Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 5, No. 3, 2017, pp. 224-231

Option pricing under the double stochastic volatility with double jump model

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Abstract In this paper, we deal with the pricing of power options when the dynamics of the risky underling asset follows the double stochastic volatility with double jump model. We prove efficiency of our considered model by fast Fourier transform method, Monte Carlo simulation and numerical results using power call options i.e. Monte Carlo simulation and numerical results show that the fast Fourier transform is correct.

Keywords. Power option, Monte Carlo, Fast Fourier Transform, Double Stochastic Volatility, Double Jump.

2010 Mathematics Subject Classification. 80G91, 91G60.

1. Introduction

Option pricing is a very important concept in financial economics and has been widely used among the traders and practitioners. A number of papers paid attention to the option pricing models. Many number of them raised to the Black-Scholes model. As known in the Black-Scholes model the volatility rate is assumed to be constant. But more observation of volatility of traded option valuations has exposed that this assumption is not coincides with reality.

A lot of literatures are proposed option pricing models with stochastic volatility models, jump-diffusion models, Markov-modulated jump-diffusion models and regime-switching models [6, 7, 8, 9].

After 1978, the most realistic and efficient model is presented by Heston. In this model the volatility and the underling asset price include a diffusion process which are correlated.

Then Christoffersen proposed the additional stochastic process (as the second volatility). So he developed the Heston model with his concerns and made the asset pricing more realistic [2].

Recently, it has been more attention paid to add more jumps and more stochastic

Received: 12 January 2017 ; Accepted: 3 July 2017.
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volatilities which yields further randomization of the volatility rate and made the models more efficient.
In this paper, we introduce a model in which the stock price follows the double stochastic volatility with double jump. We also study power options with payoffs which depend on the price of the risky underlying asset raised to power $m > 0$.
For an investor, using a power option is more useful than an ordinary option [5].
For this reason in this paper we investigate power option pricing under the double stochastic volatility with double jump. In fact we drive the characteristic function and get the option pricing via fast Fourier transform and show our analytic method efficiency by Monte Carlo simulation and numerical results.
This paper is organized as follows. In section 2, we present notations and a model in which the stock price follows the double stochastic volatility with double jump. In section 3, we investigate the characteristic function. Power option pricing using the Fast Fourier Transform is driven in section 4. Numerical results are given in section 5. The paper is concluded in section 6.

2. The model

Let $(\Omega, \mathcal{F}, P)$ be a probability space where $\{\mathcal{F}_t\}_t$ is the filtration generated by the Brownian motion and the jump process at time t, $0 \leq t \leq T$ and Q is a risk neutral probability. The underling asset price S_t at time t is given by

$$\begin{aligned} dS_t &= (r - \lambda\mu_J)S_t dt + \sqrt{V_t^{(1)}} S_t d\tilde{W}_t^{(1)} + \sqrt{V_t^{(2)}} S_t d\tilde{W}_t^{(3)} + J S_t d\tilde{N}_t, \quad (2.1) \\ dV_t^{(1)} &= k_1(\theta_1 - V_t^{(1)})dt + \sigma_{v_1}\sqrt{V_t^{(1)}} d\tilde{W}_t^{(2)} + Z d\tilde{N}_t, \\ dV_t^{(2)} &= k_2(\theta_2 - V_t^{(2)})dt + \sigma_{v_2}\sqrt{V_t^{(2)}} d\tilde{W}_t^{(4)}, \\ & d\tilde{W}_t^{(1)} d\tilde{W}_t^{(2)} = \rho_1 dt, \\ & d\tilde{W}_t^{(3)} d\tilde{W}_t^{(4)} = \rho_2 dt, \end{aligned}$$

where r is the interest rate, $\sqrt{V_t^{(i)}}$, $i = 1, 2$ is a volatility process, θ_i, $i = 1, 2$ is the long-run average of V_t, N_t represents a Poisson under the risk neutral measure with jump intensity λ, k_i, $i = 1, 2$ is the rate of mean reversion, σ_{v_i}, $i = 1, 2$ is the volatility of volatility, Z is a stochastic process which has exponential distribution with parameter μ_v, $(1 + J)|Z$ has log normal distribution with mean $\mu_s + \rho_J Z$ and variance σ_s^2 in which

$$\mu_J = \frac{exp\{\mu_s + \frac{\sigma_s^2}{2}\}}{1 - \rho_J \mu_v} - 1,$$

and $\tilde{W}_t^{(i)}$ and $\tilde{W}_t^{(i+1)}, i = 1, 3$ are two correlated Brownian motions under Q which $Cov(d\tilde{W}_t^{(1)}, d\tilde{W}_t^{(2)}) = \rho_1 dt$ and $Cov(d\tilde{W}_t^{(3)}, d\tilde{W}_t^{(4)}) = \rho_2 dt$ (ρ_1 and ρ_2 are constants).

3. Deriving the characteristic function

By Duffie, Gatheral and Zhu we drive the characteristic function [3, 4]. The log stock price and volatility processes of our considered model is

$$\begin{aligned} logS_t &= rdt - \lambda\mu_J dt - \frac{1}{2}(V_t^{(1)} + V_t^{(2)})dt + \sqrt{V_t^{(1)}}(\rho_1 d\tilde{W}_t^{(2)} + \sqrt{1-\rho_1^2}d\tilde{W}_t^{(1)}) \\ &+ \sqrt{V_t^{(2)}}(\rho_2 d\tilde{W}_t^{(4)} + \sqrt{1-\rho_2^2}d\tilde{W}_t^{(3)}) + log(1+J)d\tilde{N}_t, \end{aligned} \tag{3.1}$$

$$dV_t^{(1)} = k_1(\theta_1 - V_t^{(1)})dt + \sigma_{v_1}\sqrt{V_t^{(1)}}d\tilde{W}_t^{(2)} + Zd\tilde{N}_t, \tag{3.2}$$

$$dV_t^{(2)} = k_2(\theta_2 - V_t^{(2)})dt + \sigma_{v_2}\sqrt{V_t^{(2)}}d\tilde{W}_t^{(4)}, \tag{3.3}$$

where

$$log(1+J) \sim Normal(log(1+\mu_J) - \frac{\sigma_s^2}{2}, \sigma_s^2).$$

Denote by $H_{log(S_T)}(u)$ the characteristic function of the log stock price. So we have

$$\begin{aligned} H_{log(S_T)}(u) &= E^Q\left[exp\{iulog(S_T)\}\right] \\ &= E^Q[exp\{iu(logS_0 + rT - \lambda\mu_J T - \frac{1}{2}\int_0^T V_t^{(1)}dt - \frac{1}{2}\int_0^T V_t^{(2)}dt \\ &+ \rho_1\int_0^T \sqrt{V_t^{(1)}}d\tilde{W}_t^{(2)} + \sqrt{1-\rho_1^2}\int_0^T \sqrt{V_t^{(1)}}d\tilde{W}_t^{(1)} \\ &+ \rho_2\int_0^T \sqrt{V_t^{(2)}}d\tilde{W}_t^{(4)} + \sqrt{1-\rho_2^2}\int_0^T \sqrt{V_t^{(2)}}d\tilde{W}_t^{(3)} \\ &+ log(1+J)\int_0^T d\tilde{N}_t)\}]. \end{aligned}$$

So from (3.2) and (3.3) we have

$$\begin{aligned} H_{log(S_T)}(u) &= exp\{iu(logS_0 + rT)\}E^Q[exp\{-iu\lambda\mu_J T - \frac{iu}{2}\int_0^T V_t^{(1)}dt \\ &- \frac{iu}{2}\int_0^T V_t^{(2)}dt + \frac{iu\rho_1}{\sigma_{v_1}}[V_T^{(1)} - V_0^{(1)} - k_1\theta_1 T] + \frac{iu\rho_1}{\sigma_{v_1}}k_1\int_0^T V_t^{(1)}dt \\ &- \frac{iu\rho_1}{\sigma_{v_1}}\int_0^T Zd\tilde{N}_t + \frac{(iu)^2}{2}(1-\rho_1^2)\int_0^T V_t^{(1)}dt + \frac{iu\rho_2}{\sigma_{v_2}}[V_T^{(2)} - V_0^{(2)} - k_2\theta_2 T] \\ &+ \frac{iu\rho_2}{\sigma_{v_2}}k_2\int_0^T V_t^{(2)}dt + \frac{(iu)^2}{2}(1-\rho_2^2)\int_0^T V_t^{(2)}dt + log(1+J)\int_0^T d\tilde{N}_t\}]. \end{aligned}$$

So

$$\begin{aligned} H_{log(S_T)}(u) &= exp\{iu(logS_0 + rT) - s_2(V_0^{(1)} + k_1\theta_1 T) - s_4(V_0^{(2)} + k_2\theta_2 T)\} \\ &\times E^Q[exp\{-s_1 \int_0^T V_t^{(1)}dt + s_2 V_T^{(1)} - s_3 \int_0^T V_t^{(2)}dt + s_4 V_T^{(2)}\}] \\ &\times E^Q[exp\{-iu\lambda\mu_J T - \frac{iu\rho_1}{\sigma_{v_1}} \int_0^T Z d\tilde{N}_t + log(1+J) \int_0^T d\tilde{N}_t\}], \end{aligned}$$

when

$$\begin{aligned} s_1 &= -(\frac{-iu}{2} + \frac{iu\rho_1 k_1}{\sigma_{v_1}} + \frac{(iu)^2(1-\rho_1^2)}{2}), \\ s_3 &= -(\frac{-iu}{2} + \frac{iu\rho_2 k_2}{\sigma_{v_2}} + \frac{(iu)^2(1-\rho_2^2)}{2}), \\ s_2 &= \frac{iu\rho_1}{\sigma_{v_1}}, \\ s_4 &= \frac{iu\rho_2}{\sigma_{v_2}}. \end{aligned}$$

From [10] (Feynman-kac theorem) we have

$$\begin{aligned} E^Q[exp\{-s_1 \int_0^T V_t^{(1)}dt + s_2 V_T^{(1)}\}] &= exp\{G_T^{(1)}(u)V_0^{(1)} + G_T^{(2)}(u)\}, \\ E^Q[exp\{-s_3 \int_0^T V_t^{(2)}dt + s_4 V_T^{(2)}\}] &= exp\{G_T^{(3)}(u)V_0^{(2)} + G_T^{(4)}(u)\}, \end{aligned}$$

so that

$$\begin{aligned} H_{log(S_T)}(u) &= exp\{iu(logS_0 + rT) - s_2(V_0^{(1)} + k_1\theta_1 T) - s_4(V_0^{(2)} + k_2\theta_2 T)\} \\ &\times exp\{G_T^{(1)}(u)V_0^{(1)} + G_T^{(2)}(u) + G_T^{(3)}(u)V_0^{(2)} + G_T^{(4)}(u)\} \\ &\times E^Q[exp\{-iu\lambda\mu_J T - \frac{iu\rho_1}{\sigma_{v_1}} \int_0^T Z d\tilde{N}_t + log(1+J) \int_0^T d\tilde{N}_t\}], \end{aligned}$$

where

$$\begin{aligned} G_T^{(1)}(u) &= \frac{s_2 d_1(1+e^{-d_1 T}) - (1-e^{-d_1 T})(-\frac{iu\rho_1 k_1}{\sigma_{v_1}} + iu - (iu)^2(1-\rho_1^2))}{(1-g_1 e^{-d_1 T})(\beta_1 + d_1)}, \\ G_T^{(2)}(u) &= \frac{2k_1\theta_1}{\sigma_{v_1}^2} log(\frac{2d_1}{(1-g_1 e^{-d_1 T})(\beta_1 + d_1)} e^{\frac{1}{2}(k_1 - d_1)T}), \\ G_T^{(3)}(u) &= \frac{s_4 d_2(1+e^{-d_2 T}) - (1-e^{-d_2 T})(-\frac{iu\rho_2 k_2}{\sigma_{v_2}} + iu - (iu)^2(1-\rho_2^2))}{(1-g_2 e^{-d_2 T})(\beta_2 + d_2)}, \\ G_T^{(4)}(u) &= \frac{2k_2\theta_2}{\sigma_{v_2}^2} log(\frac{2d_2}{(1-g_2 e^{-d_2 T})(\beta_2 + d_2)} e^{\frac{1}{2}(k_2 - d_2)T}), \\ g_i &= \frac{r^{(i)}neg}{r^{(i)}pos}, i = 1, 2, \\ r^{(i)}pos/neg &= \frac{\beta_i \pm d_i}{2\gamma_i}, i = 1, 2, \end{aligned}$$

$$
\begin{gathered}
d_i = \sqrt{\beta_i^2 - 4\alpha\gamma_i}, i = 1, 2, \\
\alpha = \frac{(-u^2 - iu)}{2}, \\
\beta_i = k_i - \rho_i \sigma_{v_i} iu, i = 1, 2, \\
\gamma_i = \frac{\sigma_{v_i}}{2}, i = 1, 2.
\end{gathered}
$$

Now, it is easy to see that

$$
\begin{gathered}
(G_T^{(1)}(u) - s_2)V_0^{(1)} = r^{(1)} neg \left[\frac{1 - e^{-d_1 T}}{1 - g_1 e^{-d_1 T}} \right] V_0^{(1)} =: D_1(u,T)V_0^{(1)}, \\
(G_T^{(3)}(u) - s_4)V_0^{(2)} = r^{(2)} neg \left[\frac{1 - e^{-d_2 T}}{1 - g_2 e^{-d_2 T}} \right] V_0^{(2)} =: D_2(u,T)V_0^{(2)}, \\
G_T^{(2)}(u) - s_2 k_1 \theta_1 T = k_1 \theta_1 \left[r^{(1)} negT - \frac{2}{\sigma_{v_1}} log \left(\frac{1 - g_1 e^{-d_1 T}}{1 - g_1} \right) \right] =: C_1(u,T)\theta_1, \\
G_T^{(4)}(u) - s_4 k_2 \theta_2 T = k_2 \theta_2 \left[r^{(2)} negT - \frac{2}{\sigma_{v_2}} log \left(\frac{1 - g_2 e^{-d_2 T}}{1 - g_2} \right) \right] =: C_2(u,T)\theta_2.
\end{gathered}
$$

On the other hand it is clear that

$$
E^Q[exp\{-iu\lambda\mu_J T - \frac{iu\rho_1}{\sigma_{v_1}} \int_0^T Z d\tilde{N}_t + log(1+J) \int_0^T d\tilde{N}_t\}] =
$$

$$
E^Q[exp\{-iu\lambda\mu_J T + \lambda T((1+\mu_J)^{iu} e^{\sigma_s^2 \frac{iu}{2}(iu-1)} - 1) + e^{\lambda T(e^{\frac{iu\rho_1 Z}{\sigma_{v_1}}} - 1)}\}],
$$

so that

$$
\begin{aligned}
H_{log(S_T)}(u) \quad &= \quad exp\{iu(log(S_0) + rT) + C_1(u,T)\theta_1 + D_1(u,T)V_0^{(1)} \\
&+ \quad C_2(u,T)\theta_2 + D_2(u,T)V_0^{(2)} + P(u,T)\lambda\},
\end{aligned}
$$

where

$$
\begin{gathered}
C_1(u,T) = k_1 \left[r^{(1)} negT - \frac{2}{\sigma_{v_1}} log \left(\frac{1 - g_1 e^{-d_1 T}}{1 - g_1} \right) \right], \\
D_1(u,T) = r^{(1)} neg \left[\frac{1 - e^{-d_1 T}}{1 - g_1 e^{-d_1 T}} \right], \\
C_2(u,T) = k_2 \left[r^{(2)} negT - \frac{2}{\sigma_{v_2}} log \left(\frac{1 - g_2 e^{-d_2 T}}{1 - g_2} \right) \right], \\
D_2(u,T) = r^{(2)} neg \left[\frac{1 - e^{-d_2 T}}{1 - g_2 e^{-d_2 T}} \right], \\
P(u,T)\lambda = -T(1 + iu\mu_J) + exp\{iu\mu_s + \frac{\sigma_s^2 (iu)^2}{2}\} \times \nu \\
\nu = \frac{\beta_1 + d_1}{(\beta_1 + d_1)c - 2\mu_v \alpha} T + \frac{4\mu_v \alpha}{(d_1 c)^2 - (2\mu_v \alpha - \beta_1 c)^2} \times \\
log \left[1 - \frac{(d_1 - \beta_1)c + 2\mu_v \alpha}{2 d_1 c} (1 - e^{-d_1 T}) \right], \\
c = 1 - iu\rho_J \mu_v.
\end{gathered}
$$

4. Power option pricing using the Fast Fourier Transform

Under the risk neutral measure Q, the valuation of the m-th power call option with strike K and maturity T is as follows

$$c(t, S_T) = e^{-r(T-t)} E^Q \left[(S_T^m - K^m)^+ | \mathcal{F}_t \right], \tag{4.1}$$

where r is the constant interest rate and $(S_T^m - K^m)^+ = max\{S_T^m - K^m, 0\}$. Let $t = 0$, $X_t = lnS_t$ and $k = lnK$. We derive the power call option pricing (4.1) as a function of the log strike K rather than the terminal log asset price X_T as bellow.

$$c(T, k) = e^{-rT} \int_k^{\infty} (e^{mX_T} - e^{mk}) q_T(X_T) dX_T, \tag{4.2}$$

where $q_T(X_T)$ is the density function of X_T. Note that $c(T, k)$ converges to S_0 when k tends to $-\infty$. Carr and Madan [1] presented a modified call price function

$$C(T, k) = e^{\alpha k} c(T, k), \qquad for \quad \alpha > 0. \tag{4.3}$$

The Fourier transform of $C(T, k)$ is defined by

$$\psi_T(u) = \int_{-\infty}^{+\infty} e^{iuk} C(T, k) dk. \tag{4.4}$$

From (4.2),(4.3) and (4.4) we have

$$\begin{aligned} \psi_T(u) &= \int_{-\infty}^{+\infty} e^{iuk} e^{\alpha k} e^{-rT} \int_k^{\infty} (e^{mX_T} - e^{mk}) q_T(X_T) dX_T dk \\ &= \int_{-\infty}^{+\infty} e^{-rT} q_T(X_T) \int_{-\infty}^{X_T} (e^{mX_T + \alpha k} - e^{(\alpha+m)k}) e^{iuk} dk dX_T \\ &= \frac{m e^{-rT} H_{S_T}(u - (\alpha + m)i)}{(\alpha + iu)(m + \alpha + iu)}. \end{aligned}$$

Then the inverse transform of $\psi_T(u)$ is as follows

$$C(T, k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iuk} \psi_T(u) du. \tag{4.5}$$

So

$$c(T, k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{+\infty} e^{-iuk} \psi_T(u) du. \tag{4.6}$$

By applying the Trapezoil method in (4.6) we have

$$c(T, k) \approx \frac{e^{-\alpha k}}{\pi} \sum_{j=1}^{N} e^{-iu_j k} \psi_T(u_j) \triangle,$$

where $\triangle$ denotes the integration steps, $a = N\triangle$ and $u_j = \triangle(j-1)$.
The FFT returns N values of k and for a regular spacing size of η where N is a power of 2, the value for k is

$$k_v = -b + \eta(v - 1), \qquad for \quad v = 1, 2, ..., N, \tag{4.7}$$

where $b = \frac{N\eta}{2}$.
Equation (4.7) gives us N log strike values at regular intervals of width η, ranging from $-b$ to b.
Finally, setting $\eta\triangle = \frac{2\pi}{N}$, we get

$$c(k_v) \approx \frac{e^{-\alpha k_v}}{\pi} \sum_{j=1}^{N} e^{-i\eta\triangle(j-1)(v-1)} e^{ibu_j} \psi_T(u_j)\triangle,$$

with Simpsons method weightings, the price of power call option is as follows.

$$c(T,k) = \frac{e^{-\alpha k_v}}{\pi} \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(v-1)} e^{ibu_j} \psi_T(u_j) \frac{\triangle}{3}(3 + (-1)^j - \delta_{j-1}),$$

where δ_n is the kronecher delta function that is 1 for $n = 0$ and 0 otherwise.

5. Numerical results

In this section, we present and compare numerical results for power call option using the FFT method and the Monte Carlo simulation.
For our FFT method, we take $N = 2^{12}$, $a = 600$ and $\alpha = 0.75$. The parameters are considered as follows $r = 0.05, q = 0.06, k_1 = 0.9, \theta_1 = 0.1, \sigma_{v_1} = 0.1, \rho_1 = -0.5, v_0^{(1)} = 0.6, k_2 = 1.2, \theta_2 = 0.15, \sigma_{v_2} = 0.2, \rho_2 = -0.5, v_0^{(2)} = 0.7, \lambda_J = 0.22, \mu_s = 0.22, \sigma_s = 0.25, \rho_J = -0.4, \mu_v = 0.05, S_0 = 100$ where m, k, T are different. The numerical results are shown in Table 1. In addition, we take $N = 100,000$ simulations

Table 1. Power call option prices: FFT, Monte Carlo.

T	K	m	FFT	Monte Carlo	difference
0.5	90	1	31.4618	33.0014	1.5396
		2	1.1501×10^4	1.2024×10^4	0.0523×10^4
		3	4.3620×10^6	4.3575×10^6	-0.0045×10^6
		4	2.3358×10^9	2.1054×10^9	-0.2304×10^9
0.5	95	1	29.5257	31.2533	1.7275
		2	1.1143×10^4	1.1816×10^4	0.0673×10^4
		3	4.3123×10^6	4.5088×10^6	0.1965×10^6
		4	2.3296×10^9	2.0554×10^9	-0.2742×10^9

to price the power call option using the Monte Carlo simulation.
So numerical results proved that FFT approach is correct and more efficient than Monte Carlo simulation.

6. Conclusion

In this paper we drived the characteristic function and got the power option pricing by FFT. In the sequel, the efficiency of our analytic method by Monte Carlo simulation and numerical results is shown.

Acknowledgment

The authors would like to acknowledge the financial support of Shahrood University of Technology for this research under project No: 23096.

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