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Some new families of definite polynomials and the composition conjectures

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Abstract The planar polynomial vector fields with a center at the origin can be written as an scalar differential equation, for example Abel equation. If the coefficients of an Abel equation satisfy the composition condition, then the Abel equation has a center at the origin. Also the composition condition is sufficient for vanishing the first order moments of the coefficients. The composition conjecture and the moment vanishing problem ask for that the composition condition is a necessary condition to have the center or vanishing the moments. It is not known that if there exist examples of polynomials that satisfy the double moment conditions but don't satisfy the composition condition. In this paper we consider some composition conjectures and give some families of definite polynomials for which vanishing of the moments and the composition condition are equivalent. Our methods are based on a decomposition method for continuous functions. We give an orthogonal basis for the family of continuous functions and study the conjecture in terms of this decomposition.

Keywords. Abel equation, Composition condition, Composition conjecture, Definite polynomial, Moment.

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1. Introduction

Consider the planar system of differential equations

$$\begin{cases} \dot{x} &= P(x,y), \\ \dot{y} &= Q(x,y), \end{cases} \tag{1.1}$$

where P and Q are analytic functions in some planar region Ω. A point (x_0, y_0) is called a singular point of (1.1), if $P(x_0,y_0) = Q(x_0,y_0) = 0$. The solution $\gamma(t)$ of (1.1) is called periodic, if there exists $T > 0$ such that $\gamma(0) = \gamma(T)$. An isolated periodic solution is called limit cycle. A singular point is called center, if in a neighborhood of it, all solutions are periodic, and is called focus if in a neighborhood of it all solution

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spirals the singular point. If the origin is a center of the associated linearized system, then it would be a center or a focus of the system (1.1). Distinguishing between the centers and the foci of the nonlinear system (1.1) is an old open problem, known as Poincaré-center-focus problem, and has been remained unsolved, even for P and Q homogeneous polynomials of degree three.

Using appropriate change of variables, the system (1.1) converts to a scalar non-autonomous differential equation as

$$\dot{z} = \sum_{k=0}^{m} A_k(t) z^k, \tag{1.2}$$

such that there exists a one to one correspondence between the limit cycles surrounding the origin of (1.1) and the positive periodic solutions of (1.2), [6, 7, 10, 16]. Recall that a solution $u(t)$ of (1.2), is called periodic if for some $\omega \in \mathbb{R}$ we have $u(0) = u(\omega)$. If $m = 3$, the Eq. (1.2) is called an Abel equation.

The system

$$\begin{cases} \dot{x} &= -y + P_d(x, y), \\ \dot{y} &= x + Q_d(x, y), \end{cases} \tag{1.3}$$

where P_d, Q_d are homogeneous polynomials of degree d can be reduced to the Abel differential equation

$$\dot{x} = A(t)x^3 + B(t)x^2, \tag{1.4}$$

where $A(t), B(t)$ are polynomials in $\sin(t), \cos(t)$ of degree $2(d+1), d+1$ respectively [7]. It is said that $A(t)$ and $B(t)$ satisfy the composition condition, if there exists a periodic function $w(t)$ such that $\int A(t)dt = \tilde{A}(w(t))$, $\int B(t)dt = \tilde{B}(w(t))$, for some continuous functions $\tilde{A}$ and $\tilde{B}$. If $A(t)$ and $B(t)$ satisfy the composition condition, then the origin is a center of (1.4) and is said the equation has a CC-center [4]. So the composition conjecture is that the composition condition for $A(t)$ and $B(t)$ is necessary to have a center. This conjecture was appeared at first for trigonometric polynomials $A(t)$ and $B(t)$ in [4] and in this case it was answered negatively in [2]. The composition conjecture was considered later for polynomials $A(t)$ and $B(t)$, and it remains unsolved. One interesting problem is to construct classes of polynomials for which the conjecture is true, see for example [3].

Furthermore it is interesting to characterize the so called persistent centers and determine the relation between them and CC-centers, [3]. The Abel differential equation

$$\dot{x} = \epsilon A(t)x^3 + B(t)x^2, \tag{1.5}$$

is said to have a persistent center at the origin, if it has a center for all ϵ small enough. $z = 0$ is a persistent center for (1.5), if and only if $A(t)$ and $B(t)$ satisfy the following moment conditions

$$m_k = \int_a^b \overline{B}^k(t) A(t) dt = 0, \quad k = 0, 1, \ldots, \tag{1.6}$$

where $\overline{B}(t) = \int_a^t B(s)ds$ and $a, b \in \mathbb{R}$, [5].

In [9] it was shown that if the system (1.5) has a persistent center at $z = 0$, then A, B satisfy the moment conditions (1.6) and the following moment conditions

$$\mu_k = \int_a^b \overline{A}^k(t)B(t)dt = 0, \quad k = 0, 1, \dots, \tag{1.7}$$

where $\overline{A}(t) = \int_a^t A(s)ds$ and $a, b \in \mathbb{R}$.

The composition conjecture for the moments or the moment vanishing problem is that $A(t)$ and $B(t)$ satisfy the composition condition if and only if all moments of them equal zero. For polynomials this conjecture was solved in [12] and for trigonometric polynomials it is only partially solved in some special cases see for example [15, 1, 13]. The composition condition implies both CC-center and the moments vanishing. There are examples of trigonometric polynomials that satisfy all the moments conditions (1.6) and (1.7) but don't satisfy the composition condition [9], and there are examples of polynomials which satisfy conditions (1.6) or (1.7) but don't satisfy the composition condition [11]. Furthermore, there is no example of polynomials that satisfies all conditions (1.6) and (1.7) but does not satisfy the composition condition.

A polynomial $A(t)$ is called definite, if for all polynomials $B(t)$ the moments conditions (1.6) are equivalent to the composition condition, [14, 18].

In this paper we study these conjectures for polynomials. The organization of the paper is as follows. In Section 2, we introduce an orthogonal basis for the family of continuous functions. We give an iterative relation between the integrals of this basis. Then we introduce a special matrix and study the properties of it. In Section 3, the main results are given. Using the properties of the orthogonal basis we give some families of definite polynomials. The conclusions are made in Section 4.

2. A special matrix and its properties

Every polynomial in t can be written as a linear combination of $1-$periodic terms of the form $u_k(t) = (t^2 - t)^k$ and $v_k(t) = (2t - 1)(t^2 - t)^k$. The function $u_k(t)$ is symmetric about $t = \frac{1}{2}$ and it's behavior is similar to even functions, and $v_k(t)$ is symmetric about the point $(\frac{1}{2}, 0)$ and is similar to odd functions. The effectiveness of such decomposition is the following iterative relation.

Lemma 2.1. [3]

(1) *For any positive integer* k

$$I_k = \int_0^1 (t^2 - t)^k dt = (-1)^k \frac{(k!)^2}{(2k+1)!}.$$

(2) *The expression* I_k *satisfy the relation*

$$I_{k+1} = -\frac{k+1}{2(2k+3)} I_k.$$

Note that the families $u_k(t)$ and $v_k(t)$ defined as the above are orthogonal.

Definition 2.2. [17] Let $\{\phi_n\}$ $(n = 1, 2, \ldots)$ be a sequence of complex functions on $[a, b]$, such that

$$\int_a^b \phi_n(x)\overline{\phi_m(x)}dx = 0, \quad (n \neq m). \tag{2.1}$$

Then $\{\phi_n\}$ is said to be an orthogonal system of functions on $[a, b]$. If, in addition,

$$\int_a^b |\phi_n(x)|^2 dx = 1, \tag{2.2}$$

for all n, $\{\phi_n\}$ is said to be orthonormal.

One can see easily that $\{\frac{(2n+1)(4n+3)!(2t-1)(t^2-t)^n}{2((2n+1)!)^2}, \frac{(2n+1)!(t^2-t)^n}{(2n!)^2}\}_{n=1}^{\infty}$ is an orthonormal system of functions. More precisely for an arbitrary continuous function $f(t)$ define the Fourier coefficients as follows:

$$\begin{aligned} a_0 &= \int_0^1 f(t)dt, \\ a_n &= \frac{(2n+1)!}{(2n!)^2} \int_0^1 f(t)(t^2-t)^n dt, \\ b_n &= \frac{(2n+1)(4n+3)!}{2((2n+1)!)^2} \int_0^1 f(t)(2t-1)(t^2-t)^n dt, \end{aligned} \tag{2.3}$$

and let the Fouriere transform of $f(x)$ be as follows:

$$f(x) = a_0 + \sum_1^{\infty} a_n(t^2-t)^n + \sum_0^{\infty} b_n(2t-1)(t^2-t)^n. \tag{2.4}$$

Definition 2.3. Let $f(x)$ be a continuous function and $a_0, a_n, b_n,\ n = 1, 2, \ldots$ be the Fourier coefficients defined by (2.3), we say that f is $UV-even$, when $b_n = 0$ for $n = 1, 2, \ldots$, and f is called $UV-odd$, when $a_0 = 0$ and $a_n = 0$ for $n = 1, 2, \cdots$.

Let $C = [c_j^i]$ where

$$c_j^i = \begin{cases} 1, & i = 1, \\ \prod_{r=1}^{i-1} \frac{k_j+r}{2(k_j+r)+1}, & i \neq 1, \end{cases} \tag{2.5}$$

$i = 1, \ldots, n, j = 1, \ldots, n, k_1, \ldots, k_n \in \mathbb{N}$ and $k_1 < \ldots < k_n$. Non-singularity of the matrix C has been shown in [3]. In the next theorem we give some other useful properties of the matrix C to improve the result. In the following, when we consider two submatrices $[a_j^i], [b_j^i]$ of a matrix, we mean two submatrices are from the same rows of the matrix with the same size. Our proofs are based on a special method of calculating the determinant as Chio pivotal condensation method, see Appendix A and [8].

Theorem 2.4. *The matrix C has the following properties*

(1) *Every 2×2 submatrix of C is non-singular.*
(2) *Every arbitrary submatrix of C is non-singular, in particular C is non-singular.*
(3) *Sum of two arbitrary submatrices of C is non-singular.*
(4) *Every linear combination of submatrices of C with positive coefficients is non-singular.*

Proof. To prove (1) note that an arbitrary submatrix $B_{2\times 2}$ of C is either

$$B_1 = \begin{bmatrix} \prod_{r=1}^{l_1} \frac{k_i+r}{2k_i+2r+1} & \prod_{r=1}^{l_1} \frac{k_j+r}{2k_j+2r+1} \\ \prod_{r=1}^{l_2} \frac{k_i+r}{2k_i+2r+1} & \prod_{r=1}^{l_2} \frac{k_j+r}{2k_j+2r+1} \end{bmatrix}, \tag{2.6}$$

$(l_1 < l_2)$ or

$$B_2 = \begin{bmatrix} 1 & 1 \\ \prod_{r=1}^{l} \frac{k_i+r}{2k_i+2r+1} & \prod_{r=1}^{l} \frac{k_j+r}{2k_j+2r+1} \end{bmatrix}, \tag{2.7}$$

where $k_i < k_j$. We have

$$\det(B_1) = \mu \begin{vmatrix} 1 & 1 \\ \prod_{r=l_1+1}^{l_2} \frac{k_i+r}{2k_i+2r+1} & \prod_{r=l_1+1}^{l_2} \frac{k_j+r}{2k_j+2r+1} \end{vmatrix}, \tag{2.8}$$

where $\mu = \prod_{r=1}^{l_1} \frac{k_i+r}{2k_i+2r+1} \prod_{r=1}^{l_1} \frac{k_j+r}{2k_j+2r+1}$.

Thus it is enough to show that

$$\Delta = \begin{bmatrix} 1 & 1 \\ \prod_{r=l_1+1}^{l_2} \frac{k_i+r}{2k_i+2r+1} & \prod_{r=l_1+1}^{l_2} \frac{k_j+r}{2k_j+2r+1} \end{bmatrix}, \tag{2.9}$$

is non-singular for every $l_1, l_2 \in \{0, 1, 2, \ldots\}, l_1 < l_2$.

Let $F_s(x) = \prod_{r=l_1+1}^{l_s} \frac{x+r}{2x+2r+1}, l_1, l_s \in \{0, 1, 2, \ldots\}, l_1 < l_s$. For every $s \in \{1, 2, \ldots\}$, F_s is strictly ascending, and if $x > y$ there exists α_s such that

$$F_s(x) - F_s(y) = \tag{2.10}$$

$$exp(\alpha_s(x-y)), \tag{2.11}$$

and $\alpha_s < \alpha_{s+1},$. Since $k_i < k_j$ and F_s is strictly ascending, thus $\det(\Delta) = F_2(k_j) - F_2(k_i) > 0$.

To prove (2) let

$$\Delta = \begin{bmatrix} 1 & 1 & 1 \\ F_2(k_{i_1}) & F_2(k_{i_2}) & F_2(k_{i_3}) \\ F_3(k_{i_1}) & F_3(k_{i_2}) & F_3(k_{i_3}) \end{bmatrix}, \tag{2.12}$$

be an arbitrary 3×3 submatrix of C, where $k_{i_1} < k_{i_2} < k_{i_3}$. Using Chio pivotal condensation method we have

$$det(\Delta) = \begin{vmatrix} F_2(k_{i_2}) - F_2(k_{i_1}) & F_2(k_{i_3}) - F_2(k_{i_1}) \\ F_3(k_{i_2}) - F_3(k_{i_1}) & F_3(k_{i_3}) - F_3(k_{i_1}) \end{vmatrix}. \tag{2.13}$$

Let $\Phi(s,n) = F_s(k_{i_n}) - F_s(k_{i_1})$, thus (2.13) can be written as

$$det(\Delta) = \begin{vmatrix} \Phi(2,2) & \Phi(2,3) \\ \Phi(3,2) & \Phi(3,3) \end{vmatrix}. \tag{2.14}$$

By (2.10) we have

$$\frac{\Phi(2,2)}{\Phi(3,2)} = \frac{\exp(\alpha_2(k_{i_2} - k_{i_1}))}{\exp(\alpha_3(k_{i_2} - k_{i_1}))}, \tag{2.15}$$

and

$$\begin{aligned} &\tfrac{\Phi(2,3)}{\Phi(3,3)} = \\ &\tfrac{\exp(\alpha_2(k_{i_3}-k_{i_1}))}{\exp(\alpha_3(k_{i_3}-k_{i_1}))} = \tfrac{\exp(\alpha_2(k_{i_2}-k_{i_1}))}{\exp(\alpha_3(k_{i_2}-k_{i_1}))} \times \tfrac{\exp(\alpha_2(k_{i_3}-k_{i_2}))}{\exp(\alpha_3(k_{i_3}-k_{i_2}))}. \end{aligned} \tag{2.16}$$

Since $\frac{\exp(\alpha_2(k_{i_3}-k_{i_2}))}{\exp(\alpha_3(k_{i_3}-k_{i_2}))} < 1$, thus $det(\Delta) > 0$.

For an 4×4 submatrix let

$$\Delta = \begin{bmatrix} 1 & 1 & 1 & 1 \\ F_2(k_{i_1}) & F_2(k_{i_2}) & F_2(k_{i_3}) & F_2(k_{i_4}) \\ F_3(k_{i_1}) & F_3(k_{i_2}) & F_3(k_{i_3}) & F_3(k_{i_4}) \\ F_4(k_{i_1}) & F_4(k_{i_2}) & F_4(k_{i_3}) & F_4(k_{i_4}) \end{bmatrix}. \tag{2.17}$$

where $k_{i_1} < k_{i_2} < k_{i_3} < k_{i_4}$. Using Chio method twice we have

$$det(\Delta) = \mu \begin{vmatrix} \Psi(3,3) & \Psi(3,4) \\ \Psi(4,3) & \Psi(4,4) \end{vmatrix}, \tag{2.18}$$

where $\Psi(s,n) = (F_2(k_{i_2}) - F_2(k_{i_1}))(F_s(k_{i_n}) - F_s(k_{i_1})) - (F_2(k_{i_n}) - F_2(k_{i_1}))(F_s(k_{i_2}) - F_s(k_{i_1}))$ and $\mu = 1/(F_2(k_{i_2}) - F_2(k_{i_1})^2$.

By (2.10) we have

$$\begin{aligned} &\tfrac{\Psi(3,3)}{\Psi(4,3)} = \\ &\tfrac{\exp((\alpha_2+\alpha_3)(k_{i_2}-k_{i_1}))\left[\exp(\alpha_3(k_{i_3}-k_{i_2}))-exp(\alpha_2(k_{i_3}-k_{i_2}))\right]}{\exp((\alpha_2+\alpha_4)(k_{i_2}-k_{i_1}))\left[\exp(\alpha_4(k_{i_3}-k_{i_2}))-exp(\alpha_2(k_{i_3}-k_{i_2}))\right]}, \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} &\tfrac{\Psi(3,4)}{\Psi(4,4)} = \\ &\tfrac{\exp((\alpha_2+\alpha_3)(k_{i_2}-k_{i_1}))\left[\exp(\alpha_3(k_{i_4}-k_{i_1}))-exp(\alpha_2(k_{i_4}-k_{i_2}))\right]}{\exp((\alpha_2+\alpha_4)(k_{i_2}-k_{i_1}))\left[\exp(\alpha_4(k_{i_4}-k_{i_1}))-exp(\alpha_2(k_{i_4}-k_{i_2}))\right]}. \end{aligned} \tag{2.20}$$

Since $\alpha_3 < \alpha_4$ thus $\frac{\exp(\alpha_3(k_{i_4}-k_{i_2}))}{\exp(\alpha_4(k_{i_4}-k_{i_2}))} < 1$ and hence $det(\Delta) > 0$.

Similarly let

$$\Delta = \begin{bmatrix} 1 & 1 & \dots & 1 \\ F_2(k_{i_1}) & F_2(k_{i_2}) & \dots & F_2(k_{i_m}) \\ \vdots & \vdots & \ddots & \vdots \\ F_m(k_{i_1}) & F_m(k_{i_2}) & \dots & F_m(k_{i_m}) \end{bmatrix}, \tag{2.21}$$

be an arbitrary $m \times m$ submatrix of C, where $k_{i_1} < k_{i_2} < \ldots < k_{i_m}$. Using iterative Chio condensation method one have

$$\det(\Delta) = \mu \begin{vmatrix} \chi(m-1, m-1) & \chi(m-1, m) \\ \chi(m, m-1) & \chi(m, m) \end{vmatrix}, \tag{2.22}$$

where μ is a multiplier and χ is a polynomial. By similar argument we have

$$\frac{\chi(m-1, m)}{\chi(m, m)} = \eta \frac{\chi(m-1, m-1)}{\chi(m, m-1)}, \tag{2.23}$$

where $\eta = \frac{\exp(\alpha_{m-1}(k_{i_m} - k_{i_{m-1}}))}{\exp(\alpha_m(k_{i_m} - k_{i_{m-1}}))} < 1$, therefore $\Delta > 0$.

To prove (3) let

$$\Delta = \begin{bmatrix} 2 & \dots & 2 \\ F_2(k_{i_1}) + F_2(k_{j_1}) & \dots & F_2(k_{i_m}) + F_2(k_{j_m}) \\ \vdots & \ddots & \vdots \\ F_m(k_{i_1}) + F_m(k_{j_1}) & \dots & F_m(k_{i_m}) + F_m(k_{j_m}) \end{bmatrix}, \tag{2.24}$$

be an arbitrary sum of two submatrices of C.

We have

$$\det(\Delta) = \mu$$

$$\begin{vmatrix} \chi_i(m-1, m-1) + \chi_j(m-1, m-1) & \chi_i(m-1, m) + \chi_j(m-1, m) \\ \chi_i(m, m-1) + \chi_j(m, m-1) & \chi_i(m, m) + \chi_j(m, m) \end{vmatrix}, \tag{2.25}$$

where μ is a multiplier and χ_i, χ_j are the same polynomials as the proof of (2) in the variables $(k_{j_1}, \ldots, k_{j_m}), (k_{j_1}, \ldots, k_{j_m})$ respectively. Similar argument as in the proof of (2) shows that $\det(\Delta) \neq 0$.

(4) follows from (3) by induction. $\square$

3. Main results

Using the results of previous section we find some families of definite polynomials and solve the vanishing moment problem in special cases.

Lemma 3.1. *Every polynomial in the form $B(t) = (2t-1)(t^2-t)^m$ is definite.*

Proof. Without loose of generality we denote $\overline{B}(t)$ by $(t^2-t)^m$, hence $\overline{B}^k(t) = (t^2-t)^{mk}$. Let $A(t)$ be an arbitrary polynomial, it can be written as $A(t) = a_1(t^2-t) + \ldots + a_n(t^2-t)^n + (2t-1)P(t^2-t)$, where P is a polynomial. Consider the moment conditions (1.6), we have

$$\int_0^1 \overline{B}^k(t)A(t)dt = a_1 I_{mk+1} + \ldots + a_n I_{mk+n} = 0, \quad k = 0, 1, \cdots. \tag{3.1}$$

Consider n equation of the system (3.1). By lemma (2.1) and part (2) of theorem (2.4) we have

$$a_1 = \ldots = a_n = 0,$$

hence $A(t) = (2t-1)P(t^2-t)$ and $A(t), B(t)$ satisfy the composition condition with $w(t) = t^2 - t$. □

Lemma 3.2. *Every polynomial in the form $B(t) = (2t-1)(t^2-t)^m + (2t-1)(t^2-t)^l$ is definite.*

Proof. Without loose of generality, suppose $m < l$, $\overline{B}(t) = \frac{(t^2-t)^{m+1}}{m+1} + \frac{(t^2-t)^{l+1}}{l+1}$, that for simplicity we show it by $(t^2-t)^m + (t^2-t)^l$. Let $A(t) = a_1(t^2-t) + \ldots + a_n(t^2-t)^n + (2t-1)P(t^2-t)$. Consider the moment conditions (1.6), we have

$$\overline{B}^k(t) = \sum_{i=0}^{k} \binom{k}{i} (t^2-t)^{mk-mi+li},$$

$$\begin{aligned}
&\textstyle\int_0^1 \overline{B}^k(t)A(t)dt = \\
&a_1 \textstyle\sum_{i=0}^k \binom{k}{i} I_{mk-mi+li+1} + \ldots + a_n \sum_{i=0}^k \binom{k}{i} I_{mk-mi+li+n} = 0, \\
&k = 0, 1, \ldots,
\end{aligned} \tag{3.2}$$

and by part (4) of theorem (2.4), we have

$$a_1 = \ldots = a_n = 0,$$

and the proof is complete. □

Theorem 3.3. *Let $f(x)$ be a continuous function that is $UV-odd$ and its Fourier coefficients defined by (2.3) are non-negative, then f is definite.*

Proof. By an induction on lemma 3.2 the proof is complete. □

Example 3.4. Let $f(t) = \sum_{n=0}^{\infty} a_n(2t-1)(t^2-t)^n$, where $a_n > 0$ for all $n \geq 0$. Then $f(t)$ is definite on its convergence area. For example the function $\sum_{n=0}^{\infty} (\frac{1}{n^t})(2t-1)(t^2-t)^n$ is definite.

4. Conclusions

In this paper we decompose the polynomials as a linear combination of the functions $(t^2-t)^k$ and $(2t-1)(t^2-t)^k$. These functions are periodic and construct an orthogonal basis for the family of continuous functions. Using Fourier series in terms of these orthogonal basis, one obtain that for the expanding of continuous functions, in special for polynomials, the number of terms in the form $(2t-1)(t^2-t)^k$ determines the vanishing moment conditions and the degree of polynomial is not important.

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Appendix A

Chio pivotal condensation method. Chio pivotal condensation method is a method of calculating the determinant [8](pp. 129-134). In this method an $n\times n$ matrix reduce to an $(n-1)\times(n-1)$ matrix.

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} =$$

$$\frac{1}{a_{11}^{n-1}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \dots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \dots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{31} & a_{3n} \end{vmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{n1} & a_{n3} \end{vmatrix} & \dots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} \end{vmatrix}.$$

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