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Existence of triple positive solutions for boundary value problem of nonlinear fractional differential equations

Kamal Shah*
Department of Mathematics, University of Malakand,
Chakadara Dir(L), Khyber Pakhtunkhwa, Pakistan.
E-mail: kamalshah408@gmail.com

Salman Zeb
Department of Mathematics, University of Malakand,
Chakadara Dir(L), Khyber Pakhtunkhwa, Pakistan.
E-mail: salmanzeb@uom.edu.pk

Rahmat Ali Khan
Department of Mathematics, University of Malakand,
Chakadara Dir(L), Khyber Pakhtunkhwa, Pakistan.
E-mail: rahmat_alipk@yahoo.com

Abstract This article is devoted to the study of existence and multiplicity of positive solutions to a class of nonlinear fractional order multi-point boundary value problems of the type

$$-D^q_{0+}u(t) = f(t, u(t)),\ 1 < q \leq 2, \quad 0 < t < 1,$$

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \delta_i u(\eta_i),$$

where D^q_{0+} represents standard Riemann-Liouville fractional derivative, $\delta_i, \eta_i \in (0, 1)$ with $\sum\limits_{i=1}^{m-2} \delta_i \eta_i^{q-1} < 1$, and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is a continuous function. We use some classical results of fixed point theory to obtain sufficient conditions for the existence and multiplicity results of positive solutions to the problem under consideration. In order to show the applicability of our results, we provide some examples.

Keywords. Fractional differential equations; Boundary value problems; Positive solutions; Green's function; Fixed point theorem.

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1. Introduction

This article is concerned with the existence triple positive solutions to multi-point boundary value problems with nonlinear fractional order differential equations of the

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∗ Corresponding author.

form

$$\begin{aligned} &-D^q_{0+}u(t) = f(t, u(t)),\ 1 < q \leq 2, \quad 0 < t < 1 \\ &u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \delta_i u(\eta_i), \end{aligned} \tag{1.1}$$

where D^q_{0+} is the standard Riemann-Liouville fractional derivative of order q and $\delta_i, \eta_i \in (0, 1)$ with $\sum_{i=1}^{m-2} \delta_i \eta_i^{q-1} < 1$, and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous. Differential equations of fractional order is one of the fast growing area of research in the field of mathematics and have recently been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, one can find numerous applications of fractional order differential equations in viscoelasticity, electro-chemistry, control theory, movement through porous media, electromagnetics, etc (see [2, 6, 10, 11, 18, 19, 21, 22]).

The theory of existence of solutions to boundary value problems associated with fractional differential equations have recently been attracted the attention of many researchers, see for example [1, 3, 4, 5, 8, 12, 13, 16, 23] and the references therein. In these cited references, existence of at least one solution is studied with the tools of classical fixed point theory.

In [23], Rehman and Khan investigated multi-point boundary value problem for fractional order differential equation

$$\begin{aligned} &D^\alpha_t y(t) = f(t, y(t), D^\beta_t y(t)); \quad t \in (0, 1), \\ &y(0) = 0, \quad D^\beta_t y(1) - \sum_{i=1}^{m-2} \zeta_i D^\beta_t y(\xi_i) = y_0, \end{aligned}$$

where $1 < \alpha \leq 2$, $0 < \beta < 1$, $0 < \xi_i < 1$, $\zeta_i \in [0, +\infty)$ with $\sum_{i=1}^{m-2} \zeta_i \xi^{\alpha-\beta-1} < 1$ and obtained sufficient conditions for existence and uniqueness of nontrivial solutions via Schauder fixed point theorem, and contraction mapping principle.

As for the existence of multiple positive solutions are concerned, few papers can be found in the literature dealing with the existence and multiplicity of positive solutions to multi-point boundary value problems for fractional differential equations [7, 24, 29]. Zhang et. al [29] studied existence of positive solutions for the eigenvalue problem corresponding to a class of fractional differential equation

$$\begin{aligned} &-D^\alpha_t x(t) = \lambda f(t, x(t), D^\beta_t x(t)); \quad t \in (0, 1), \\ &D^\beta_t x(0) = 0, \quad D^\gamma_t x(1) = \sum_{j=1}^{p-2} a_j D^\gamma_t x(\xi_j), \end{aligned}$$

where λ is a parameter, $1 < \alpha \leq 2$, $\alpha - \beta > 1$, $0 < \beta \leq \gamma < 1$, $0 < \xi_1 < \xi_2 < ... < \xi_{p-2} < 1$, $a_j \in [0, +\infty)$ with $c = \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\gamma-1} < 1$, and D_t is the standard Riemann-Liouville derivative.

Recently existence of positive solutions for single and system of boundary value problems of nonlinear fractional order differential equations corresponding to different boundary conditions are also studied by many authors for which we refer some recent works in [9, 17, 20, 26, 27, 28].

Inspired from the above works, in this paper, we study a different problem and obtain sufficient conditions for existence, uniqueness as well as conditions for existence of triple positive solutions to the BVP (1.1). For the applicability of our results, we include some examples.

2. Preliminaries

This section contains some necessary definitions and lemmas taken from fractional calculus [18, 21] and functional analysis books . These definitions and lemmas will be used frequently in the forthcoming section.

Definition 2.1. The fractional integral of order $q > 0$ of a function $y : (0, \infty) \to R$ is given by

$$I^q_{0+} y(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s)\, ds,$$

provided that the integral converges.

Definition 2.2. The fractional derivative of order $q > 0$ of a continuous function $y : (0, \infty) \to R$ is given by

$$D^q_{0+} y(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1} y(s)\, ds,$$

where $n = [q] + 1$, provided that the right side is pointwise defined on $(0, \infty)$.

Definition 2.3. A map θ is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E provided that $\theta : P \to [0, \infty)$ is continuous and

$$\theta(tx + (1-t)y) \geq t\theta(x) + (1-t)\theta(y),$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

The next two lemmas provide an important rule for obtaining the equivalent integral equation of BVP (1.1).

Lemma 2.4. [25] *If we assume $u \in C(0,1) \cap L(0,1)$, then the fractional differential equation of order $q > 0$*

$$D^q_{0+} u(t) = 0,$$

has a unique solution of the form

$$u(t) = C_1 t^{q-1} + C_2 t^{q-2} + ... + C_N t^{q-N}, \quad C_i \in R, \quad i = 1, 2, ..., N.$$

The following law of composition can be easily deduced from Lemma 2.4.

Lemma 2.5. *Assume that $u \in C(0,1) \cap L(0,1)$, with a fractional derivative of order q that belongs to $C(0,1) \cap L(0,1)$, then*

$$I^q_{0+} D^q_{0+} u(t) = u(t) + C_1 t^{q-1} + C_2 t^{q-2} + ... + C_N t^{q-N}, \quad C_i \in R, \quad i = 1, 2, ..., N.$$

Lemma 2.6. [14] *Let E be a Banach space, $P \subseteq E$ a cone, and Ω_1, Ω_2 be the two bounded open balls of E centered at the origin with $\overline{\Omega}_1 \subset \Omega_2$. Suppose that $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator such that either*

(i) $\|Tu\| \leq \|u\|, u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|, u \in P \cap \partial\Omega_2$, *or*
(ii) $\|Tu\| \geq \|u\|, u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|, u \in P \cap \partial\Omega_2$

holds. Then T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 2.7. [15] *Let P be a cone in a real Banach space E, $P_c = \{u \in P : \|u\| \leq c\}$, θ a nonnegative continuous concave function on P such that $\theta(u) \leq \|u\|$ for all $u \in \overline{P_c}$, and $P(\theta, b, d) = \{u \in P : b \leq \theta(u), \|u\| \leq d\}$. Suppose $T : \overline{P_c} \to \overline{P_c}$ is completely continuous and there exist constants $0 < a < b < d \leq c$ such that*

(i) $\{u \in P(\theta, b, d) \mid \theta(u) > b\} \neq \emptyset$, *and* $\theta(Tu) > b$ *for* $u \in P(\theta, b, d)$
(ii) $\|Tu\| < a$ *for* $u \leq a$
(iii) $\theta(Tu) > b$ *for* $u \in P(\theta, b, c)$ *with* $\|Tu\| > d$.

Then T has at least three fixed points u_1, u_2, u_3 with $\|u_1\| < a$, $b < \theta(u_2)$, $a < \|u_3\|$ with $\theta(u_3) < b$.

3. Main Results

In this section we develop sufficient conditions on the nonlinear function f, under which the BVP (1.1) has at least one solution and also we study sufficient conditions leading to multiplicity of positive solutions. Denote by $E = C[0, 1]$ the Banach space of all continuous real-valued functions on [0,1] with norm $\|u\| = \sup_{0 \leq t \leq 1} |u(t)|$ and a cone by P such that

$$P = \{u \in E : u(t) \geq 0, t \in [0, 1]\}.$$

Define nonnegative continuous concave functional θ on the cone P as follow

$$\theta(u) = \min_{\eta_{i-2} \leq t \leq \eta_i} |u(t)|. \tag{3.1}$$

Lemma 3.1. *For $y(t) \in C[0, 1]$, the linear BVP*

$$\begin{aligned} &D^q_{0+} u(t) + y(t) = 0; \quad 0 < t < 1, 1 < q \leq 2, \\ &u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \delta_i u(\eta_i), \end{aligned} \tag{3.2}$$

has a unique solution of the form $u(t) = \int_0^1 G(t, s)y(s)\, ds$, where the Green function $G(t, s)$ is given by

$$G(t,s) = \frac{1}{\Gamma(q)} \begin{cases} \frac{t^{q-1}}{1-\lambda} \left[(1-s)^{q-1} - \sum\limits_{j=i}^{m-2} \delta_j (\eta_j - s)^{q-1} \right] - (t-s)^{q-1}; & s \leq t, \ \eta_{i-1} < s \leq \eta_i, \\ & i = 1, 2, ..., m-1, \\ \frac{t^{q-1}}{1-\lambda} \left[(1-s)^{q-1} - \sum\limits_{j=i}^{m-2} \delta_j (\eta_j - s)^{q-1} \right]; & t \leq s, \ \eta_{i-1} < s \leq \eta_i, \\ & i = 1, 2, ..., m-1. \end{cases} \tag{3.3}$$

Proof. In view of Lemma (2.5), we obtain

$$u(t) = -I^q_{0+}y(t) + C_1 t^{q-1} + C_2 t^{q-2}, \tag{3.4}$$

for some $C_1, C_2 \in R$. The boundary condition $u(0) = 0$ implies $C_2 = 0$ and the condition $u(1) = \sum_{i=1}^{m-2} \delta_i u(\eta_i)$, yields $C_1 = \frac{1}{1-\lambda}\left[I^q_{0+}y(1) - \sum_{i=1}^{m-2} \delta_i I^q_{0+}y(\eta_i)\right]$, where $\lambda = \sum_{i=1}^{m-2} \delta_i \eta_i^{q-1} < 1$. Hence, (3.4) takes the form

$$u(t) = -I^q_{0+}y(t) + \frac{t^{q-1}}{1-\lambda}\left[I^q_{0+}y(1) - \sum_{i=1}^{m-2} \delta_i I^q_{0+}y(\eta_i)\right]. \tag{3.5}$$

We discuss several cases: for $0 \le t \le \eta_1$, we write (3.5) as

$$\begin{aligned} u(t) \;\; = \;\; & \int_0^t \big[-\frac{(t-s)^{q-1}}{\Gamma(q)} + \frac{t^{q-1}}{\Gamma(q)(1-\lambda)}\big((1-s)^{q-1} - \sum_{j=1}^{m-2} \delta_j(\eta_j - s)^{q-1}\big)\big] y(s)\,ds \\ & + \frac{t^{q-1}}{\Gamma(q)(1-\lambda)} \int_t^{\eta_1} \big((1-s)^{q-1} - \sum_{j=1}^{m-2} \delta_j(\eta_j - s)^{q-1}\big) y(s)\,ds \\ & + \sum_{i=2}^{m-2} \int_{\eta_{i-1}}^{\eta_i} \big((1-s)^{q-1} - \sum_{j=1}^{m-2} \delta_j(\eta_j - s)^{q-1}\big) y(s)\,ds + \int_{\eta_{m-2}}^1 (1-s)^{q-1} y(s)\,ds. \end{aligned}$$

For $\eta_{l-1} \le t \le \eta_l, 2 \le l \le m-2$, we write (3.5) as

$$\begin{aligned} u(t) \;\; & = \int_0^{\eta_1} \big[-\tfrac{(t-s)^{q-1}}{\Gamma(q)} + \tfrac{t^{q-1}}{\Gamma(q)(1-\lambda)}\big((1-s)^{q-1} - \sum_{j=1}^{m-2} \delta_j(\eta_j - s)^{q-1}\big)\big] y(s)\,ds \\ & + \sum_{i=2}^{m-2} \int_{\eta_{i-1}}^{\eta_i} \big[-\tfrac{(t-s)^{q-1}}{\Gamma(q)} + \tfrac{t^{q-1}}{\Gamma(q)(1-\lambda)}\big((1-s)^{q-1} - \sum_{j=i}^{m-2} \delta_j(\eta_j - s)^{q-1}\big) + (1-s)^{q-1}\big] y(s)\,ds \\ & + \int_{\eta_{l-1}}^t \big[-\tfrac{(t-s)^{q-1}}{\Gamma(q)} + \tfrac{t^{q-1}}{\Gamma(q)(1-\lambda)}\big((1-s)^{q-1} - \sum_{j=l}^{m-2} \delta_j(\eta_j - s)^{q-1}\big)\big] y(s)\,ds \\ & + \tfrac{t^{q-1}}{\Gamma(q)(1-\lambda)}\big[\int_t^{\eta_l} \big((1-s)^{q-1} - \sum_{j=l}^{m-2} \delta_j(\eta_j - s)^{q-1}\big)\big] y(s)\,ds \\ & + \sum_{i=l+1}^{m-2} \int_{\eta_{i-1}}^{\eta_i} \big((1-s)^{q-1} - \sum_{j=i}^{m-2} \delta_j(\eta_j - s)^{q-1}\big) y(s)\,ds \\ & + \int_{\eta_{m-2}}^1 (1-s)^{q-1} y(s)\,ds. \end{aligned}$$

For $\eta_{m-2} \le t \le 1$, we write (3.5) as

$$\begin{aligned} u(t) \;\; = \;\; & \int_0^{\eta_1} \big[-\frac{(t-s)^{q-1}}{\Gamma(q)} + \frac{t^{q-1}}{\Gamma(q)(1-\lambda)}\big((1-s)^{q-1} - \sum_{j=1}^{m-2} \delta_j(\eta_j - s)^{q-1}\big)\big] y(s)\,ds \\ & + \sum_{i=2}^{m-2} \int_{\eta_{i-1}}^{\eta_i} \big[-\frac{(t-s)^{q-1}}{\Gamma(q)} + \frac{t^{q-1}}{\Gamma(q)(1-\lambda)}\big((1-s)^{q-1} - \sum_{j=i}^{m-2} \delta_j(\eta_j - s)^{q-1}\big)\big] y(s)\,ds \\ & + \frac{t^{q-1}}{\Gamma(q)(1-\lambda)}\big[\int_{\eta_{m-2}}^t (1-\lambda)(t-s)^{q-1} + (1-s)^{q-1} y(s)\,ds + \int_t^1 (1-s)^{q-1} y(s)\,ds\big]. \end{aligned}$$

Therefore, the unique solution of the BVP (1.1) is given by $u(t) = \int_0^1 G(t,s)y(s)\,ds$, where

$$G(t,s) = \frac{1}{\Gamma(q)} \begin{cases} \frac{t^{q-1}}{1-\lambda}\left[(1-s)^{q-1} - \sum_{j=i}^{m-2} \delta_j(\eta_j - s)^{q-1}\right] - (t-s)^{q-1}; & s \le t,\ \eta_{i-1} < s \le \eta_i, \\ & i = 1,2,...,m-1, \\ \frac{t^{q-1}}{1-\lambda}\left[(1-s)^{q-1} - \sum_{j=i}^{m-2} \delta_j(\eta_j - s)^{q-1}\right]; & t \le s,\ \eta_{i-1} < s \le \eta_i, \\ & i = 1,2,...,m-1. \end{cases}$$

□

Lemma 3.2. *The Green's function defined by* (3.3) *satisfies the following conditions:*
(i) $G(t,s) > 0$, *for* $t, s \in (0,1)$;
(ii) There exists a positive function $\gamma \in C(0,1)$ *such that*

$$\min_{\eta_{i-1} \le t \le \eta_i} G(t,s) \ge \gamma(s) \max_{0 \le t \le 1} G(t,s) = \gamma(s)G(s,s), \ \textit{for}\ 0 < s < 1. \tag{3.6}$$

Proof. (i): If $\eta_{j-1} < s \le \eta_j$,

$$G(t,s) = \frac{t^{q-1}[(1-s)^{q-1} - \sum_{j=i}^{m-2} \delta_j(\eta_j - s)^{q-1}] - (1-\lambda)(t-s)^{q-1}}{\Gamma(q)(1-\lambda)}.$$

For $t < \eta_j$

$$G(t,s) > \frac{t^{q-1}[(1-s)^{q-1} - \sum_{j=i}^{m-2} \delta_j(\eta_j - s)^{q-1}]}{\Gamma(q)(1-\lambda)} > 0,$$

and for $t \ge \eta_j$

$$G(t,s) \ge \frac{t^{q-1}(1-s)^{q-1} - (t-s)^{q-1}}{\Gamma(q)(1-\lambda)} > 0.$$

For $\eta_j \le s \le t \le 1$,

$$G(t,s) = \frac{t^{q-1}(1-s)^{q-1} - (1-\lambda)(t-s)^{q-1}}{\Gamma(q)(1-\lambda)} \ge \frac{t^{q-1}(1-s)^{q-1} - (t-s)^{q-1}}{\Gamma(q)(1-\lambda)} > 0.$$

Therefore for each $\eta_j \le s \le t \le 1$ and $0 \le t \le s, s \ge \eta_j$. Hence the expression for $G(t,s)$ in (3.3) shows that $G(t,s) > 0$, for $s,t \in (0,1)$.
(ii) Let us denote

$$g_1(t,s) = \frac{t^{q-1}}{\Gamma(q)(1-\lambda)}\big((1-s)^{q-1} - \sum_{j=i}^{m-2} \delta_j(\eta_j - s)^{q-1}\big) - \frac{(t-s)^{q-1}}{\Gamma(q)},$$

$$g_2(t,s) = \frac{t^{q-1}}{\Gamma(q)(1-\lambda)}\big((1-s)^{q-1} - \sum_{j=i}^{m-2} \delta_j(\eta_j - s)^{q-1}\big),$$

then, $\frac{\partial g_1(t,s)}{\partial t} < 0$ and $\frac{\partial g_2(t,s)}{\partial t} > 0$ for $s \le t$, which implies that $g_1(t,s)$ is decreasing and $g_2(t,s)$ is an increasing function for $s \le t$. It follows that $G(t,s)$ is decreasing with respect to t for $s \leq t$ and increasing with respect to t for $t \leq s$. Consequently,

$$\min_{\eta_{i-1}\leq t\leq \eta_i} G(t,s) = \begin{cases} g_1(\eta_i, s), & s \in (0, \eta_{i-1}], \\ \min\{g_1(\eta_i, s), g_2(\eta_{i-1}, s)\}, & s \in [\eta_{i-1}, \eta_i], \\ g_2(\eta_{i-1}, s), & s \in [\eta_i, 1), \end{cases}$$

which implies that

$$\min_{\eta_{i-1}\leq t\leq \eta_i} G(t,s) = \begin{cases} g_1(\eta_i, s), & s \in (0, r], \\ g_2(\eta_{i-1}, s), & s \in [r, 1), \end{cases}$$

where $\eta_{i-1} < r < \eta_i$. Further, we note that

$$\max_{0\leq t\leq 1} G(t,s) = G(s,s) = \frac{s^{q-1}}{\Gamma(q)(1-\lambda)}\big[(1-s)^{q-1} - \sum_{j=i}^{m-2} \delta_i(\eta_i - s)^{q-1}\big],$$

for $s \in (0, 1)$. As a result

$$\gamma(s) = \begin{cases} \left(\frac{\eta_i}{s}\right)^{q-1} - \frac{(\eta_i - s)^{q-1}}{\frac{s^{q-1}}{(1-\lambda)}\left[(1-s)^{q-1} - \sum\limits_{j=i}^{m-2} \delta_j(\eta_j - s)^{q-1}\right]}, & s \in (0, r], \\ \left(\frac{\eta_{i-1}}{s}\right)^{q-1}, & s \in [r, 1). \end{cases}$$

□

In view of Lemma (3.1), the BVP (1.1) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s) f(s, u(s)) ds, \tag{3.7}$$

and by a solution of the BVP (1.1), we mean a solution of the integral equation (3.7), that is a fixed point of the operator $T : P \to P$ defined by

$$Tu(t) = \int_0^1 G(t,s) f(s, u(s)) ds. \tag{3.8}$$

Lemma 3.3. *For nonnegative real-valued functions m, $n \in L[0, 1]$ such that*

$$f(t, u) \leq m(t) + n(t)u, \text{ for almost every } t \in [0, 1], \text{ and all } u \in [0, \infty), \tag{3.9}$$

the operator T defined by (3.8) is completely continuous.

Proof. Due to nonnegativity and continuity of $G(t, s)$ and $f(t, s)$, the operator T is continuous. For each $u \in \Omega = \{u \in P : \|u\| \leq M, M > 0\}$, we have

$$\begin{aligned} |Tu(t)| = \left| \int_0^1 G(t,s) f(s, u(s)) ds \right| &\leq \int_0^1 G(t,s)(m(s) + n(s)u(s)) ds \\ &\leq \int_0^1 G(s,s) m(s) ds + M \int_0^1 G(s,s) n(s) ds = l, \end{aligned}$$

which implies that $T(\Omega)$ is bounded.

For equicontinuity of $T : P \to P$, take $t_1, t_2 \in [0, 1]$ such that $t_1 < t_2$ with $t_2 - t_1 < \delta$. Then for $u \in \Omega$, we have

$$|Tu(t_2) - Tu(t_1)| = \left| \int_0^1 [G(t_2, s) f(s, u(s)) - G(t_1, s) f(s, u(s))] ds \right|$$

$$\leq \int_0^1 |G(t_2,s)-G(t_1,s)|(m(s)+n(s)M)ds,$$

which in view of the continuity of $G(t,s)$ and $K = \max\{m(s)+n(s)M : s \in [0,1]\}$ implies that

$$\begin{aligned} |Tu(t_2)-Tu(t_1)| \quad &\leq \quad \frac{1}{\Gamma(q)}\left|t_2^{q-1}-t_1^{q-1}\right|\int\limits_0^1 \frac{1}{1-\lambda}\left[(1-s)^{q-1}-\sum_{j=1}^{m-2}\delta_i(\eta_i-s)^{q-1}\right]Kds \\ &\leq \frac{1}{\Gamma(q)}\left|t_2^{q-1}-t_1^{q-1}\right|\frac{K}{q(1-\lambda)}. \end{aligned}$$

Using mean value theorem on $|t_2^{q-1}-t_1^{q-1}|$ and the choice $|t_2-t_1|<\delta$, we have

$$|Tu(t_2)-Tu(t_1)| \leq \frac{K}{\Gamma(q)q(1-\lambda)}(q-1)C^{q-2}|t_2-t_1| \leq \frac{K(q-1)}{\Gamma(q)q(1-\lambda)}\delta^{q-1} \leq \epsilon$$

where $C^{q-2} \leq \delta^{q-2}$, $\delta = \left(\frac{\epsilon}{Aq}\right)^{\frac{1}{q-1}}$ and $A = \frac{K}{\Gamma(q)q(1-\lambda)}$. Hence $T : P \to P$ is equicontinuous. By Arzela-Ascoli theorem, we conclude that the operator $T : P \to P$ is completely continuous. □

Now, we show existence of at least one solution of the BVP (1.1). Fix $M = \left(\int_0^1 G(s,s)ds\right)^{-1}$ and $N = \left(\int_{\eta_{i-1}}^{\eta_i} \gamma(s)G(s,s)ds\right)^{-1}$.

Theorem 3.4. *Assume that the condition* (3.9) *of Lemma* (3.3) *holds and there exist two positive constants $r_2 > r_1 > 0$ such that*

$$f(t,u) \leq Mr_2,\ \textit{for}\ (t,u) \in [0,1]\times[0,r_2]\ \textit{and}\ f(t,u) \geq Nr_1,\ \textit{for}\ (t,u) \in [0,1]\times[0,r_1]. \tag{3.10}$$

Then BVP (1.1) *has at least one positive solution u such that $r_1 \leq \|u\| \leq r_2$.*

Proof. In view of (3.8) and Lemma (3.9), T is completely continuous. By Schauder fixed point theorem, T has a fixed point, $Tu = u$.

Define $\Omega_2 = \{u \in P : \|u\| < r_2\}$. For $u \in \partial\Omega_2$, we have $0 \leq u(t) \leq r_2$ for all $t \in [0,1]$ and using (3.10) and Lemma (3.2), it follows that

$$\|Tu\| = \max_{0\leq t\leq 1}\int_0^1 G(t,s)f(s,u(s))ds \leq Mr_2\int_0^1 G(s,s)ds = r_2 = \|u\|.$$

Similarly, define $\Omega_1 = \{u \in P : \|u\| < r_1\}$. For $u \in \partial\Omega_1$, we have $0 \leq u(t) \leq r_1$ for all $t \in [0,1]$ and for $t \in [\eta_{i-i},\eta_i]$, we have

$$\begin{aligned} \|Tu\| = \int_0^1 G(t,s)f(s,u(s))ds &\geq \int_0^1 \gamma(s)G(s,s)f(s,u(s))ds \\ &\geq Nr_1\int_{\eta_{i-1}}^{\eta_i}\gamma(s)G(s,s)ds = r_1 = \|u\|. \end{aligned}$$

□

Example 1. For the BVP

$$\begin{aligned} & D_{0+}^{\frac{3}{2}}u(t) + u^2 + \frac{\cos t}{4} + \frac{1}{5} = 0,\ 0 < t < 1, \\ & u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \delta_i u(\eta_i) = \frac{1}{5}, \end{aligned} \tag{3.11}$$

$M = \left(\int_0^1 G(s,s)ds\right)^{-1} \approx 1.088$ and $N = \left(\int_{\frac{1}{4}}^{\frac{3}{4}} \gamma(s)G(s,s)\,ds\right)^{-1} = 4.914$. Choosing $r_1 = \frac{1}{25}$, $r_2 = \frac{3}{4}$, we have

$$f(t,u) = u^2 + \frac{\cos t}{4} + \frac{1}{5} \le 1.151 \le Mr_2, for\ (t,u) \in [0,1] \times \left[0, \frac{3}{4}\right],$$

$$f(t,u) = u^2 + \frac{\cos t}{4} + \frac{1}{5} \ge \frac{1}{5} \ge Nr_1, for\ (t,u) \in [0,1] \times \left[0, \frac{1}{25}\right].$$

By Theorem 3.2, the BVP (3.11) has at least one solution u such that $\frac{1}{25} \le \|u\| \le \frac{3}{4}$.

Theorem 3.5. *Let there exists $h(t) \in L[0,1]$ such that $\int_0^1 G(s,s)h(s)ds < 1$ and*

$$|f(t,u) - f(t,v)| \le h(t)|u - v|,$$

for almost $t \in L[0,1]$, and $u, v \in [0,\infty)$, then the BVP (1.1) has a unique positive solution.

Proof. For u, $v \in P$, we have

$$\begin{aligned} |Tu(t) - Tv(t)| &= \left| \int_0^1 G(t,s)(f(s,u(s) - f(s,v(s))ds \right| \\ &\le \int_0^1 G(t,s)|(f(s,u(s)) - f(s,v(s))|ds, \\ &\le \int_0^1 G(s,s)h(s)|u(s) - v(s)|ds \le \int_0^1 G(s,s)h(s)ds\|u - v\| \\ &= \alpha\|u - v\|, \end{aligned}$$

where $\alpha = \int_0^1 G(s,s)h(s)ds < 1$. Therefore, by Banach contraction principle, The BVP has a unique solution. □

Example 2. Consider the BVP

$$\begin{aligned} & D_{0+}^{\frac{3}{2}}u(t) + \frac{e^{2t}u}{4(1+e^{2t})(1+u)} + \cos^2 t + 1 = 0,\ 0 < t < 1, \\ & u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \delta_i u(\eta_i) = \frac{1}{5}. \end{aligned} \tag{3.12}$$

Here, $f(t,u) = \frac{e^{2t}u}{4(1+e^{2t})(1+u)} + \cos^2 t + 1$ and taking $h(t) = \frac{e^{2t}}{4(1+e^{2t})}$, then

$$\begin{aligned} \int_0^1 G(s,s)h(s)ds &\leq \int_0^1 \frac{\sqrt{s}}{5} + \frac{2\sqrt{s(1-s)}}{\sqrt{\pi}} \frac{e^{2s}}{4(1+e^{2s})} ds \\ &\leq \int_0^1 \frac{\sqrt{s}}{5} + \frac{2\sqrt{s(1-s)}}{\sqrt{\pi}} ds = 0.576446 \end{aligned}$$

and

$$|f(t,u) - f(t,v)| \leq h(t)|u-v|, \quad for \quad (t,u),(t,v) \in [0,1] \times [0,\infty).$$

By Theorem (3.5), the BVP has a unique positive solution.

Theorem 3.6. *Assume that the condition (3.9) of Lemma (3.3) holds and there exist positive constants* $0 < a < b < c$ *such that*

(i) $f(t,u) < Ma$, *for* $(t,u) \in [0,1] \times [0,a]$
(ii) $f(t,u) \geq Nb$, *for* $(t,u) \in [\eta_{i-1}, \eta_i] \times [b,c]$
(iii) $f(t,u) \leq Mc$, *for* $(t,u) \in [0,1] \times [0,c]$,

then the BVP (1.1) has at least three positive solutions u_1, u_2, *and* u_3 *such that*

$$\max_{0\leq t\leq 1} |u_1(t)| < a,\ b < \min_{\eta_{i-1}\leq t\leq \eta_i} |u_2(t)| \leq c, a < \max_{0\leq t\leq 1} |u_3(t)| < c,\ \min_{\eta_{i-1}\leq t\leq \eta_i} |u_3(t)| < b. \tag{3.13}$$

Proof. For $u \in \overline{P_c}$, the relation

$$\|Tu\| = \max_{0\leq t\leq 1} \left| \int_0^1 G(t,s)f(s,u(s))ds \right| \leq \int_0^1 G(s,s)f(s,u)s))ds \leq \int_0^1 G(s,s)Mcds \leq c$$

follows from item (iii) and the completely continuity of $T : \overline{P}_c \to P_c$ follows from Lemma (3.3). Choose $u(t) = \frac{b+c}{2}, 0 \leq t \leq 1$. Then using (3.1), we have $u(t) = \frac{b+c}{2} \in P(\theta,b,c), \theta(u) = \theta(\frac{(b+c)}{2}) > b$ implies that $\{u \in P(\theta,b,c)\ |\theta(u) > b\} \neq \emptyset$. Hence, if $u \in P(\theta,b,c)$, then $b \leq u(t) \leq c$ for $\eta_{i-1} \leq t \leq \eta_i$. Also, from assumption (ii), we have $f(t,u(t)) \geq Nb$, for $\eta_{i-1} \leq t \leq \eta_i$ and

$$\theta(Tu) = \min_{\eta_{i-1}\leq t\leq \eta_i} |(T(u))| \geq \int_0^1 \gamma(s)G(s,s)f(s,u(s))ds > \int_{\eta_{i-1}}^{\eta_i} \gamma(s)G(s,s)Nbds = b,$$

which implies that

$$\theta(Tu) > b,\ for\ all\ u \in P(\theta,b,c).$$

Hence by Lemma (2.7), the BVP (1.1) has at least three positive solutions u_1, u_2,and u_3 satisfying (3.13). □

Example 3. For the problem

$$D_{0+}^{\frac{3}{2}} u(t) + f(t,u) = 0, 0 < t < 1,\ u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \delta_i u(\eta_i) = \frac{1}{5}, \tag{3.14}$$

where

$$f(t,u) = \begin{cases} \frac{\cos t}{100} + u^2; & u \leq 1 \quad , \\ 3 + \frac{\cos t}{100} + u; & u > 1 \quad , \end{cases}$$

we find that $M \approx 1.088$ and $N = 4.914$. Choosing $a = \frac{1}{10},\ b = \frac{3}{4}$ and $c = 44$, we have

$$f(t,u) = \frac{\cos t}{100} + u^2 \leq 0.02 < Ma \approx 0.1088,\ for\ (t,u) \in [0,1] \times \big[0, \frac{1}{10}\big],$$

$$f(t,u) = 3 + \frac{\cos t}{100} + u \leq 47.01 \geq Nb \approx 3.6855, for\ (t,u) \in \big[\frac{1}{4}, \frac{3}{4}\big] \times \big[\frac{3}{4}, 44\big],$$

$$f(t,u) = 3 + \frac{\cos t}{100} + u \leq 47.01 \leq Mc \approx 47.872, for\ (t,u) \in [0,1] \times [0,44].$$

Hence, by Theorem (3.6), the BVP (3.14) has at least three positive solutions u_1, u_2, and u_3 with

$$0 < \max_{0 \leq t \leq 1} |u_1(t)| < \tfrac{1}{10},\ 1 < \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} |u_2(t)| \leq 44,$$

$$\tfrac{1}{10} < \max_{0 \leq t \leq 1} |u_3(t)| < 44, \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} |u_3(t)| < \tfrac{3}{4}.$$

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