation areinvestigated using the Schauder fixed point theorem in semilinear Banach space.

Keywords. Implicit fuzzy integro-differential equation, Semilinear Banach space, Schauder fixed point theorem, Generalized differentiability.

2010 Mathematics Subject Classification. 45JXX, 94DXX.

1. Introduction

One of the interesting subjects related to fuzzy mathematics is differential and integral equations with fuzziness. Uncertainty of some parameters in the governing equations of engineering systems and fuzziness of the pertaining boundary or initial conditions arisen by the nature of such systems motivated researchers to study different kinds of fuzzy differential or integro-differential equations (see e.g. [1]-[5], [7]-[10], [15], [16], [19]).

Recently, different forms of fuzzy integro-differential equations have been studied theoretically and numerically. There is considerable amount of study dealing with the numerical solution of fuzzy integro-differential equations (see e.g. [19, 3, 4]). At the same time, significant investigations can be found in the literature on the existence results of such equations. The existence and uniqueness results for nonlinear form of fuzzy integro-differential equations has been done by Balasubramaniam et al. [8] using Banach fixed point theorem. Afterward, the same authors studied the existence and uniqueness result of semilinear fuzzy integro-differential equations with nonlocal initial condition via Banach fixed point theorem [7]. After introducrtion of generalized differentiability of fuzzy functions by Bede in [9], fuzzy integro-differential equations nuder generalized differentiability gained much attention of researchers (see e.g. [2, 5]). All researches in the existence and uniqueness of solution for fuzzy integro-differential equations are by the help of various fixed point theorems. A generalization

Received: 23 February 2017 ; Accepted: 3 July 2017.

of Schauder fixed point theorem for semilinear Banach space has been done by Agarwal et al. [1]. Since $\mathbb{R}_{\mathcal{F}}$ is a semilinear Banach space, this theorem can be used in fuzzy differential equations. The advantage of this fixed point theorem is that it's required conditions are weaker than Banach fixed point theorem. In this work, this theorem is used to prove the existence solution for the equation understudy.

Here, an implicit kind of fuzzy integro-differential equation is considered. Throughout this work, the term "implicit" is used for the typical nonlinear equation $G(x, h(x), h'(x), ..., h^{(n)}(x)) = 0$ which is not convertible to $h^{(n)}(x) = H(x, h(x), h'(x), ..., h^{(n-1)}(x))$. In spite of extensive study dealing with the explicit fuzzy integro-differential equations, it is difficult to find an investigation about the so called implicit kind of such equations. This kind of equations appear frequently in different engineering problems such particle dynamic in fluid mechanics (see e.g. [6, 13, 17]). Recently, the existence and uniqueness of solution for the crisp kind of this equation has been studied in [14]. In the present work, our interest goes to the fully fuzzy form of the same equation in which all the pertaining parameters, functions and initial conditions include uncertainty based on their physical concepts.

This paper is organized as follows: In section 2, some elementary definitions and theorems are given which are necessary for the remaining parts of the paper. The existence results for the problem under study is presented in section 3. The problem is studied in two separate cases including (i)-differentiability (case 1) and (ii)-differentiability (case 2) of the solution. Finally in the section 4, the uniqueness results are discussed.

2. Preliminaries

In the current section, some elementary definitions and theorems are stated which will be required in the remaining parts of this paper.

Definition 2.1. (see [12].) A fuzzy number is a fuzzy convex and normal fuzzy subset in $\mathbb{R}$ with upper semicontinuous membership function and compact support. The set of all fuzzy numbers is denoted by $\mathbb{R}_{\mathcal{F}}$.

For $0 \leq r \leq 1$, the r-cut of fuzzy number u is defined as

$$[u]^r = \begin{cases} \{x \in \mathbb{R} : u(x) \geq r\}, & 0 < r \leq 1, \\ \overline{\{x \in \mathbb{R} : u(x) > 0\}}, & r = 0. \end{cases}$$

It is clear that u is a fuzzy number if and only if it's r-cuts are closed and bounded intervals and $[u]^1 \neq \emptyset$ (see [11]).

Definition 2.2. (See [9].) Let $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$. The sum, H-difference and scalar product are defined levelwise as follows:

$$[u + v]^r = [u]^r + [v]^r,$$
$$[u \ominus v]^r = [u^r_- - v^r_-, u^r_+ - v^r_+],$$
$$[\lambda.u]^r = \lambda[u]^r,$$

where $[u]^r = [u^r_-, u^r_+]$ and $[v]^r = [v^r_-, v^r_+]$.

The space $\mathbb{R}_{\mathcal{F}}$ can be metricized by the metric $D(u,v) = sup_{r\in[0,1]} max\{|u_-^r - v_-^r|, |u_+^r - v_+^r|\}$, where $[u]^r = [u_-^r, u_+^r]$ and $[v]^r = [v_-^r, v_+^r]$ are r-cuts of $u, v \in \mathbb{R}_{\mathcal{F}}$. It is well known that $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space with the following properties:
(i) $D(u+v, u+w) = D(v,w)$,
(ii) $D(k.u, k.v) = |k|D(u,v)$,
(iii) $D(u+v, w+e) \leq D(u,w) + D(v,e)$.
Let $\mathbb{R}_{\mathcal{F}}^c$ be the space of fuzzy numbers $u \in \mathbb{R}_{\mathcal{F}}$ such that the function $\alpha \to [u]^\alpha$ is continuous. It is well known that $(\mathbb{R}_{\mathcal{F}}^c, D)$ is a complete metric space. Consequently, $(C([a,b], \mathbb{R}_{\mathcal{F}}^c), d)$ is a complete metric space where

$$d(y,z) = \sup_{t\in[a,b]} D(y(t), z(t)).$$

Definition 2.3. The subset $A \subseteq \mathbb{R}_{\mathcal{F}}^c$ is called compact supported if for some compact set $K \subseteq \mathbb{R}$ we have

$$[y]^0 \subseteq K, \quad \forall y \in A.$$

Definition 2.4. Let $A \subseteq \mathbb{R}_{\mathcal{F}}^c$ and $\alpha_0 \in [0,1]$. The subset A is called level-equicontinuous at α_0 if for every $\epsilon > 0$ there exists $\delta > 0$ such that:

$$|\alpha - \alpha_0| < \delta \Rightarrow d_H([y]^\alpha, [y]^{\alpha_0}) < \epsilon, \quad \forall y \in A,$$

where d_H is the Hausdorff distance

$$d_H([y]^\alpha, [y]^{\alpha_0}) = max\{|y_-^\alpha - y_-^{\alpha_0}|, |y_+^\alpha - y_+^{\alpha_0}|\}.$$

Theorem 2.5. *(see [1]) Assume A is a compact-supported subset of $\mathbb{R}_{\mathcal{F}}^c$. Then, A is relatively compact subset of $(\mathbb{R}_{\mathcal{F}}^c, D)$ if and only if A is level-equicontinuous on $[0,1]$.*

Definition 2.6. (see [18]) Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$. The function f is called Riemann integrable if for every $\epsilon > 0$ there exists $\delta > 0$ such that for any partition Δ_n : $a = x_0 < x_1 < ... < x_n = b$ of $[a,b]$ with $\max\limits_{1\leq i\leq n}(x_i - x_{i-1}) < \delta$ and for any $\psi_i \in [x_{i-1}, x_i], \forall i = 1,..,n$, we have

$$D\left(\sum_{i=1}^{n}(x_i - x_{i-1}).f(\psi_i), A\right) \leq \epsilon,$$

for some $A \in \mathbb{R}_{\mathcal{F}}$. Then, A is called Riemann integral of f and it is denoted by $A = \int_a^b f(t)dt$.

Definition 2.7. (see [9].) The function $f : (a,b) \to \mathbb{R}_{\mathcal{F}}$ is called strongly generalized differentiable at $x_0 \in (a,b)$ if there exists $f'(x_0) \in \mathbb{R}_{\mathcal{F}}$ such that for sufficiently small $h > 0$
(i) there exist H-differences $f(x_0+h) \ominus f(x_0)$ and $f(x_0) \ominus f(x_0-h)$ such that

$$\lim_{h\searrow 0} D\left(\frac{f(x_0+h)\ominus f(x_0)}{h}, f'(x_0)\right) = \lim_{h\searrow 0} D\left(\frac{f(x_0)\ominus f(x_0-h)}{h}, f'(x_0)\right) = 0.$$

or
(ii) there exist H-differences $f(x_0) \ominus f(x_0+h)$ and $f(x_0-h) \ominus f(x_0)$ such that

$$\lim_{h\searrow 0} D\left(\frac{f(x_0)\ominus f(x_0+h)}{-h}, f'(x_0)\right) = \lim_{h\searrow 0} D\left(\frac{f(x_0-h)\ominus f(x_0)}{(-h)}, f'(x_0)\right) = 0.$$

or

(iii) there exist H-differences $f(x_0+h)\ominus f(x_0)$ and $f(x_0-h)\ominus f(x_0)$ such that

$$\lim_{h\searrow 0} D\left(\frac{f(x_0+h)\ominus f(x_0)}{h}, f'(x_0)\right) = \lim_{h\searrow 0} D\left(\frac{f(x_0-h)\ominus f(x_0)}{-h}, f'(x_0)\right) = 0.$$

or

(iv) there exist H-differences $f(x_0)\ominus f(x_0+h)$ and $f(x_0)\ominus f(x_0-h)$ such that

$$\lim_{h\searrow 0} D\left(\frac{f(x_0)\ominus f(x_0+h)}{-h}, f'(x_0)\right) = \lim_{h\searrow 0} D\left(\frac{f(x_0)\ominus f(x_0-h)}{h}, f'(x_0)\right) = 0.$$

Lemma 2.8. *(see [9]) Let $x_0 \in \mathbb{R}$ and $f : \mathbb{R}\times\mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ be continuous. Then, the fuzzy differential equation $y' = f(x, y), y(x_0) = y_0 \in \mathbb{R}_{\mathcal{F}}$ is equivalent to one of the integral equations:*

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt, \forall x \in [x_0, x_1],$$

if y is considered to be (i)-differentiable, or

$$y_0 = y(x) + (-1)\int_{x_0}^{x} f(t, y(t))dt, \forall x \in [x_0, x_1],$$

if y is considered to be (ii)-differentiable on some interval $(x_0, x_1) \subset \mathbb{R}$.

Lemma 2.9. *(see [18]) Let $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ be continuous. Then $F(t) = \int_a^t f(s)ds$ is (i)-differentiable and $F'(t) = f(t)$.*

In the remaining part of the present section, semilinear Banach spaces and Schauder fixed point theorem in these spaces are introduced which has been proven by Agarwal et al. [1].

Definition 2.10. (see [1]) A semilinear space S equipped with a metric $d : S \times S \to \mathbb{R}_+$ is called semilinear metric space if

- d is translation invariant, that is, $d(u+v, w+v) = d(u, w)$ for any $u, v, w \in S$,
- d is positively homogeneous, that is, $d(\lambda u, \lambda w) = \lambda d(u, w)$ for any $u, w \in S$ and $\lambda \geq 0$.

In this space, a norm is defined by $\|x\| = d(x, 0)$. S is called a semilinear Banach space if it be simultaneously a semiliniear and complete metric space. Therefore, although the set of real fuzzy numbers is not a Banach space, it can be a semiliniear Banach space because it is both semilinear and complete metric space.

Theorem 2.11. *([1], Schauder Fixed point theorem for semilinear spaces) Let B be a nonempty, closed, bounded and convex subset of a semilinear Banach space S having the cancelation property, and suppose that $P : B \to B$ is a compact operator. Then P has at least one fixed point in B.*

Remark 2.12. The space of fuzzy numbers $\mathbb{R}^c_{\mathcal{F}}$ is a semilinear Banach space having the cancelation property. Therefore, the Schauder fixed point theorem holds true for $\mathbb{R}^c_{\mathcal{F}}$.

3. Problem description

The present section is devoted to study the following implicit form of fuzzy integro-differential equation with initial condition

$$h'(t) = g\big(t, h(t)\big) + \int_0^t H\Big(t, s, h(s), h'(s)\Big)ds, \quad t \in I = [0, t_f], \tag{3.1}$$

$$h(0) = h_0, \tag{3.2}$$

where g and H are continuous functions and h_0 is a fuzzy number. It is worth to mention that this equation is general form of the equation governing the motion of a particle submerged in a viscous fluid medium including some uncertain parameters.

Based on the kind of generalized differentiability of the solution, the existence result of the initial value problem (3.1)-(3.2) is investigated in two separate forms: (i)-differentiable solution (case 1) and (ii)-differentiable solution (case 2).

Case 1. Suppose that h is (i)-differentiable. The following lemma converts the initial value problem (3.1)-(3.2) into an equivalent integral equation. This lemma allows us to use the equivalent integral equation instead of the initial value problem in our theoretical investigations.

Lemma 3.1. *Consider the nonlinear integral equation*

$$v(t) = g\Big(t, h_0 + \int_0^t v(z)dz\Big) + \int_0^t H\Big(t, s, h_0 + \int_0^s v(z)dz, v(s)\Big)ds. \tag{3.3}$$

The equation (3.3) has a unique continuous solution v if and only if (3.1)-(3.2) has a unique (i)-differentiable solution h such that

$$h(t) = h_0 + \int_0^t v(z)dz.$$

Proof. First, it is shown that the existence of a (i)-differentiable solution to (3.1)-(3.2) is equivalent to the existence of a continuous solution to (3.3). Suppose that (3.1)-(3.2) has a (i)-differentiable solution h. Let me to define

$$v(t) := h'(t). \tag{3.4}$$

Considering Lemma 2.8, the equality (3.4) along with the initial condition $h(0) = h_0$ is equivalent to

$$h(t) = h_0 + \int_0^t v(z)dz. \tag{3.5}$$

Replacing (3.4) and (3.5) in (3.1), it holds

$$v(t) = g\Big(t, h_0 + \int_0^t v(z)dz\Big) + \int_0^t k\Big(t, s, h_0 + \int_0^s v(z)dz, v(s)s\Big)ds.$$

Thus the integral equation (3.3) has a solution v. To prove the inverse, suppose that (3.3) has a continuous solution v and let

$$h(t) := h_0 + \int_0^t v(z)dz. \tag{3.6}$$

Since v is continuous, from Lemma 2.9, $h(t)$ is (i)-differentiable and

$$h'(t) = v(t). \tag{3.7}$$

Replacing (3.6) and (3.7) in (3.3), it holds

$$h'(t) = g\Big(t, h(t)\Big) + \int_0^t H\Big(t, s, h(s), h'(s)\Big)ds. \tag{3.8}$$

Furthermore, (3.6) leads to $h(0) = h_0$. This fact along with (3.8) mean that (3.1)-(3.2) has a (i)-differentiable solution h.

Now, it is shown that the uniqueness of the solution to (3.1)-(3.2) is equivalent to the uniqueness of the solution to (3.3). In a proof by contradiction, it is assumed that (3.1)-(3.2) has a unique (i)-differentiable solution h and (3.3) has two different solutions v_1 and v_2. From (3.6), the functions $h_1(t) = h_0 + \int_0^t v_1(z)dz$ and $h_2(t) = h_0 + \int_0^t v_2(z)dz$ are solutions of (3.1)-(3.2). Uniqueness of the solution of the initial value problem (3.1)-(3.2) implies that $h_1(t) = h_2(t)$, and therefore, $h_1'(t) = h_2'(t)$. Thus, (3.7) leads to $v_1(t) = v_2(t)$. Inversely, let (3.3) has a unique continuous solution v and (3.1)-(3.2) has two different solutions h_1 and h_2. From (3.4), $v_1(t) = h_1'(t)$ and $v_2(t) = h_2'(t)$ are the solutions of (3.3). Since this equation has a unique solution, it can be concluded $v_1 = v_2$. Taking integral from both sides of this equality and considering the fact that h_1 and h_2 are (i)-differentiable, it holds

$$h_0 + \int_0^t v_1(s)ds = h_0 + \int_0^t v_2(s)ds \quad \Rightarrow \quad h_1(t) = h_2(t),$$

which completes the proof. □

Theorem 3.2. *Let $R > 0$ and*

$$G_0 := \Big\{(t,s)|t \in [0,t_f], s \in [0,t]\Big\},$$

$$G_1 := \Big\{(t,h) \in [0,t_f] \times \mathbb{R}_{\mathcal{F}}^c : D(h,\hat{0}) \le R\Big\},$$

$$G_2 := \Big\{(t,s,h,w) \in [0,t_f] \times [0,t_f] \times \mathbb{R}_{\mathcal{F}}^c \times \mathbb{R}_{\mathcal{F}}^c : D(h,\hat{0}) \le R \ \&\ D(w,\hat{0}) \le \|h_0\| + t_f R\Big\}.$$

If

(i) *g is continuous and compact on G_1 and, $M_g = \sup_{(t,h)\in G_1} D(g(t,h),\hat{0})$,*

(ii) *H is continuous and compact on G_2 and, $M_H = \sup_{(t,s,h,v)\in G_2} D(H(t,s,h,v),\hat{0})$,*

(iii) *$M_g + t_f M_H \le R$.*

Then (3.1)-(3.2) has at least one global (i)-differentiable solution.

Proof. Define the operator $\Phi : \Lambda \to \Lambda$ as

$$\Phi v(t) = g\Big(t, h_0 + \int_0^t v(z)dz\Big) + \int_0^t H\Big(t, s, h_0 + \int_0^s v(z)dz, v(s)\Big)ds,$$

where

$$\Lambda := \Big\{v \in C([0,t_f], \mathbb{R}_{\mathcal{F}}^c); d(v,\hat{0}) \le R\Big\}.$$

From Lemma 3.1, $v \in C([0,t_f], \mathbb{R}_{\mathcal{F}}^c)$ is a solution of (3.3) if and only if $h = h_0 + \int_0^t v(z)dz$ is a (i)-differentiable solution of (3.1)-(3.2). So, to prove (3.1)-(3.2) has a (i)-differentiable solution, it is enough to prove (3.3) has a continuous solution. Clearly, a fixed point of Φ is a solution of (3.3). Thus, in order to show the existence of a solution of (3.3), it is enough to show Φ has a fixed point.

At the first, it is needed to prove Φ is well-defined. To this end, suppose that $v \in \Lambda$ and $t_1, t_2 \in [0, t_f]$ $(t_1 < t_2)$

$$D\big(\Phi v(t_1), \Phi v(t_2)\big) \leq D\Big(g\big(t_1, h_0 + \int_0^{t_1} v(z)dz\big), g\big(t_2, h_0 + \int_0^{t_2} v(z)dz\big)\Big) +$$
$$\int_0^{t_1} D\Big(H\big(t_1, s, h_0 + \int_0^s v(z)dz, v(s)\big), H\big(t_2, s, h_0 + \int_0^s v(z)dz, v(s)\big)\Big)ds$$
$$+ \int_{t_1}^{t_2} D\Big(\hat{0}, H\big(t_2, s, h_0 + \int_0^s v(z)dz, v(s)\big)\Big)ds.$$

Since g and H are continuous and H is bounded, it can be easily concluded that $t_1 \to t_2$ implies $D(\Phi v(t_1), \Phi v(t_2)) \to 0$. Thus $\Phi v \in C([0, t_f], \mathbb{R}^c_{\mathcal{F}})$. Now, to complete the proof of well-definedness of ϕ, it is required and sufficient to show $d(\Phi v, \hat{0}) \leq R$. From the conditions (i), (ii) and (iii), it follows

$$\begin{aligned} D\big(\Phi v(t), \hat{0}\big) \leq & D\Big(g\big(t, h_0 + \int_0^t v(z)dz\big), \hat{0}\Big) + \int_0^t D\Big(H\big(t, s, h_0 + \int_0^s v(z)dz, v(s)\big), \hat{0}\Big)ds \\ \leq & M_g + \int_0^t M_H ds \leq M_g + t_f M_H \leq R. \end{aligned}$$

Therefore,

$$d(\Phi v, \hat{0}) = \sup_{t \in [0, t_f]} D(\Phi v(t), \hat{0}) \leq R.$$

Thus, Φ maps Λ to itself.

Now, it is demonstrated that Φ is a compact operator. Since the processes of this proof is long, in order to avoid confusion and to clarify the proof procedure, the following flowchart is proposed. In this flowchart, each rectangular contains an assertion which is proved by proving all the next assertions inside the different rectangulars. Some assertions are demonstrated directly and some others are proven by using the theorems which are mentioned in the middle of lines connecting the rectangulars.

By the definition of compact operator, it is enough to demonstrate $\Phi(\Lambda)$ is relatively compact. According to Arzela-Ascoli theorem, relatively compactness of $\Phi(\Lambda)$ is satisfied if $\Phi(\Lambda)$ is equicontinuous and $\Phi(\Lambda)(t)$ is relatively compact for each $t \in [0, t_f]$.

In order to prove equicontinuity of $\Phi(\Lambda)$, let $v \in \Lambda$ and ϵ is given. Since g is continuous, there exists some $\delta_1 > 0$ such that for $(t_1, \nu_1), (t_2, \nu_2) \in [0, t_f] \times \mathbb{R}_{\mathcal{F}}$

$$\max\Big\{|t_1 - t_2|, D(\nu_1, \nu_2)\Big\} \leq \delta_1 \quad \rightarrow \quad D\Big(g(t_1, \nu_1), g(t_2, \nu_2)\Big) \leq \frac{\epsilon}{3}. \tag{3.9}$$

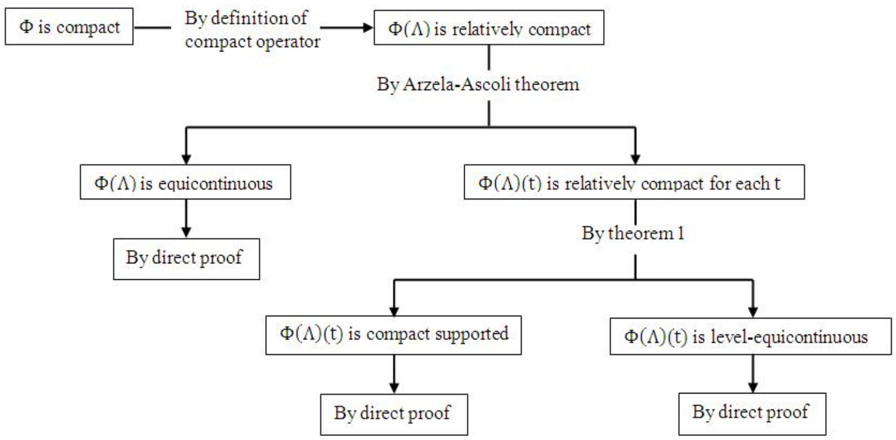
By a simple calculation, for $|t_1 - t_2| \leq \delta_1/R$, it can be proved that

$$D\Big(h_0 + \int_0^{t_1} v(z)dz, h_0 + \int_0^{t_2} v(z)dz\Big) \leq \delta_1. \tag{3.10}$$

By combining (3.9) and (3.10), if $|t_1 - t_2| \leq \delta_2$ in which $\delta_2 = \min\{\delta_1, \delta_1/R\}$ then

$$D\bigg(g\Big(t_1, h_0 + \int_0^{t_1} v(z)dz\Big), g\Big(t_2, h_0 + \int_0^{t_2} v(z)dz\Big)\bigg) \leq \frac{\epsilon}{3}. \tag{3.11}$$

FIGURE 1. The flowchart representing procedure for proving " Φ is compact".



Furthermore, since H is continuous, there exists some $\delta_3 > 0$ such that $|t_1 - t_2| \leq \delta_3$ implies

$$D\Big(H\Big(t_1, s, h_0 + \int_0^s v(z)dz, v(s)\Big), H\Big(t_2, s, h_0 + \int_0^s v(z)dz, v(s)\Big)\Big) \leq \frac{\epsilon}{3t_f}. \tag{3.12}$$

Let $\delta = \min\{\delta_2, \delta_3, \frac{\epsilon}{3M_H}\}$ and $|t_1 - t_2| \leq \delta$. Then, (3.11) and (3.12) yields

$$\begin{aligned} &D\big(\Phi v(t_1), \Phi v(t_2)\big) \leq D\Big(g\Big(t_1, h_0 + \int_0^{t_1} v(z)dz\Big), g\Big(t_2, h_0 + \int_0^{t_2} v(z)dz\Big)\Big) \\ &+ \int_0^{t_1} D\Big(H\Big(t_1, s, h_0 + \int_0^s v(z)dz, v(s)\Big), H\Big(t_2, s, h_0 + \int_0^s v(z)dz, v(s)\Big)\Big)ds \\ &+ \int_{t_1}^{t_2} D\Big(\hat{0}, H\Big(t_2, s, h_0 + \int_0^s v(z)dz, v(s)\Big)\Big)ds \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \end{aligned}$$

which means $\Phi(\Lambda)$ is equicontinuous.

In order to prove relatively compactness of $\Phi(\Lambda)(t)$, based on theorem 2.5 it is enough to prove $\Phi(\Lambda)(t)$ is compact supported and level-equicontinuous. To this end, let me to define the set $\hat{\Lambda}$ as

$$\hat{\Lambda} := \Big\{w \in C([0, t_f], \mathbb{R}^c_{\mathcal{F}}) \mid w(t) = h_0 + \int_0^t v(z)dz \ \&\ v \in \Lambda\Big\}.$$

Clearly $\hat{\Lambda}$ is bounded. By using compactness of g, one can concluded $g([0, t_f], \hat{\Lambda})$ is relatively compact and consequently by theorem 2.5, it is level-equicontinuous. By a similar reasoning, $H(G_0, \hat{\Lambda}, \Lambda)$ is level-equicontinuous as well. So, for $\epsilon > 0$ there exists $\delta > 0$ such that $|\alpha - \beta| < \delta$ implies

$$d_H\Big(\Big[g\big(t, h_0 + \int_0^t v(z)dz\big)\Big]^\alpha, \Big[g\big(t, h_0 + \int_0^t v(z)dz\big)\Big]^\beta\Big) \leq \frac{\epsilon}{2} \qquad \forall t \in [0, t_f], v \in \Lambda \tag{3.13}$$

and for all $(t, s) \in G_0$ and $v \in \Lambda$

$$d_H\Big(\Big[H\big(t, s, h_0 + \int_0^s v(z)dz, v(s)\big)\Big]^\alpha, \Big[H\big(t, s, h_0 + \int_0^s v(z)dz, v(s)\big)\Big]^\beta\Big) \leq \frac{\epsilon}{2t_f}. \tag{3.14}$$

Hence, from (3.13) and (3.14)

$$\begin{aligned}
&d_H\Big(\Big[\Phi(\Lambda)(t)\Big]^\alpha, \Big[\Phi(\Lambda)(t)\Big]^\beta\Big) \leq d_H\Big(\Big[g\big(t, h_0 + \int_0^t v(z)dz\big)\Big]^\alpha, \Big[g\big(t, h_0 + \int_0^t v(z)dz\big)\Big]^\beta\Big) \\
&+ \int_0^t d_H\Big(\Big[H\big(t, s, h_0 + \int_0^s v(z)dz, v(s)\big)\Big]^\alpha, \Big[H\big(t, s, h_0 + \int_0^s v(z)dz, v(s)\big)\Big]^\beta\Big)ds \\
&\leq \frac{\epsilon}{2} + \int_0^t \frac{\epsilon}{2t_f}ds \leq \epsilon.
\end{aligned}$$

Therefor, $\Phi(\Lambda)(t)$ is level-equicontinuous in $\mathbb{R}^c_{\mathcal{F}}$. Finally, it is necessary to show $\Phi(\Lambda)(t)$ is compact-supported. Let $v \in \Lambda$, then

$$\begin{aligned}
\big[\Phi(v)(t)\big]^0 &= g\Big(t, [h_0]^0 + \int_0^t [v(s)]^0 ds\Big) \\
&\quad + \int_0^t H\Big(t, s, [h_0]^0 + \int_0^s [v(s)]^0 ds, [v(s)]^0\Big)ds \\
&\subseteq g\Big([0, t_f], [\hat{\Lambda}]^0\Big) + \int_0^t H\Big(G_0, [\hat{\Lambda}]^0, [\Lambda]^0\Big)ds,
\end{aligned}$$

where

$$[\hat{\Lambda}]^0 = \Big\{[v]^0 | v \in \hat{\Lambda}\Big\} \qquad \& \qquad [\Lambda]^0 = \Big\{[v]^0 | v \in \Lambda\Big\}.$$

Since Λ and $\hat{\Lambda}$ are bounded, it follows that $[\Lambda]^0$ and $[\hat{\Lambda}]^0$ are bounded. Considering compactness of g and H, there exist compact sets K_1 and K_2 such that $g\big([0, t_f], [\hat{\Lambda}]^0\big) \subseteq K_1$ and $H\big(G_0, [\hat{\Lambda}]^0, [\Lambda]^0\big) \subseteq K_2$. Hence $\Phi(\Lambda)(t)$ is compact supported. Therefore, ϕ is a compact operator.

Now theorem 2.11 shows that the equation (3.3) has at least one solution v. From Lemma 3.1, the initial value problem (3.1)-(3.2) has at least one (i)-differentiable solution h and

$$h(x) = h_0 + \int_0^t v(z)dz.$$

□

Case 2. Suppose that h is (ii)-differentiable. In the same spirit of case 1, in order to facilitate the investigation of existence and uniqueness results the following Lemma is proposed which converts the initial value problem (3.1)-(3.2) to an equivalent integral equation under the assumption of (ii)-differentiability of the solution.

Lemma 3.3. *Consider the following nonlinear integral equation*

$$v(t) = g\Big(t, h_0 \ominus (-1)\int_0^t v(z)dz\Big) + \int_0^t H\Big(t, s, h_0 \ominus (-1)\int_0^s v(z)dz, v(s)\Big)ds \quad (3.15)$$

The equation (3.15) has a continuous unique solution v if and only if (3.1)-(3.2) has a unique (ii)-differentiable solution h, such that

$$h(x) = h_0 \ominus (-1)\int_0^t v(z)dz.$$

Proof. Similar to Lemma 3.3. □

The most challenge in this case is the existence of H-difference $h_0 \ominus (-1)\int_0^t v(z)dz$. Part **(iv)** of the following theorem ensures the existence of the H-difference.

Theorem 3.4. *Let $R > 0$ and G_0, G_1 and G_2 are the same as in theorem 3.2. Consider the assumption* **(i)**-**(iii)** *of theorem 3.2 along with the following assumption,* **(iv)** *Let there exist the constants $C_1 > 0$, $C_2 < 0$, $D_1 > 0$ and $D_2 < 0$ such that*

$$\frac{\partial}{\partial r}{h_0}^r_- \geq C_1, \qquad \frac{\partial}{\partial r}{h_0}^r_+ \leq C_2,$$
$$\frac{\partial}{\partial r}v^r_-(t) \leq D_1, \qquad \frac{\partial}{\partial r}v^r_+(t) \geq D_2, \qquad \forall t \in [0, t_f]$$

and for all $t \in [0, t_f]$ it holds that $v^1_-(t) = v^1_+(t)$. Then, the initial value problem (3.1)-(3.2) has at least one local (ii)-differentiable solution.

Proof. In a similar way of Lemma 2.2 of ref. [10], I have to show for all $r \in [0, 1]$, $\big[{h_0}^r_- + \int_0^t v^r_+(t)dt, {h_0}^r_+ + \int_0^t v^r_-(t)dt\big]$ are r-cuts of a fuzzy number. This sentence is equivalent to show

$${h_0}^1_- + \int_0^t v^1_+(t)dt \leq {h_0}^1_+ + \int_0^t v^1_-(t)dt,$$
$${h_0}^r_- + \int_0^t v^r_+(t)dt \text{ is increasing with respect to } r,$$
$${h_0}^r_+ + \int_0^t v^r_-(t)dt \text{ is decreasing with respect to } r.$$

It can be proven that the above assertions hold true for all $t \in [0, h]$, in which $h = \min\{-\frac{C_1}{D_2}, -\frac{C_2}{D_1}\}$. For details of proof see Lemma 2.2 of ref. [10]. I use this h in the definition of the Λ i.e.

$$\Lambda := \Big\{v \in C([0, h], \mathbb{R}^c_{\mathcal{F}}); d(v, \hat{0}) \leq R\Big\}.$$

In the same way of theorem 3.2, the operator $\Phi : \Lambda \to \Lambda$ can be defined as

$$\Phi v(t) = g\Big(t, h_0 \ominus (-1)\int_0^t v(z)dz\Big) + \int_0^t H\Big(t, s, h_0 \ominus (-1)\int_0^s v(z)dz, v(s)\Big)ds.$$

From Lemma 3.3, $v \in C([0, t_f], \mathbb{R}^c_{\mathcal{F}})$ is a solution of (3.15) if and only if $h = h_0 \ominus (-1)\int_0^t v(z)dz$ is a (ii)-differentiable solution of (3.1)-(3.2). So, to prove (3.1)-(3.2) has a (ii)-differentiable solution, it is enough to prove (3.15) has a continuous solution.

Clearly, a fixed point of Φ is a solution of (3.15). Thus, in order to show the existence of a solution of (3.15), it is enough to show Φ has a fixed point. For this purpose, according to the Schauder fixed point theorem, it is enough to prove Φ is a compact operator. The details of demonstration of this result is very similar to the theorem 3.2. □

4. Uniqueness

Uniqueness of the solution of initial value problem (3.1)-(3.2) is also can be obtained applying some strong condition such as Lipschitz conditions.

Theorem 4.1. *Let g and H are Lipschitz continuous, i.e. there exist some real positive numbers L, L_1 and L_2 such that*

$$D\Big(g(t,h_1),g(t,h_2)\Big) \leq LD(h_1,h_2)$$
$$D\Big(H(t,s,h_1,v_1),H(t,s,h_2,v_2)\Big) \leq L_1D(h_1,h_2)+L_2D(v_1,v_2)$$

Then the initial value problem (3.1)-(3.2) has a unique solution both in (i)-differentiable and (ii)-differentiable cases.

Proof. Taking into account Lemma 3.1, the initial value problem (3.1)-(3.2) has a unique (i)-differentiable solution iff the integral equation (3.3) has a unique continuous solution. So in order to prove uniqueness of (3.1)-(3.2), it is enough to prove (3.3) has a unique solution. To prove uniqueness of solution of (3.3), let me assume v_1, v_2 are two solutions of (3.3) then

$$\begin{aligned}
&D\big(v_1(t),v_2(t)\big) \leq LD\Big(h_0+\int_0^t v_1(z)dz, h_0+\int_0^t v_2(z)dz\Big)+\\
&\int_0^t \Big(L_1D\Big(h_0+\int_0^s v_1(z)dz, h_0+\int_0^s v_2(z)dz\Big)+L_2D\Big(v_1(s),v_2(s)\Big)\Big)ds\\
&\leq L\int_0^t D\big(v_1(z),v_2(z)\big)dz+\int_0^t \Big(L_1\int_0^s D\big(v_1(z),v_2(z)\big)dz+L_2D\Big(v_1(s),v_2(s)\Big)\Big)ds\\
&= L\int_0^t D\big(v_1(z),v_2(z)\big)dz+L_1\int_0^t\int_0^s D(v_1(z),v_2(z))dzds+L_2\int_0^t D(v_1(s),v_2(s))ds\\
&= L\int_0^t D\big(v_1(z),v_2(z)\big)dz+L_1\int_0^t (t-z)D\big(v_1(z),v_2(z)\big)dz+L_2\int_0^t D\big(v_1(s),v_2(s)\big)ds\\
&\leq (L+t_fL_1+L_2)\int_0^t D\big(v_1(s),v_2(s)\big)ds.
\end{aligned}$$

By using Gronwall's lemma, I obtain that $D\big(v_1(t),v_2(t)\big)=0$ on $[0,t_f]$ which yields $v_1=v_2$.

In a similar way, one can prove the initial value problem (3.1)-(3.2) has a unique (ii)-differentiable solution. □

Remark 4.2. The fuzzy differential equations based on generalized Hukuhara derivative have some applicable limitations. In fact, if the time freely be increased, the solution diameter dramatically increase or decrease. This leads to incorrect solution in which the boundary lines diverge from each other under (i)-differentiability or cross each other under (ii)-differentiability. This shortcoming has been examined in [2]. They introduced mixed solutions as a possible way to tackle the problem. When the kind of differentiability of the solution is changed within the domain (from (i) to (ii) or inverse) the solution is called "mixed solution". By using the same strategy for the solutions of (3.1)-(3.2), the equivalent integral equations will be

$$v(t) = \begin{cases} g(t, h_0 + \int_0^t v(s)ds) + \int_0^t H(t, s, h_0 + \int_0^s v(\tau)d\tau, v(s))ds, & t \in [0, t_s], \\ g(t, h(t_s) \ominus (-1) \int_{t_s}^t v(s)ds) \\ \quad + \int_0^t H(t, s, h(t_s) \ominus (-1) \int_{t_s}^s v(\tau)d\tau, v(s))ds, & t \in [t_s, t_f], \end{cases} \tag{4.1}$$

or

$$v(t) = \begin{cases} g(t, h_0 \ominus (-1) \int_0^t v(s)ds) \\ + \int_0^t H(t, s, h_0 \ominus (-1) \int_0^s v(\tau)d\tau, v(s))ds, & t \in [0, t_s], \\ g(t, h(t_s) + \int_{t_s}^t v(s)ds) + \int_0^t H(t, s, h(t_s) + \int_{t_s}^s v(\tau)d\tau, v(s))ds, & t \in [t_s, t_f]. \end{cases} \tag{4.2}$$

The point t_s is called switching point. Since for $t \in [0, t_s]$, $v(t)$ coincides with case 1 in (4.1) or case 2 in (4.2) and for $t \in [t_s, t_f]$, $v(t)$ coincides with case 2 in (4.1) or case 1 in (4.2), the details of the proofs for existence and uniqueness results of these integral equations are avoided.

5. Examples

Example 1. (**Case 1.**) Consider the following implicit form of fuzzy integro-differential equation under (i)-differentiability

$$\begin{aligned} &h'(t) = \cos(t).A \ominus \frac{h^2(t)}{2} + \int_0^t h(s)h'(s)ds, \quad t \in I = [0, \frac{\pi}{5}], \\ &h(0) = 0, \end{aligned}$$

in which $A = (0, 1, 2)$ and the exact solution is $h(t) = \sin(t).A$. In this example, we have

$$\begin{cases} g\big(t, h(t)\big) := \cos(t).A \ominus \frac{h^2(t)}{2}, \\ H\big(t, s, h(s), h'(s)\big) := h(s)h'(s), \\ h_0 := 0. \end{cases} \tag{5.1}$$

Applying Lemma 3.1 along with (5.1), the equivalent integro-differential equation of this problem is

$$v(t) = \cos(t).A \ominus \frac{1}{2}.\left(\int_0^t v(z)dz\right)^2 + \int_0^t \int_0^s v(z)v(s)dzds$$

with the exact solution $v(t) = h'(t) = \cos(t).A$.

Example 2. (**Case 2.**) Consider the following implicit form of fuzzy integro-differential equation under (ii)-differentiability

$$\begin{aligned} &h'(t) = e^{-t}.(0, 2, 4) + (te^{-t} - 1).(0, 1, 2) + \int_0^t sh'(s)ds, \quad t \in I = [0, 1], \\ &h(0) = A, \end{aligned}$$

in which $A = (-2, -1, 0)$ and the exact solution is $h(t) = e^{-t}.A$. In this example, we have

$$\begin{cases} g\big(t, h(t)\big) := e^{-t}.(0,2,4) + (te^{-t} - 1).(0,1,2), \\ H\big(t, s, h(s), h'(s)\big) := sh'(s), \\ h_0 := A. \end{cases} \tag{5.2}$$

Applying Lemma 3.3 and (5.2), the equivalent integro-differential equation of this problem is

$$v(t) = e^{-t}.(0,2,4) + (te^{-t} - 1).(0,1,2) + \int_0^t sv(s)ds$$

with the exact solution $v(t) = h'(t) = e^{-t}.(0,1,2)$.

6. Conclusion

I demonstrate a lemma which converts the implicit form of fuzzy integro-differential equation to an equivalent fuzzy nonlinear integral equation. Using the recently published Schauder fixed point theorem in semilinear Banach space instead of using the classical Banach fixed point theorem, I studied the existence of the solutions for the proposed fuzzy implicit integro-differential equation. This requires weaker conditions than the classical Banach fixed point theorem. I considered (i) and (ii) kinds of generalized differentiability.

Acknowledgment

The author acknowledges the financial support of university of Tabriz which is based on a research project contract. Also, I would like to thank Ghiyam Eslami for his kind consultation on the application of the proposed equation in the field of fluid mechanics.

References

[1] R. P. Agarwal, S. Arshad, D. O'Regan, and V. Lupulescu, *A Schauder fixed point theorem in semilinear spaces and applications*, Fixed Point Theory and Applications, 2013.

[2] R. Alikhani, F. Bahrami, and A. Jabbari, *Existence of global solutions to nonlinear fuzzy Volterra integro-differential equations*, Nonlinear Anal.: Theory, Methods & Applications, *75* (2012), 1810–1821.

[3] T. Allahviranloo, S. Abbasbandy, and S. Hashemzehi, *Approximating the Solution of the Linear and Nonlinear Fuzzy Volterra Integrodifferential Equations Using Expansion Method*, Abstract and Applied Analysis, *2014* (2014), 7 pages.

[4] T. Allahviranloo, S. Abbasbandy, O. Sedaghgatfar, and P. Darabi, *A new method for solving fuzzy integro-differential equation under generalized differentiability*, Neural Comput & Applic, *21* (2012), 191–196.

[5] O. A. Arqub, S. Momani, S. Al-Mezel, and M. Kutbi, *Existence, Uniqueness, and Characterization Theorems for Nonlinear Fuzzy Integrodifferential Equations of Volterra Type*, Mathematical Problems in Engineering, *2015* (2015), 13 pages.

[6] A. Ashyralyev, *Well-posedness of the Basset problem in spaces of smooth functions*, Applied Mathematics Letters, *24* (2011), 1176–1180.

[7] P. Balasubramaniam and S. Muralisankar, *Existence and Uniqueness of Fuzzy Solution for Semilinear Fuzzy Integrodifferential Equations with Nonlocal Conditions*, Computers and Mathematics with Applications, *47* (2004), 1115–1122.

[8] P. Balasubramaniam and S. Muralisankar, *Existence and Uniqueness of Fuzzy Solution for the Nonlinear Fuzzy Integrodifferential Equations*, Applied Mathematics Letters, *14* (2001), 455–462.

[9] B. Bede and S. G. Gal, *Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations*, Fuzzy Sets Syst., *151* (2005), 581–599.

[10] B. Bede and S. G. Gal, *Solutions of fuzzy differential equations based on generalized differentiability*, Commun. Math. Anal., *9* (2010), 22–41.

[11] R. Goetschel and W. Voxman, *Elementary fuzzy calculus*, Fuzzy Sets Syst., *18* (1986), 31–43.

[12] L. T. Gomes, L. C. Barros, and B. Bede, *Fuzzy differential equations in various approaches*, Springer, 2015.

[13] E. Loth and A. J. Dorgan, *An equation of motion for particles of finite Reynolds number and size*, Environ Fluid Mech, *9* (2009), 187–206.

[14] B. G. Pachpatte, *Implicit type Volterra integro-differential equation*, Tamkang journal of mathematics, *41* (2010), 97–107.

[15] N. Van Hoa, *The initial value problem for interval-valued second-order differential equations under generalized H-differentiability*, Information Science, *311* (2015), 119–148.

[16] E. J. Villamizar-Roa, V. Angulo-Castillo, and Y. Chalco-Cano, *Existence of solutions to fuzzy differential equations with generalized Hukuhara derivative via contractive-like mapping principles*, Fuzzy Sets Syst., *265* (2015), 24–38.

[17] Ye. V. Visitskii, A. G. Petrov, and M. M. Shunderyuk, *The motion of a particle in a viscous fluid under gravity, vibration and Basset's force*, Journal of Applied Mathematics and Mechanics, *73* (2009), 548–557.

[18] C. Wu and Z. Gong, *On Henstock integral of fuzzy-number-valued functions I*, Fuzzy Sets Syst., *120* (2001), 523–532.

[19] M. Zeinali, S. Shahmorad, and K. Mirnia, *Fuzzy integro-differential equations: discrete solution and error estimation*, Iranian Journal of Fuzzy System, *10* (2013), 107–122.