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Solution of Troesch's problem through double exponential Sinc-Galerkin method

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Abstract Sinc-Galerkin method based upon double exponential transformation for solving Troesch's problem is given in this study. Properties of the Sinc-Galerkin approach are utilized to reduce the solution of nonlinear two-point boundary value problem to same nonlinear algebraic equations, also, the matrix form of the nonlinear algebraic equations is obtained.The error bound of the method is found. Moreover, in order to illustrate the accuracy of presented method, the obtained results compare with numerical results in the open literature. The demonstrated results confirm that proposed method is considerably efficient and accurate.

Keywords. Sinc Function, Galerkin method, Double exponential transformation, Nonlinear Troesch's problem, BVP.

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1. Introduction

Sinc approximation methods have been proposed and studied by F. Stenger since 1974 [18]. These methods have been recognized as powerful tools for solving a wide range of linear and nonlinear problems arising from scientific and engineering applications. It is well known that the approximation by Sinc approaches has the order of accuracy $O\Big(\exp(-k\sqrt{n})\Big)$ where C is a positive constant and n is the number of node or bases functions used in the method [12, 19].

In 2002, Sugihara composed Sinc function by double exponential transformation, discovered by Mori [22], and found that the error of new method is $O\Big(exp(-k'n/logn)\Big)$ with some positive k' [20, 21].

In the current study, application of Sinc-Galerkin method based on double exponential transformation (DE) to solve Troesch's problem is developed.

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Troesch's problem is a special nonlinear two point boundary value problem, defined as

$$\begin{cases} y''(x) = m\sinh(m\ y(x)),\ 0 < x < 1 \\ \\ y(0) = 0,\ y(1) = 1, \end{cases} \tag{1.1}$$

where $'m'$ is a positive constant. This problem arises in an investigation of the confinement of a plasma column by radiation pressure [25] as well as in the theory of gas porous electrodes [7, 13].
The closed form solution to this problem in terms of the Jacobian elliptic function has been given [5] as

$$y(x) = \frac{2}{m}\sinh^{-1}\Big\{\frac{y'(0)}{2}Sc\Big(m\ x|1-\frac{1}{4}y'(0)^2\Big)\Big\}, \tag{1.2}$$

where $y'(0)$ (the derivation of y at $x=0$) is given by expression $y'(0) = 2\sqrt{1-t}$, with t being the solution of the transcendental equation

$$\frac{\sinh(\frac{m}{2})}{\sqrt{1-t}} = Sc(m|t), \tag{1.3}$$

where the Jacobian elliptic function $Sc(m|t)$ is defined by $Sc(m|t) = \tan\phi$, where ϕ, m are related through the integral below

$$m = \int_0^{\phi} \frac{1}{\sqrt{1-t-\sin^2\theta}}d\theta. \tag{1.4}$$

It has been shown that $y'(x)$ has a singularity located approximately at [15, 24]

$$\chi_s = \frac{1}{m}\ln\Big(\frac{8}{y'(0)}\Big), \tag{1.5}$$

which implies that the singularity lies the integration range of $y'(0) > 8e^{-\eta}$.

Many of researches have been conducted to study and solved this problem by approximation methods. Chang [1] applied shooting method, Xinlong et al [5] used an modified homotopy perturbation method, Zarebnia et al. [26] developed Sinc-Galerkin method based on single exponential (SE) transformation. They found that the rate of convergence by using Sinc-Galerkin method based on SE transformation is $O\Big(\exp(-k\sqrt{n})\Big)$. EL-Gamel [6] applied Sinc-collocation method based on SE transformation. Deeba et al. [3] proposed decomposition approximation method. Saadatmandi et al. [16] applied Christov rational functions and collocation points, Khuri et al. [5] developed B-Spline method, Scott [17] used an invariant imbedding method.

The rest of the paper is organized into five sections. A brief introduction to the Sinc function, definitions, theorems and notations is presented in Section 2. In Section 3, Sinc-Galerkin method based on DE transformation to solve Troesch's problem was developed and also find the error of the approach. The obtained numerical results are compared with numerical ones of the existing methods in Section 4. In Section 5, the conclusions of study is made.

2. SINC FUNCTION PROPERTIES

In this section, some properties of the Sinc function, quadrature as well as its notations to be reviewed.

The Sinc function is defined on the whole real line, $-\infty < x < \infty$, by

$$\mathrm{Sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

For any $h > 0$, the translated Sinc function with evenly spaced nodes are given as

$$\mathrm{S}(j,h)(x) = \mathrm{Sinc}\Big(\frac{x - jh}{h}\Big), \; j = 0, \pm 1, \pm 2, ... \tag{2.1}$$

The $S(j,h)$ is called the **j**th Sinc function with step size h at x.

Lemma 2.1. [12] *Let $S(k,h)(x)$ is the **k**th Sinc function with step h, so*

$$\delta^{(0)}_{jk} = S(j,h)(kh) = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases}$$

$$\delta^{(1)}_{jk} = h\frac{d}{dz}[S(j,h)(z)](kh) = \begin{cases} 0, & j = k, \\ \frac{(-1)^{k-j}}{k-j}, & j \neq k, \end{cases}$$

$$\delta^{(2)}_{jk} = h^2\frac{d^2}{dz^2}[S(j,h)(z)](kh) = \begin{cases} \frac{-\pi^2}{3}, & j = k, \\ \frac{-2(-1)^{k-j}}{(k-j)^2}, & j \neq k. \end{cases}$$

For the assembly of the discrete system, it is convenient to define the following matrices

$$I^{(l)} = [\delta^{(l)}_{jk}] \qquad l = 0, 1, 2, \tag{2.2}$$

where $\delta^{(l)}_{jk}$ denotes the (j,k)th element of the matrix $I^{(l)}$. The matrix $I^{(0)}$ is the $m \times m$ identity matrix. The matrix $I^{(1)}$ is the skew symmetric Toeplitz matrix and $I^{(2)}$ is the symmetric Toeplitz matrix.

The following notation will be necessary for writing down the system. Let $D(g)$ be the $m \times m$ diagonal matrix

$$D\Big(g(x)\Big) = \begin{pmatrix} g(-Nh) & 0 & 0 & \dots & 0 \\ 0 & g((-N+1)h) & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & 0 & & g((Nh)) \end{pmatrix}.$$

If the function f is defied on the real line, then for $h > 0$ the series

$$\mathrm{C}(f,h)(x) = \sum_{j=-\infty}^{\infty} \mathrm{f}(jh)\mathrm{Sinc}\Big(\frac{x - jh}{h}\Big),$$

is called the Whittaker cardinal expansion of f where this series converges [19]. These properties are derived in the infinite strip D_d of the complex plane, where for $d > 0$

$$D_d = \Big\{ w = \xi + i\eta : |\eta| < d < \frac{\pi}{2} \Big\}.$$

To state the decay property of functions precisely, we introduce the following function space. Let $H^1(D_d)$ be a function space defined as

$$H^1(D_d) = \{ f : D_d \to C | f \ is \ analytic \ on \ D_d \ and \ N^1(f, D_d) < \infty \},$$

where

$$N^1(f, D_d) \equiv \lim_{\varepsilon \to 0} \int_{\partial D_d(\varepsilon)} |f(t)| \ |dt|,$$

$$D_d(\varepsilon) = \{ \ t \in \mathbf{C} | \ |Re \ t| \leq 1/\varepsilon, \ |Im \ t| \leq d(1-\varepsilon) \}.$$

Theorem 2.2. [23] *Assume that a function f satisfies*

1)$f \in H^1(D_d)$,

2)$\forall x \in \boldsymbol{R} : | \ f(x) \ | \leq A \exp(-B \ \ \exp(\gamma \ \ | \ x \ |))$,

for positive constants A, B, γ and d where $\gamma d \leq \frac{\pi}{2}$. Then there exists a constant C independent of N, such that

$$\sup_{-\infty < x < \infty} \Big| \ f(x) - \sum_{k=-N}^{N} f(kh) S(k,h)(x) \Big| \leq C \exp\Big(- \frac{\pi d \gamma N}{log(\pi d \gamma N/B)} \Big),$$

where

$$h = \frac{\log(\pi d \gamma N / B)}{\gamma N}.$$

Theorem 2.3. [20] *For $d > 0$, let f be a holomorphic function on D_d such that*

$$N(f,d) \equiv \lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} (| \ f(x + i(d-\varepsilon)) \ | + | \ f(x - i(d-\varepsilon)) \ |) dx < \infty, \tag{2.3}$$

$$\lim_{\varepsilon \to 0^+} \int_{(d-\varepsilon)}^{-(d-\varepsilon)} | \ f(x+iy) \ | \ dy = 0, \tag{2.4}$$

for arbitrary ε with $0 < \varepsilon < 1$, and

$$\forall x \epsilon R : | \ f(x) \ | \leq A \exp(-B \ \ \exp(\gamma \ \ | \ x \ |)), \tag{2.5}$$

for constants A, $B > 0$ and $\gamma > 0$ with $\gamma d \leq \frac{\pi}{2}$. Then there exists a constant C independent of N, such that

$$\Big| \int_{-\infty}^{\infty} f(x) dx - h \sum_{k=-N}^{N} f(kh) \Big| \leq C \exp\Big(- \frac{2\pi d \gamma N}{log(2\pi d \gamma N/B)} \Big), \tag{2.6}$$

where

$$h = \frac{\log(2\pi d \gamma N/B)}{\gamma N}. \tag{2.7}$$

3. DE-Sinc-Galerkin Method for Troesch's Problem

Consider Troesch's problem as

$$\begin{cases} L(y(x)) \equiv y''(x) - m\ \sinh(m\ y(x)) = 0, & 0 < x < 1, \\ \\ y(0) = 0,\ y(1) = 1. \end{cases} \tag{3.1}$$

Before illustrating Sinc-Galerkin method based on DE transformation, we need to convert the nonhomogeneous boundary conditions to homogeneous ones. For this purpose the change of variable $u(x) = y(x) - x$ was considered. So by applying this change of variable the above problem is converted to

$$\begin{cases} L(u(x)) \equiv u''(x) - m\ \sinh(m\ u(x) + m\ x) = 0,\ 0 < x < 1, \\ \\ u(0) = 0,\ u(1) = 0. \end{cases} \tag{3.2}$$

Since the suitable domain for Sinc function is $(-\infty, \infty)$, so for problems with another domain, there are two points of view. The first is to change of variables in the problem so that, in new variables, the problem has a domain corresponding to $(-\infty, \infty)$. The second procedure is to move the numerical process and study it on original domain of problem. The former approach is chosen here. For our problem with domain $(0, 1)$, the appropriate transformation is following conformal map

$$x = \psi(t) = \frac{1}{2}\tanh\left(\frac{\pi}{2}\sinh(t)\right) + \frac{1}{2}, \tag{3.3}$$

$$t = \phi(x) = \psi^{-1}(x) = \log\left[\frac{1}{\pi}\log(\frac{x}{1-x}) + \sqrt{1 + \left(\frac{1}{\pi}\log(\frac{x}{1-x})\right)^2}\right], \tag{3.4}$$

which is called double exponential (DE) transformation and Sinc-Galerkin method based on this transformation is called DE-Sinc-Galerkin method. This DE transformation [22, 23] maps $\mathbf{R}$ onto $(0, 1)$ and maps D_d onto the domain

$$\psi(D_d) = \left\{z\epsilon\mathcal{C} : \left|\arg\left(\frac{1}{\pi}\log(\frac{z}{1-z}) + \sqrt{1 + \left(\frac{1}{\pi}\log(\frac{z}{1-z})\right)^2}\right)\right| < d\right\}.$$

By applying ψ to problem (3.2), this problem is transformed to new one on $(-\infty, \infty)$ as follows

$$\begin{cases} L(u(\psi(t))) \equiv \frac{d^2}{dx^2}u(\psi(t)) - m\ \sinh(m\ u(\psi(t)) + m\ \psi(t)) = 0, \\ \\ \lim_{t\to\pm\infty} u(\psi(t)) = 0. \end{cases} \tag{3.5}$$

Set $v(t) = u(\psi(t))$, so by chain rule of differentiation we obtain

$$\frac{d}{dx}u(\psi(t)) = \frac{1}{\psi'(t)}v'(t), \tag{3.6}$$

$$\frac{d^2}{dx^2}(u(\psi(t))) = \left(\frac{1}{\psi'(t)}\right)^2 v''(t) - \frac{\psi''(t)}{(\psi'(t))^3}v'(t). \tag{3.7}$$

By using the formulas (3.6) and (3.7) in eq (3.5) and multiply by $\Big(\psi'(t)\Big)$ we have

$$\begin{cases} L(v(t)) \equiv \frac{1}{\psi'(t)}v''(t) + \Big(-\frac{\psi''(t)}{(\psi'(t))^2}\Big)v'(t) - m\Big(\psi'(t)\Big)\sinh\Big(mv(t)+m\psi(t)\Big) = 0, \\ \\ \lim_{t\to\pm\infty} v(t) = 0. \end{cases} \tag{3.8}$$

To approximate the solution of problem (3.8) the Sinc approximation by the following formula was considered

$$v_r(t) = \sum_{j=-N}^{N} c_j S_j(t), \;\; r = 2N+1, \tag{3.9}$$

where the bases Sinc functions $S_j(t) = S(j,h)(t)$ are defined in (2.1) and the unknown coefficients $\{c_j\}_{k=-N}^{N}$ need to be determined. Notice that the v_r satisfies the boundary conditions because of $\lim_{t\to\pm\infty} S_j(t) = 0$.

Our purpose is to apply the Sinc-Galerkin method based on DE transformation on the problem (3.8). At first, the following form of inner product for arbitrary function f and g was considered as

$$\prec f, g \succ = \int_{-\infty}^{+\infty} f(t)g(t)w(t)dt, \tag{3.10}$$

where $w(t)$ is a weight function. The weight function in the Sinc-Galerkin inner product may be chosen based on a variety of reasons. For instance, Lund [12] used a kind of w for symmetrization of system in a self-adjoint linear problems, but the selection that we make here is due to the requirement of vanishing the boundary terms when using the integration by part in inner product. In this case, we use $w(t) = \psi'(t)$. For more information about different choices of the weight function see [12, 14, 18].

The coefficients $\{c_j\}_{j=-N}^{N}$ are determined by orthogonalizing the residual Lv with respect to the functions $\Big\{S_k(x)\Big\}_{k=-N}^{N}$, in other words

$$\prec Lv, S_k \succ = 0, \;\; k = -N, -N+1, ..., N. \tag{3.11}$$

So we have

$$\prec \frac{1}{\psi'(t)}v''(t), S_k(t) \succ - \prec \frac{\psi''(t)}{(\psi'(t))^2}v'(t), S_k(t) \succ \\ - \prec \psi'(t) m \sinh(mv(t) + m\psi(t), S_k(t) \succ = 0,$$

$$k = -N, -N+1, ..., N. \tag{3.12}$$

By considering the inner product in (3.10), the above inner products are calculated as follow

$$\prec \psi'(t) m \sinh(mv(t)+m\psi(t)), S_k(t) \succ = \int_{-\infty}^{+\infty} \Big(\psi'(t)\Big)^2 m \sinh(mv(t) + m\psi(t)) S_k(t) dt. \quad (3.13)$$

By applied quadrature formula in Theorem 2.3, we obtain

$$\Big| \int_{-\infty}^{+\infty} \Big(\psi'(t)\Big)^2 m \sinh(mv(t) + m\psi(t)) S_k(t) dt - h\Big(\Big(\psi'(kh)\Big)^2 m \sinh(mc_k + m\psi(kh))\Big)\Big| \leq C_0 \exp\Big(-\frac{k'N}{log(k'N/B)}\Big), \quad (3.14)$$

where h is defined in (2.7) and $k' = 2\pi d\gamma$.

Another inner product is

$$\prec \frac{\psi''(t)}{(\psi'(t))^2} v'(t), S_k(t) \succ = \int_{-\infty}^{+\infty} \psi'(t) \frac{\psi''(t)}{(\psi'(t))^2} v'(t) S_k(t) dt \quad (3.15)$$

$$= v(t) \frac{\psi''(t)}{(\psi'(t)} S_k(t)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} v(t) \Big[\Big(\frac{\psi''(t)}{(\psi'(t))}\Big)' S_k(t) + \frac{\psi''(t)}{(\psi'(t))} S_k'(t)\Big] dt,$$

but $\lim_{t\to\pm\infty} S_k(t) = 0$, $\lim_{t\to\pm\infty} v(t) = 0$, $\lim_{t\to\pm\infty} \frac{\psi''}{\psi'}(t) = 0$. So the integration of (3.15) is equal to

$$-\int_{-\infty}^{+\infty} v(t) \Big[\Big(\frac{\psi''(t)}{(\psi'(t))}\Big)' S_k(t) + \frac{\psi''(t)}{(\psi'(t))} S_k'(t)\Big] dt. \quad (3.16)$$

Based on quadrature formulas in Theorem 2.3 we have

$$\Big| \int_{-\infty}^{+\infty} v(t) \Big[\Big(\frac{\psi''(t)}{(\psi'(t))}\Big)' S_k(t) + \frac{\psi''(t)}{(\psi'(t))} S_k'(t)\Big] dt - h \sum_{j=-N}^{N} v(jh) \Big[\Big(\frac{\psi''(jh)}{(\psi'(jh))}\Big)' S_k(jh) + \frac{\psi''(jh)}{(\psi'(jh))} S_k'(jh)\Big]\Big| \quad (3.17)$$

$$\leq C_1 \exp\Big(-\frac{k'N}{log(k'N/B)}\Big).$$

So

$$\prec \frac{1}{\psi'(t)} v''(t), S_k(t) \succ = \int_{-\infty}^{+\infty} v''(t) S_k(t) dt. \quad (3.18)$$

By twice integration by part we can evaluate above integration as follows

$$\prec \frac{1}{\psi'(t)} v''(t), S_k(t) \succ = v'(t) S_k(t) - v(t) S_k'(t) + \int_{-\infty}^{+\infty} v(t) S_k''(t) dt. \quad (3.19)$$

According to the boundary conditions and $\lim_{t\to\pm\infty} S_k(t) = 0$, and based on quadrature formulas in Theorem 2.3 we get

$$\Big| \int_{-\infty}^{+\infty} v(t) S_k''(t) dt - h \sum_{j=-N}^{N} v(jh) S_k''(jh) \Big| \leq C_2 \exp\Big(- \frac{k'N}{log(k'N/B)} \Big). \tag{3.20}$$

To find the discrete system for equation (3.11) considering (3.12), (3.14), (3.17), (3.20) we have

$$\begin{aligned}
&\Big| \prec Lv, S_k \succ -h \sum_{j=-N}^{N} v(jh) \Big\{ S_k''(jh) + \Big(\frac{\psi''(jh)}{(\psi'(jh))} \Big) S_k'(jh) + \\
&\Big(\frac{\psi''(jh)}{(\psi'(jh))} \Big)' S_k(jh) \Big\} - h \Big(\psi'(kh) \Big)^2 m \sinh(mv(kh) + m\psi(kh)) \Big| \\
&\leq \Big| \prec \frac{1}{\psi'(t)} v''(t), S_k(t) \succ -h \sum_{j=-N}^{N} v(jh) S_k''(jh) \Big| + \\
&\Big| \prec \frac{\psi''(t)}{(\psi'(t))^2} v'(t), S_k(t) \succ -h \sum_{j=-N}^{N} v(jh) \Big[\Big(\frac{\psi''(jh)}{(\psi'(jh))} \Big)' S_k(jh) + \\
&\frac{\psi''(jh)}{(\psi'(jh))} S_k'(jh) \Big] \Big| + \Big| \prec \psi'(t) m \sinh(mv(t) + m\psi(t)), S_k(t) \succ - \\
&h \Big(\Big(\psi'(kh) \Big)^2 m \sinh(mv(kh) + m\psi(kh)) \Big) \Big| \\
&\leq \Big(C_0 + C_1 + C_2 \Big) \exp\Big(\frac{-k'N}{\log(k'N/B)} \Big).
\end{aligned} \tag{3.21}$$

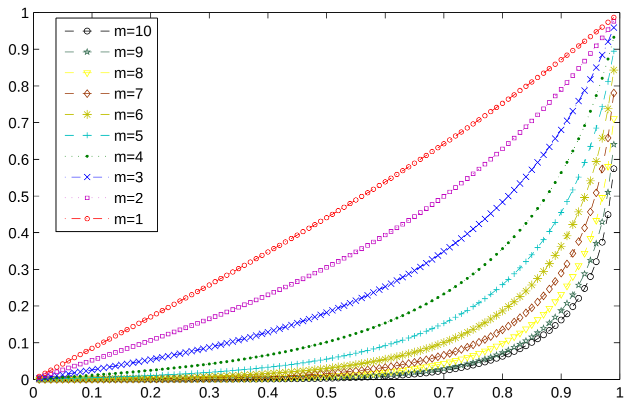
Deleting the error term of order $O\Big(\exp\Big(- \frac{k'N}{\log(k'N/B)} \Big) \Big)$, replacing $v(jh)$ by c_j, dividing by h, and by considering the definition of $\delta_{kj}^{(l)}$ we have the following nonlinear algebraic system

$$\begin{aligned}
&\sum_{j=-N}^{N} c_j \Big\{ \frac{1}{h^2} \delta_{kj}^{(2)} + \frac{1}{h} \Big(\frac{\psi''(jh)}{(\psi'(jh))} \Big) \delta_{kj}^{(1)} + \Big(\frac{\psi''(jh)}{(\psi'(jh))} \Big)' \delta_{kj}^{(0)} \Big\} - \\
&\qquad \Big(\psi'(kh) \Big)^2 m \sinh(mc_k + m\psi(kh)) = 0, \\
&\qquad k = -N, -N+1, ..., N-1, N.
\end{aligned} \tag{3.22}$$

For convenience, by recalling the notations in Section 2, we can writing down the nonlinear algebraic system of (3.22) as matrix-vector form. Let C be the r-vector with j^{th} component given by c_j, and $\sinh(m\ C + m\ \psi)$ be the r-vector with j^{th} component given by $\sinh(m\ c_j + m\ \psi(jh))$. Thus, the nonlinear algebraic system is rewriting as follow

$$AC + B \sinh\Big(m\ C + m\ \psi \Big) = 0, \tag{3.23}$$

Figure 1. Approximation solution for $m = 1, 2, ..., 10$ for Troesch's Problem in this study.



where

$$A = \frac{1}{h^2} I^{(2)} + \frac{1}{h} I^{(1)} D\Big(\Big(\frac{\psi''(jh)}{\psi'(jh)}\Big)\Big) + I^{(0)} D\Big(\Big(\frac{\psi''(jh)}{\psi'(jh)}\Big)'\Big), \tag{3.24}$$

$$B = -m\ D\Big((\psi')^2\Big). \tag{3.25}$$

Now, we have a nonlinear system of $r = 2N + 1$ equations and $'r'$ unknown coefficients $\{c_j\}_{j=-N}^{N}$. By solving this system, we can find the unknown coefficients $\{c_j\}_{j=-N}^{N}$ and calculate the Sinc approximation solution by (3.9).

For solving system (3.23), the Newton's method was used by starting an initial guess C_0 as

$$C_{k+1} = C_k - J^{-1}(C_k)\ \Big\{F(C_k)\Big\}, \tag{3.26}$$

where

$$F(C) = AC + B \sinh\Big(m\ C + m\ \psi\Big),$$

and

$$J(C) = A + m\ B\ D\Big(\cosh\Big(m\ C + m\ \psi\Big)\Big).$$

Here, C_k is the current iteration and C_{k+1} is the new iteration. A common numerical rule to stop the Newton iteration is whenever the distance between two iterates is less than a given tolerance, i.e. where $\|C_{k+1} - C_k\| < \varepsilon$, where the Euclidean norm is used. By solving this system and obtaining the $C = (c_{-N}, ..., c_N)^T$, we can calculate the $v_r(t)$ in (3.9) as numerical solution.

FIGURE 2. Approximation solution for $m = 12,\ 14,\ 16,\ 18,\ 20$ for Troesch's Problem in this study.

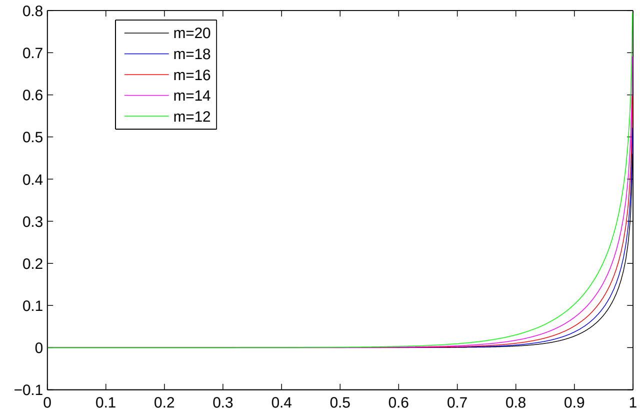
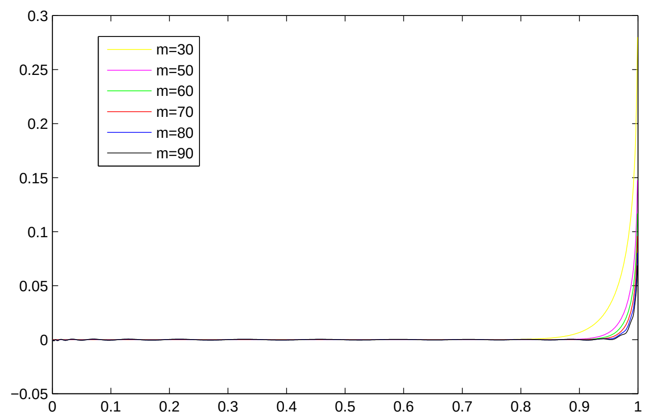


FIGURE 3. Approximation solution for $m = 30,\ 50,\ 60,\ 70,\ 80,\ 90$ for Troesch's Problem in this study.



4. NUMERICAL RESULTS

In this section, the DE-Sinc-Galerkin method (DESG) was applied to solve Troesch's problem for several ms. Besides, the obtained results are compared with the results of other methods in the literatures [3, 5, 6, 26].

To apply the DE-Sinc -Galerkin method, we supposed that $d = \frac{\pi}{4}$, $\gamma = 1$, $B = 1$ therefore for different values of N, $h = \ln\left(\frac{\pi N}{4}\right)/N$. For solving nonlinear system (3.23), the Newton's method was used. In Newton's method, we start with an initial guess C_0 as zero vector then use the Newton iteration (3.27). The last row in these tables is Maximum Error (MR) on top of that column.

In Tables 1 and 2, the exact solutions for $m = 0.5$ and $m = 1$, and the numerical results of the present method are compared with some other existing methods including; decomposition method approximation (DMA) [3], modified homotopy perturbation method (HPM) [5], Sinc-Galerkin based on single exponential transformation (SESG) [26], and the Sinc-collocation method based on single exponential transformation (SESC) [6]. It is interesting that inaccurate tabulated *exact* solutions are given in [3, 5, 6, 26]. If authors consider the exact solutions which is reported here and some other works [4, 8, 11], as a basis of comparison, they would have found that their approximation solutions were much more accurate than they realized in comparisons. In compare with the numerical results reported in some other work, the accuracy of the presented method is noticeable.

In Tables 3 and 4, absolute error in the solution of the Troesch's Problem at $x_j = 0.1, 0.2, ..., 0.9$ with $m = 0.5$ and $m = 1$ are presented respectively. In these Tables, results of proposed method (DESG) for $N = 5, 10, 15$ and results of Sinc-collocation method based on single exponential (SESC) [6] for $N = 5, 10, 15$ are compared. These tables showed that the obtained results are more accurate than the results reported in Reference [6].

In Table 5 and 6, the comparison of our method with $N = 5, 10, 15$ and Sinc-Galerkin method based on single exponential transformation (SESG) [26] for $N = 5, 10, 20$ are tabulated. The results show that result of presented method are more accurate than Reference [26].

Table 7 shows the maximum absolute error in the solution at point $x_j = 0.1, 0.2, 0.3, ..., 0.9$. Results of this table are demonstrated convergency of the method by increasing value of $'N'$.

Figure 1 displays solutions of DESG method for $m = 1, 2, 3, ..., 10$.

For $m > 1$ the decomposition method referenced in [3], Laplace decomposition method presented in [10] and B-Spline method in [9] do not yield a good approximation. Although Khuri et al. [9] by using a few mesh points obtained appropriate results by B-Spline method, but using Sinc method without any changes given acceptable results. In Table 8 the numerical solution is calculated by DESG for $m = 5$ and compared with numerical approximate of the exact solution given by a FORTRAN code called TWDBVP, the numerical solution of SESG [6] and B-spline method [9]

Beside, Figure 2 shows solution of DESG method for $m = 12, 14, 16, 18, 20$.

In Table 9 the numerical solution of DESG method by consideration $N = 80$ mesh points and adaptive collocation method given in [9] by $N = 330$ and numerical results obtained by Chang and Chang [2], and those obtained by Scott [17] for $m = 10$ are compared.

Figure 3 contains the graphs of the problem for $m = 30, 50, 60, 70, 80, 90$.

TABLE 1. Results of Troesch's Problem at $m = 0.5$.

x	Exact Solution	DESG	DMA [3]	HPM [5]	SESG [26]	SESC [6]
		$N = 15$			$N = 20$	$N = 20$
0.1	0.0959443493	0.09594434932	0.09593835	0.09593956	——-	0.0959445348
0.2	0.1921287477	0.19212874768	0.19211805	0.19211932	0.19212882	0.1921287458
0.3	0.2887944009	0.28879440094	0.28878032	0.28878069	——-	0.2887947251
0.4	0.3861848464	0.38618484638	0.38616870	0.38616754	0.38618437	0.3861843754
0.5	0.4845471647	0.48454716477	0.48453029	0.48452741	——-	0.4845471259
0.6	0.5841332484	0.58413324848	0.58411697	0.58411278	0.58413371	0.5841336720
0.7	0.6852011483	0.68520114831	0.68518684	0.68518224	——-	0.6852009802
0.8	0.7880165227	0.78801652269	0.78800556	0.78800183	0.78801652	0.7880164746
0.9	0.8928542161	0.89285421616	0.89284802	0.89284621	——-	0.8928542003
ME		$8.60\ E-011$	$1.68\ E-005$	$2.04\ E-005$	4.76 E-007	4.71 E-007

TABLE 2. Results of Troesch's Problem at $m = 1$.

x	Exact Solution	DESG	DMA [3]	HPM [5]	SESG [26]	SESC [6]
		$N = 15$			$N = 20$	$N = 20$
0.1	0.0846612565	0.084661256642	0.08424876	0.0843817004	——-	0.0846618450
0.2	0.1701713582	0.170171358273	0.16943070	0.1696207644	0.1701715950	0.1701714089
0.3	0.2573939080	0.257393908175	0.25641450	0.2565929224	——-	0.2573949705
0.4	0.3472228551	0.347222855224	0.34608572	0.3462107378	0.3472212611	0.3472212935
0.5	0.4405998351	0.440599835276	0.43940198	0.4394422743	——-	0.4405997991
0.6	0.5385343980	0.538534398177	0.53736570	0.5373300622	0.5385360476	0.5385359466
0.7	0.6421286091	0.642128609348	0.64108380	0.6410104651	——-	0.6421279331
0.8	0.7526080939	0.752608094135	0.75178800	0.7517335467	0.7526080999	0.7526080112
0.9	0.8713625196	0.871362519949	0.87090870	0.8708835371	——-	0.8713623820
ME		$3.49\ E-010$	0.0012	0.0012	1.59 E-006	1.56 E -006

TABLE 3. Absolute error in the solution of Troesch's Problem for $m = 0.5$.

x	DESG $N=5$	DESG $N=10$	DESG $N=15$	SESC [6] $N=5$	SESC [6] $N=10$	SESC [6] $N=15$
0.1	$2.2592\ E-05$	$1.8115\ E-08$	$2.5048\ E-11$	$2.3572\ E-04$	1.2862 E-05	1.2652 E-06
0.2	$2.5630\ E-05$	$1.6427\ E-08$	$1.7526\ E-11$	$9.2361\ E-05$	1.5730 E-05	2.1638 E-06
0.3	$2.7692\ E-05$	$2.3845\ E-08$	$4.3191\ E-11$	$1.2592\ E-04$	7.3417 E-06	9.2352 E-07
0.4	$2.8415\ E-05$	$1.7477\ E-08$	$1.5468\ E-11$	$1.3288\ E-04$	1.8795 E-05	2.2513 E-06
0.5	$3.0812\ E-05$	$2.5432\ E-08$	$7.4792\ E-11$	$4.1831\ E-06$	3.9774 E-06	3.8266 E-08
0.6	$3.3690\ E-05$	$3.1650\ E-08$	$8.6008\ E-11$	$1.2856\ E-04$	1.1479 E-05	2.4579 E-06
0.7	$3.5689\ E-05$	$2.4687\ E-08$	$1.0786\ E-11$	$1.7766\ E-04$	7.2266 E-06	4.2468 E-07
0.8	$3.7312\ E-05$	$3.4867\ E-08$	$7.2489\ E-12$	$1.4500\ E-04$	5.2517 E-06	7.6033 E-07
0.9	$4.0225\ E-05$	$3.0881\ E-08$	$6.4269\ E-11$	$9.1385\ E-05$	9.8091 E-07	9.2200 E-07

TABLE 4. Absolute error in the solution of Troesch's Problem for $m = 1$.

x	DESC $N=5$	DESC $N=10$	DESC $N=15$	SESC [6] $N=5$	SESC [6] $N=10$	SESC [6] $N=15$
0.1	$8.3786\ E-05$	$6.2704\ E-08$	$1.4252\ E-10$	$8.4343\ E-04$	1.2862 E-05	4.2466 E-06
0.2	$9.1528\ E-05$	$6.1456\ E-08$	$7.3548\ E-11$	$3.1020\ E-04$	1.5730 E-05	7.2176 E-06
0.3	$9.8042\ E-05$	$7.3691\ E-08$	$1.7571\ E-10$	$4.8911\ E-04$	7.3417 E-06	2.7879 E-06
0.4	$1.0222\ E-04$	$7.7978\ E-08$	$1.2443\ E-10$	$4.9541\ E-04$	1.8795 E-05	8.0623 E-06
0.5	$1.1019\ E-04$	$8.5391\ E-08$	$1.7609\ E-10$	$1.8040\ E-05$	3.9774 E-06	3.3665 E-07
0.6	$1.2175\ E-04$	$9.6812\ E-08$	$1.7736\ E-10$	$5.4415\ E-04$	1.1479 E-05	9.0358 E-06
0.7	$1.3277\ E-04$	$1.0412\ E-07$	$2.4831\ E-10$	$7.1169\ E-04$	7.2266 E-06	8.1195 E-07
0.8	$1.4259\ E-04$	$1.1892\ E-07$	$2.3567\ E-10$	$4.9203\ E-04$	5.2517 E-06	3.5125 E-06
0.9	$1.5766\ E-04$	$1.2622\ E-07$	$3.4981\ E-10$	$2.3468\ E-04$	9.8091 E-07	3.9632 E-06

TABLE 5. Absolute error of DESG method for solution of Troesch's Problem for $m = 0.5$.

x	DESG $N = 5$	DESG $N = 10$	DESG $N = 15$	SESG [26] $N = 5$	SESG [26] $N = 10$	SESG [26] $N = 20$
0.2	2.5630 $E-05$	1.6427 $E-08$	1.7526 $E-11$	1.7798 $E-05$	1.7132 E-05	7.2300 E-08
0.4	2.8415 $E-05$	1.7477 $E-08$	1.5468 $E-11$	1.5723 $E-04$	1.6366 E-05	4.7640 E-07
0.6	3.3690 $E-05$	3.1650 $E-08$	8.6008 $E-11$	1.1858 $E-04$	1.5162 E-05	4.6160 E-07
0.8	3.7312 $E-05$	3.4867 $E-08$	7.2489 $E-12$	5.0917 $E-05$	5.1927 E-06	2.7000 E-09

TABLE 6. Absolute error in the solution of Troesch's Problem for $m = 1$.

x	DESG $N = 5$	DESC $N = 10$	DESC $N = 15$	SESG [26] $N = 5$	SESG [26] $N = 10$	SESG [26] $N = 20$
0.2	9.1528 $E-05$	6.1456 $E-08$	7.3548 $E-11$	5.9688 $E-05$	6.0723 E-05	2.3680 E-07
0.4	1.0222 $E-04$	7.7978 $E-08$	1.2443 $E-10$	5.7321 $E-04$	5.9727 E-05	1.5940 E-06
0.6	1.2175 $E-04$	9.6812 $E-08$	1.7736 $E-10$	5.2725 $E-04$	6.0328 E-05	1.6496 E-06
0.8	1.4259 $E-04$	1.1892 $E-07$	2.3567 $E-10$	1.9061 $E-04$	2.6693 E-05	6.0000 E-09

TABLE 7. Maximum Absolute Error in the solution for DE Sinc Galerkin method

x	$N = 2$	$N = 5$	$N = 7$	$N = 9$	$N = 11$	$N = 13$	$N = 15$
$m = 0.5$	6.87 $E-03$	4.02 $E-05$	2.09 $E-6$	1.13 $E-07$	8.02 E-09	6.09 E-10	8.60 E-11
$m = 1$	2.75 $E-02$	1.57 $E-04$	8.20 $E-6$	4.72 $E-07$	3.08 E-08	2.54 E-09	3.49 E-10

TABLE 8. Numerical solution of Troesch's Problem for $m = 5$

x	DESG	FORT.code [6]	SESC [6]	B-Spline [9]
0.2	0.01075340	0.01075342	0.00762552	0.01002027
0.4	0.03320049	0.03320051	0.03817903	0.03099793
0.8	0.25821648	0.25821664	0.23252435	0.24170496
0.9	0.45506002	0.45506034	0.44624551	0.42461830

TABLE 9. Numerical solution of Troesch's Problem for $m = 10$

x	DESG	B-Spline [9]	y(x) [17]	y(x) [2]
0.100	4.210951717852 E-5	4.207335076936 E-5	4.211183679705 E-5	4.211281725912 E-5
0.200	1.299647099690 E-4	1.298517430190 E-4	1.299639238293 E-4	1.299669488791 E-4
0.300	3.589779987567 E-4	3.586905006808 E-4	3.589778855481 E-4	3.589862266738 E-4
0.400	9.778997633100 E-4	9.771828520268 E-4	9.779014227050 E-4	9.779240889443 E-4
0.500	2.659019959670 E-3	2.657239728995 E-3	2.659017178062 E-3	2.659078455376 E-3
0.600	7.289348154660 E-3	7.224571491931 E-3	7.228924695208 E-3	7.229088828840 E-3
0.700	1.966406491387 E-2	1.965351623938 E-2	1.966406025665 E-2	1.966449243994 E-2
0.800	5.373032972891 E-2	5.370517634226 E-2	5.373032958567 E-2	5.373151474481 E-2
0.900	1.521140817724 E-1	1.520568160982 E-1	1.521140787863 E-1	1.521177213775 E-1
0.925	2.020016774993 E-1	2.019932942876 E-1	2.020016854925 E-1	2.020068749831 E-1
0.950	2.762677386212 E-1	2.761735277379 E-1	2.762677349042 E-1	2.762758626603 E-1
0.970	3.722643366229 E-1	3.721473843742 E-1	3.722643330645 E-1	3.722780156753 E-1
0.980	4.482330343929 E-1	4.481030070094 E-1	4.482330386284 E-1	4.482533078571 E-1
0.990	5.740765092106 E-1	5.739404061656 E-1	5.740764982493 E-1	5.741148407302 E-1
0.995	6.901149480694 E-1	6.900157731379 E-1	6.901149417369 E-1	6.901835930585 E-1
0.997	7.657697281787 E-1	7.657004105106 E-1	7.657697261350 E-1	7.658699971015 E-1
0.998	8.180328186471 E-1	8.179769073631 E-1	8.180328282850 E-1	8.181629530055 E-1
0.999	8.889931264395 E-1	8.889496654105 E-1	8.889931171768 E-1	8.891729462281 E-1

5. Conclusions

In this investigation, the Sinc-Galerkin method based on double exponential transformation developed to solve the nonlinear two point boundary value problem that applied to Troesch's equation. By utilizing the Sinc Galerkin method, the solution of nonlinear boundary value problem converted to solution of some nonlinear system of algebraic equations in which solved comfortable by Newton iteration due to the obtained matrix form. Besides, the obtained results, could solve effectively the such proposed problems. Moreover, a comparison between the numerical results of the presented approach in the literature, illustrated that the proposed technique has more accurate and reliable to solve this type of problems.

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