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Sinc operational matrix method for solving the Bagley-Torvik equation

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Abstract The aim of this paper is to present a new numerical method for solving the Bagley-Torvik equation. This equation has an important role in fractional calculus. The fractional derivatives are described based on the Caputo sense. Some properties of the sinc functions required for our subsequent development are given and are utilized to reduce the computation of solution of the Bagley-Torvik equation to some algebraic equations. It is well known that the sinc procedure converges to the solution at an exponential rate. Numerical examples are included to demonstrate the validity and applicability of the technique.

Keywords. Bagley-Torvik equation, Sinc functions, Operational matrix, Caputo derivative, Numerical methods.

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1. Introduction

A history of fractional calculus, i.e. the theory of derivatives and integrals of fractional (non-integer) order, can be found in [11, 14, 16]. Both differential equations and fractional differential equations have been used to model physical and engineering processes such as electromagnetic, acoustics, viscoelasticity, electroanalytical chemistry, neuron modeling, diffusion processing and material sciences (see for example [2, 5, 7, 13, 20] and the references therein). The analytic results on existence and uniqueness of solutions to fractional differential equations have been investigated by many authors [7, 16]. In general, most of the fractional differential equations do not have exact solutions. Recently increased attention has turned to comparing numerical methods for solving fractional differential equations, fractional partial differential equations, fractional integro-differential equations and dynamic system containing fractional derivative (see for example [3, 8, 12, 18, 21, 22, 23]).

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In this study, we consider the Bagley-Torvik equation

$$A_1 y^{(2)} + A_2 y^{(3/2)} + A_3 y = f(x), \qquad x \in [0, 1], \tag{1.1}$$

with the boundary conditions

$$y(0) = a, \quad y(1) = b, \tag{1.2}$$

where A_1, A_2, A_3, a, b are real constants and $y(x)$ is an unknown function. The Bagley-Torvik equation arises in modeling of the motion of a thin rigid plate immersed in a Newtonian fluid [6, 16]. This equation has been numericaly solved by using the hybridizable discontinuous Galerkin methods [6], pseudo-spectral scheme [3], Bessel collocation method [28], generalized Taylor collocation method [1], Haar wavelet, [17] and hybrid functions [10].

In the present paper we intend to extend the application of sinc mthods to solve the Bagley-Torvik equation. Sinc function properties are discussed thoroughly in [9, 26] and it is widely used for solving a wide range of problems arising from scientific and engineering applications including Hallen's integral equation [25], third-order boundary value problems [24], squeezing flow [19], fractional convection-diffusion equations [23], differential-algebraic equations [27] and Thomas-Fermi equation [15].

Our method consists of reducing the problem to the solution of algebraic equations by expanding the required approximate solution as the elements of the sinc functions with unknown coefficients. The properties of sinc functions are then utilized to evaluate the unknown coefficients.

The organization of the rest of this paper is as follows: In Section 2, we introduce some necessary definitions and mathematical preliminaries of sinc functions, fractional calculus and Gauss-Jacobi quadrature. In Section 3, the new method proposed in the current work is presented. As a result a set of algebraic equations is formed and a solution of the considered problem is introduced. In Section 4, several numerical results are given to show the efficiency of our methods. In Section 5, we give a brief conclusion.

2. Preliminaries and notations

2.1. A short overview on sinc functions.

The goal of this section is to recall properties and definition of the sinc function. These are discussed thoroughly in [9, 26]. The sinc function is defined on the whole real line, $-\infty < x < \infty$, by

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

For $h > 0$, and $k = 0, \pm 1, \pm 2, \ldots$, the translated sinc functions with evenly spaced nodes are given by

$$S(k,h)(x) = \operatorname{sinc}\left(\frac{x - kh}{h}\right) = \begin{cases} \frac{\sin[\frac{\pi}{h}(x-kh)]}{\frac{\pi}{h}(x-kh)}, & x \neq kh, \\ 1, & x = kh. \end{cases} \tag{2.1}$$

The sinc function for the interpolating points $x_j = jh$ is given by

$$S(k,h)(jh) = \delta_{kj} = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases}$$

If a function $f(x)$ is defined on the real axis, then for $h > 0$ the series

$$C(f,h)(x) = \sum_{k=-\infty}^{\infty} f(kh)\ \mathrm{Sinc}\left(\frac{x-kh}{h}\right),$$

is called the Whittaker cardinal expansion of f whenever this series converges. The properties of Whittaker cardinal expansion have been extensively studied in [9]. These properties are derived in the infinite strip D_S of the complex w-plane, where for $d > 0$,

$$D_S = \left\{ w = t + is\ : |s| < d \leq \frac{\pi}{2} \right\}.$$

Approximations can be constructed for infinite, semi-infinite and finite intervals. To construct approximations on the interval $(0,1)$, which is used in this paper, the eye-shaped domain in the z-plane

$$D_E = \left\{ z = x + iy\ : \left| arg\left(\frac{z}{1-z}\right)\right| < d \leq \frac{\pi}{2} \right\},$$

is mapped conformally onto the infinite strip D_S via

$$w = \phi(z) = \ln\left(\frac{z}{1-z}\right).$$

The basis functions on $(0,1)$ are taken to be the composite translated sinc functions,

$$S_k(x) = S(k,h) \circ \phi(x) = \mathrm{sinc}\left(\frac{\phi(x) - kh}{h}\right), \tag{2.2}$$

where $S(k,h) \circ \phi(x)$ is defined by $S(k,h)(\phi(x))$. The inverse map of $w = \phi(z)$ is

$$z = \phi^{-1}(w) = \frac{\exp(w)}{1 + \exp(w)}.$$

Thus we may define the inverse images of the real line and of the evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ as

$$\Gamma = \{\psi(t) \in D_E\ : -\infty < t < \infty\} = (0,1),$$

and

$$x_k = \phi^{-1}(kh) = \frac{e^{kh}}{1 + e^{kh}}, \qquad k = 0, \pm 1, \pm 2, \ldots \tag{2.3}$$

respectively.

The class of functions such that the known exponential error estimates exist for sinc interpolation is denoted by $B(D_E)$ and is defined in the following.

Definition 2.1. Let $B(D_E)$ be the class of functions F which are analytic in D_E, satisfy

$$\int_{\psi(t+L)} |F(z)dz| \longrightarrow 0, \qquad t \longrightarrow \pm\infty,$$

where $L = \left\{iv \ : |v| < d \leq \frac{\pi}{2}\right\}$, and on the boundary of D_E, (denoted ∂D_E), satisfy

$$N(F) = \int_{\partial D_E} |F(z)dz| < \infty.$$

Interpolation for function in $B(D_E)$ are defined in the following theorem whose proof can be found in [26].

Theorem 2.2. *If $\phi' F \in B(D_E)$ then for all $x \in \Gamma$*

$$\begin{aligned}\left|F(x) - \sum_{k=-\infty}^{\infty} F(x_k) S(k,h) \circ \phi(x)\right| &\leq \frac{N(F\phi')}{2\pi d \sinh(\pi d/h)} \\ &\leq \frac{2N(F\phi')}{\pi d} e^{-\pi d/h}.\end{aligned}$$

Moreover, if $|F(x)| \leq Ce^{-\alpha|\phi(x)|}$, $x \in \Gamma$, for some positive constants C and α, and if the selection $h = \sqrt{\pi d/\alpha N} \leq 2\pi d/\ln 2$, then

$$\left|F(x) - \sum_{k=-N}^{N} F(x_k) S(k,h) \circ \phi(x)\right| \leq C_2 \sqrt{N} \exp(-\sqrt{\pi d \alpha N}), \quad x \in \Gamma,$$

where C_2 depends only on F, d and α.

The above expressions show sinc interpolation on $B(D_E)$ converge exponentially [26]. We also require derivatives of composite sinc functions evaluated at the nodes. The expressions required for the present discussion are [23].

$$\delta_{k,j}^{(0)} = [S(k,h) \circ \phi(x)]|_{x=x_j} = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases} \tag{2.4}$$

$$\delta_{k,j}^{(1)} = h\frac{d}{d\phi}[S(k,h) \circ \phi(x)]|_{x=x_j} = \begin{cases} 0, & k = j, \\ \frac{(-1)^{j-k}}{j-k}, & k \neq j. \end{cases} \tag{2.5}$$

$$\delta_{k,j}^{(2)} = h^2\frac{d^2}{d\phi^2}[S(k,h) \circ \phi(x)]|_{x=x_j} = \begin{cases} \frac{-\pi^2}{3}, & k = j, \\ \frac{-2(-1)^{j-k}}{(j-k)^2}. & k \neq j. \end{cases} \tag{2.6}$$

2.2. **The fractional derivative in the Caputo sense.**
There are various definitions of fractional integration and differentiation of order $\gamma > 0$, and not necessarily equivalent to each other [11, 14]. We recall here some classical definitions which will be useful in the sequel.

Definition 2.3. Caputo's definition of the fractional-order derivative is defined as

$$D^{\beta} f(x) = \begin{cases} \frac{1}{\Gamma(n-\beta)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{\beta+1-n}} dt, & n-1 < \beta < n, \quad n \in \mathbb{N}, \\ \\ \frac{d^n}{dx^n} f(x), & \beta = n \in \mathbb{N}. \end{cases} \tag{2.7}$$

where $\beta > 0$ is the order of the derivative, $\Gamma(.)$ is the Gamma function and $n = [\beta]+1$, with $[\beta]$ denoting the integer part of β.

For the Caputo's derivative we have [14],

$$D^{\beta}C = 0, \quad (C \text{ is a constant}), \tag{2.8}$$

$$D^{\beta}x^{\gamma} = \begin{cases} 0, & \text{for } \gamma \in \mathbb{N} \cup \{0\} \text{ and } \gamma < \lceil \beta \rceil, \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\beta)} x^{\gamma-\beta}, & \text{for } \gamma \in \mathbb{N} \cup \{0\} \text{ and } \gamma \geq \lceil \beta \rceil. \end{cases} \tag{2.9}$$

We use the ceiling function $\lceil \beta \rceil$ to denote the smallest integer greater than or equal to β. Similar to integer-order differentiation, Caputo's fractional differentiation is a linear operator:

$$D^{\beta}(c_1 f(x) + c_2 g(x)) = c_1 D^{\beta} f(x) + c_2 D^{\beta} g(x), \tag{2.10}$$

where c_1 and c_2 are constants.

2.3. **Gauss-Jacobi quadrature.**

Let $\lambda, \mu > -1$. The Jacobi polynomials $P_m^{(\lambda,\mu)}(x), m = 0, 1, 2, ..., x \in (-1, 1)$ are defined by

$$P_m^{(\lambda,\mu)}(x) = \frac{(-1)^m}{2^m m!}(1-x)^{-\lambda}(1+x)^{-\mu}\frac{d^m}{dx^m}[(1-x)^{\lambda+m}(1+x)^{\mu+m}]. \tag{2.11}$$

They have the following orthogonality relation

$$\int_{-1}^{1} P_n^{(\lambda,\mu)}(x) P_m^{(\lambda,\mu)}(x)(1-x)^{\lambda}(1+x)^{\mu}dx = \begin{cases} \frac{2^{\lambda+\mu+1}}{\lambda+\mu+2n+1}\frac{\Gamma(\lambda+n+1)\Gamma(\mu+n+1)}{n!\Gamma(\lambda+\mu+n+1)}, & n = m, \\ 0, & n \neq m. \end{cases}$$

As a result, all the zeros of $P_m^{(\lambda,\mu)}(x)$ are simple and belong to the interval $(-1, 1)$. For a given positive integer m, we denote the Gauss-Jacobi points with parameters λ and μ, by $\{\xi_i^{(\lambda,\mu)}\}_{i=1}^m$ which is the set of m roots of $P_m^{(\lambda,\mu)}(x)$.

The Gauss-Jacobi quadrature rule, with parameters λ and μ, is based on Gauss-Jacobi points $\{\xi_i^{(\lambda,\mu)}\}_{i=1}^m$ and can be used to approximate the integral of a function over the range $[-1, 1]$ with weight $(1-x)^{\lambda}(1+x)^{\mu}$ as

$$\int_{-1}^{1} f(x)(1-x)^{\lambda}(1+x)^{\mu}dx \approx \sum_{i=1}^{m} \omega_i^{(\lambda,\mu)} f(\xi_i^{(\lambda,\mu)}), \tag{2.12}$$

where the Gauss-Jacobi weights $\{\omega_i^{(\lambda,\mu)}\}_{i=1}^m$ are given by [4]

$$\omega_i^{(\lambda,\mu)} = \frac{\Gamma(\lambda+m+1)\Gamma(\mu+m+1)}{m!\Gamma(\lambda+\mu+m+1)} \frac{2^{\lambda+\mu+1}}{\left(1 - \left(\xi_i^{(\lambda,\mu)}\right)^2\right)\left[P_m^{(\lambda,\mu)\prime}(\xi_i^{(\lambda,\mu)})\right]^2}. \tag{2.13}$$

Gauss-Jacobi quadrature can be used to approximate integrals with singularities at the end points. Also it is well known that Gauss-Jacobi quadrature is exact for all polynomials of degree $2m-1$.

3. Description of the method

First of all, we reformulate the problem (1.1)-(1.2) by applying the following transformation that makes the boundary conditions become homogeneous

$$u(x) = y(x) + (a-b)x - a.$$

Therefore, we consider the following Bagley-Torvik equation

$$A_1 u^{(2)} + A_2 u^{(3/2)} + A_3 u = g(x), \qquad x \in [0,1], \tag{3.1}$$

with homogeneous boundary conditions

$$u(0) = 0, \qquad u(1) = 0, \tag{3.2}$$

where $g(x) = f(x) + A_3((a-b)x - a)$. Now, we approximate solution for $u(x)$, in Eq. (3.1) as

$$u(x) \approx u_M(x) = \sum_{k=-N}^{N} u_k S_k(x), \tag{3.3}$$

where $u_k = u(x_k)$ and $M = 2N+1$. It is worth pointing out that $u_M(x) = 0$ when x tends to 0 or 1. The first derivative of Eq. (2.2) is given by

$$\frac{d}{dx}\left[S(k,h)\circ\phi(x)\right] = \phi'(x)\frac{d}{d\phi}\left[S(k,h)\circ\phi(x)\right].$$

Thus, using Eq. (7) we get

$$\left.\frac{d}{dx}S_k(x)\right|_{x=x_j} = \frac{1}{h}\phi'(x_j)\delta_{kj}^{(1)}. \tag{3.4}$$

Similarly by taking the second derivative from Eq. (2.2) and using Eqs.(7) and (8) we obtain

$$\left.\frac{d^2}{dx^2}S_k(x)\right|_{x=x_j} = \frac{1}{h}\phi''(x_j)\delta_{kj}^{(1)} + \frac{1}{h^2}[\phi'(x_j)]^2\delta_{kj}^{(2)}. \tag{3.5}$$

Therefore, the approximations of the first and second derivatives at the sinc nodes x_j take the form

$$u_M'(x_j) = \sum_{k=-N}^{N} u_k \left\{\frac{1}{h}\phi'(x_j)\delta_{kj}^{(1)}\right\}, \tag{3.6}$$

$$u_M''(x_j) = \sum_{k=-N}^{N} u_k \left\{\frac{1}{h}\phi''(x_j)\delta_{kj}^{(1)} + \frac{1}{h^2}[\phi'(x_j)]^2\delta_{kj}^{(2)}\right\}. \tag{3.7}$$

The approximations (3.6) and (3.7) are more conveniently recorded by defining the vector $\overrightarrow{u} = [u_{-N}, ..., u_N]^T$. Then define the $M\times M$ Toeplitz matrices $\mathbf{I}^{(q)} = [\delta_{kj}^{(q)}], q = 0, 1, 2.$ i.e., the matrix whose kj-entry is given by $\delta_{kj}^{(q)}$. Also define the diagonal matrix $\mathbf{E}(p) = \mathrm{diag}[p(x_{-N}), ..., p(x_N)]$. The matrix $\mathbf{I}^{(0)}$ is an identity matrix. Note that the matrix $\mathbf{I}^{(2)}$ is a symmetric matrix, i.e., $\mathbf{I}^{(2)}_{kj} = \mathbf{I}^{(2)}_{jk}$ and the matrix $\mathbf{I}^{(1)}$ is a skew-symmetric matrix, i.e., $\mathbf{I}^{(1)}_{kj} = -\mathbf{I}^{(1)}_{jk}$. They take the form

$$\mathbf{I}^{(1)} = \begin{pmatrix} 0 & -1 & \dots & \frac{(-1)^{M-1}}{M-1} \\ 1 & & \dots & \vdots \\ \vdots & \vdots & \ddots & -1 \\ \frac{(-1)^{1-M}}{1-M} & \dots & \dots & 0 \end{pmatrix}_{M\times M},$$

$$\mathbf{I}^{(2)} = \begin{pmatrix} \frac{-\pi^2}{3} & 2 & \dots & \frac{-2(-1)^{M-1}}{(M-1)^2} \\ 2 & & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-2(-1)^{M-1}}{(M-1)^2} & \dots & \dots & \frac{-\pi^2}{3} \end{pmatrix}_{M\times M}.$$

Then approximations (3.6) and (3.7) can be written as

$$\overrightarrow{u}' \approx \left\{\frac{1}{h}\mathbf{I}^{(1)}\mathbf{E}(\phi')\right\}\overrightarrow{u} \equiv \mathbf{D}^{(1)}\overrightarrow{u}. \tag{3.8}$$

$$\overrightarrow{u}'' \approx \left\{\frac{1}{h}\mathbf{I}^{(1)}\mathbf{E}(\phi'') + \frac{1}{h^2}\mathbf{I}^{(2)}\mathbf{E}(\phi'^2)\right\}\overrightarrow{u} \equiv \mathbf{D}^{(2)}\overrightarrow{u}. \tag{3.9}$$

Also, the fractional derivative of order β for $S_k(x)$ at the sinc nodes x_j is given by

$$D^\beta(S_k(x))\big|_{x=x_j} = \frac{1}{\Gamma(2-\beta)}\int_0^{x_j}(x_j - t)^{1-\beta}S_k^{(2)}(t)dt, \quad 1 < \beta < 2. \tag{3.10}$$

In order to use the Gauss-Jacobi quadrature formula for Eq. (3.10), we transfer the t-interval $[0, x_j]$ into τ-interval $[-1, 1]$ by means of the transformation

$$\tau = \frac{2}{x_j}t - 1.$$

Eq. (3.10), may then be restated as

$$D^\beta(S_k(x))\big|_{x=x_j} = \frac{(\frac{x_j}{2})^{2-\beta}}{\Gamma(2-\beta)}\int_{-1}^{1}(1-\tau)^{1-\beta}\, S_k^{(2)}\left(\frac{x_j}{2}(1+\tau)\right)d\tau. \tag{3.11}$$

Using the Gauss-Jacobi quadrature rule (2.13), with parameters $\lambda = 1-\beta$ and $\mu = 0$, we obtain

$$D^\beta(S_k(x))\big|_{x=x_j} \approx \frac{(\frac{x_j}{2})^{2-\beta}}{\Gamma(2-\beta)}\sum_{i=1}^{m}\omega_i^{(1-\beta,0)}S_k^{(2)}\left(\frac{x_j}{2}(1+\xi_i^{(1-\beta,0)})\right). \tag{3.12}$$

Thus, the approximation of the fractional derivative of order β at the sinc nodes x_j takes the form

$$u_M^\beta(x_j) = \sum_{k=-N}^{N} u_k\left\{\delta_{kj}^{(\beta)}\right\}, \tag{3.13}$$

where $\delta_{kj}^{(\beta)}$ is given by

$$\delta_{kj}^{(\beta)} = \frac{(\frac{x_j}{2})^{2-\beta}}{\Gamma(2-\beta)} \sum_{i=1}^{m} \omega_i^{(1-\beta,0)} S_k^{(2)} \left(\frac{x_j}{2} (1 + \xi_i^{(1-\beta,0)}) \right). \tag{3.14}$$

Now, define the $M \times M$ matrix $\mathbf{D}^{(\beta)} = [\delta_{kj}^{(\beta)}]$, i.e., the matrix whose kj-entry is given by $\delta_{kj}^{(\beta)}$. Then, the approximation of the fractional derivative of order β can be written as

$$\overrightarrow{u}^{(\beta)} \approx \mathbf{D}^{(\beta)} \overrightarrow{u}. \tag{3.15}$$

Applying Eqs. (3.9) and (3.15) in Eq. (3.1), the vector of unknowns $\overrightarrow{u}$ is related to the known vector $\overrightarrow{g} = [g(x_{-N}), ..., g(x_N)]^T$ by

$$(A_1 \mathbf{D}^{(2)} + A_2 \mathbf{D}^{(3/2)} + A_3 \mathbf{I}^{(0)}) \overrightarrow{u} = \overrightarrow{g}. \tag{3.16}$$

Eq. (3.16) gives M linear algebraic equations. Therefore these M algebraic equations can be solved for the unknown vector $\overrightarrow{u}$. Consequently $u_M(x)$ given in Eq. (3.3) can be calculated.

4. Numerical results

In this section, we present some examples to show the efficiency of method for solving the Bagley-Torvik equation. In all examples we choose $\alpha = 1/2$ and $d = \pi/2$ which leads to $h = \pi/\sqrt{N}$. Also, we choose $m = 10$.

Example 1. In this example, we consider the Bagley-Torvik equation [28]

$$y^{(2)} + y^{(3/2)} + y = 1 + x, \qquad x \in [0, 1],$$

with the boundary conditions $y(0) = 1$ and $y(1) = 2$. By using the sinc method with $N = 2$ we obtain $y(x) = x + 1$, which is the exact solution of this problem.

Example 2. Let us solve the following Bagley-Torvik equation [10]

$$y^{(2)} + \frac{8}{17} y^{(3/2)} + \frac{13}{51} y = \frac{x^{-1/2}}{89250\sqrt{\pi}} \left(48p(x) + 7\sqrt{\pi x} q(x) \right), \qquad x \in [0, 1],$$

where $p(x) = 16000x^4 - 32480x^3 + 21280x^2 - 4746x + 189$ and $q(x) = 3250x^5 - 9425x^4 + 264880x^3 - 448107x^2 + 233262x - 34578$. Here, the boundary conditions are $y(0) = 0$ and $y(1) = 0$. It can be easily verified that the exact solution is

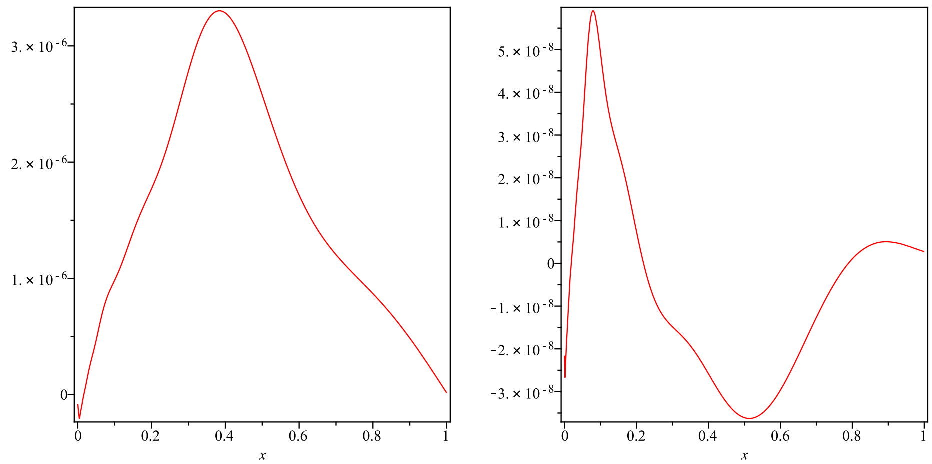
$$y(x) = x^5 - \frac{29}{10}x^4 + \frac{76}{25}x^3 - \frac{339}{250}x^2 + \frac{27}{125}x.$$

The absolute errors are obtained in Table 1 for different values of N using the presented method. Also, Figure 1 shows the plot of absolute error with $N = 40, 50$.

Example 3. In our third example, we consider the equation [6]

$$y^{(2)} + y^{(3/2)} + y = 2 + \sqrt{x/\pi} + x^2, \qquad x \in [0, 1],$$

FIGURE 1. Plot of the absolute error with $N = 40$ (left) and $N = 50$ (right) for Example 2.



with the boundary conditions $y(0) = 0$ and $y(1) = 1$. The exact solution of this problem is given by $y(x) = x^2$. Figure 2 shows the plot of absolute error with $N = 32$ and $N = 64$ using the presented method. Of course the accuracy of our method can be improved by increasing N.

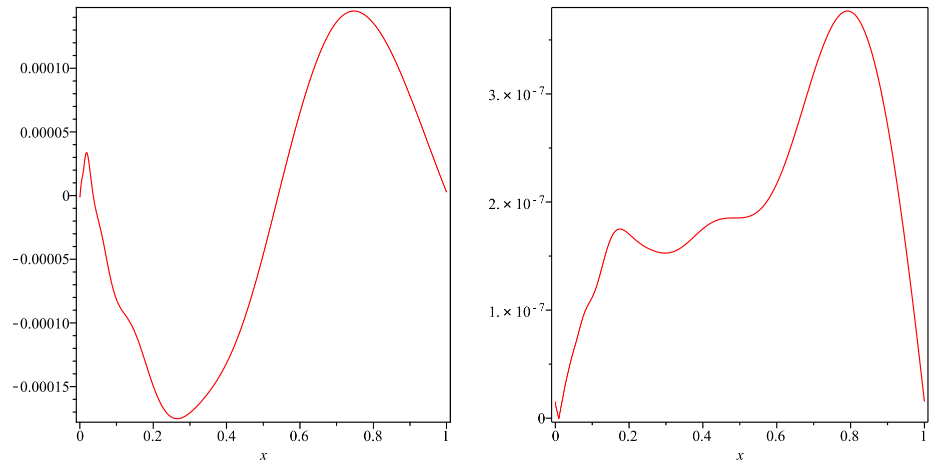
TABLE 1. Absolute errors for different values of N for Example 2.

x	$N = 8$	$N = 16$	$N = 32$	$N = 64$
0.1	2.90×10^{-3}	2.96×10^{-4}	1.36×10^{-6}	3.92×10^{-9}
0.2	9.92×10^{-4}	4.78×10^{-4}	4.17×10^{-6}	4.59×10^{-9}
0.3	2.18×10^{-4}	6.26×10^{-4}	1.36×10^{-6}	4.16×10^{-9}
0.4	3.83×10^{-4}	9.04×10^{-4}	5.30×10^{-6}	4.12×10^{-9}
0.5	1.36×10^{-3}	1.13×10^{-3}	3.41×10^{-6}	3.87×10^{-9}
0.6	1.88×10^{-3}	1.18×10^{-3}	3.19×10^{-7}	3.82×10^{-9}
0.7	1.83×10^{-3}	1.03×10^{-3}	2.17×10^{-6}	4.54×10^{-9}
0.8	1.32×10^{-3}	7.44×10^{-4}	2.74×10^{-6}	4.78×10^{-9}
0.9	6.17×10^{-4}	3.79×10^{-4}	1.65×10^{-6}	3.16×10^{-9}

5. Conclusion

In this work, we employed the sinc method for solving the Bagley-Torvik equation. Properties of the sinc function are utilized to reduce the computation of this problem to some algebraic equations. The method is computationally attractive and applications are demonstrated through illustrative examples. The obtained results showed that this approach can solve the problem effectively.

FIGURE 2. Plot of the absolute error with $N = 32$ (left) and $N = 64$ (right) for Example 3.



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