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Numerical solution of the forced Duffing equations using Legendre multiwavelets

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Abstract A numerical technique based on the collocation method using Legendre multiwavelets are presented for the solution of forced Duffing equation. The operational matrix of integration for Legendre multiwavelets is presented and is utilized to reduce the solution of Duffing equation to the solution of linear algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the new technique.

Keywords. Forced Duffing equations, Multiwavelet, Operational matrix of integration, Collocation method.
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1. Introduction

In this paper, we consider the following forced Duffing equation [4]

$$u''(t) + \sigma u'(t) + f(t,u) = 0, \qquad 0 < t < 1, \quad \sigma \in R - 0, \tag{1.1}$$

with integral boundary conditions

$$\begin{aligned} \mu_1 u(0) - \mu_2 u'(0) &= \textstyle\int_0^1 h_1(s)u(s)ds, \\ \mu_3 u(1) - \mu_4 u'(1) &= \textstyle\int_0^1 h_2(s)u(s)ds, \end{aligned} \tag{1.2}$$

where $f : [0,1] \times R \to R$ and μ_i are nonnegative constant.

The Duffing equation is a well known nonlinear equation of applied science which is used as a powerful tools to discuss some important practical phenomena such as orbit extraction, nonuniformity caused by an infinite domain, nonlinear mechanical oscillators, ets. An important application of the Duffing equation is in the field of the prediction of diseases. The numerical solutions of the forced Duffing equations with two-point boundary conditions have been widely investigated [18, 21, 24]. However, there are few references on the forced Duffing equation with integral boundary conditions [19].

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The existence and uniqueness of the solution of the forced Duffing equation with integral boundary conditions are presented by means of a constructive method [6]. Dehghan presented some effective methods for solving problems with nonlocal conditions [12–17].

Wavelets theory is a relatively new and emerging area in mathematical research. It has been applied in a wide range of engineering disciplines; in particular, wavelets are very successfully used in signal analysis for waveform representation and segmentations, time-frequency analysis and fast algorithms for easy implementation [8, 9].

Wavelets permit the accurate representation of a variety of functions and operators. Moreover wavelets establish a connection with fast numerical algorithms [5]. Publications on integral equation methods have shown a marked preference for orthogonal wavelets [22].

Different variations of wavelet bases (orthogonal, biorthogonal, multiwavelets) have been presented and the design of the corresponding wavelet and scaling functions has been addressed [7, 10, 11, 20]. Multiwavelets are generated by more than one scaling function [1, 20]. Multiwavelets have some advantages in comparison to single wavelets. For example, such features as short support, orthogonality, symmetry and vanishing moments are known to be important in signal processing and numerical methods. A single wavelets cannot possess all these properties at the same time. On the other hand, a multiwavelets system can have all of them simultaneously [25]. This suggests that multiwavelets could perform better in various applications.

In this paper, we use Legendre (Alpert) multiwavelets for solving forced Duffing equation with integral boundary conditions. These multiwavelets constructed in [1] and also considered in [2] and [3]. Our method consists of reducing forced Duffing equation equation to a set of algebraic equations by expanding unknown function as Legendre multiwavelets with unknown coefficients. The properties of these multiwavelets are then utilized to evaluate the unknown coefficients.

The paper is organized as follows: Section 2 is devoted to the basic formulation of the Legendre multiwavelets required for our subsequent development. In Section 3 the proposed method is used to approximate the Duffing equation. In Section 4, we report our numerical finding and demonstrate the accuracy of the proposed numerical scheme by considering numerical examples. Section 5, ends this paper with a brief conclusion

2. Legendre multiwavelets systems

2.1. **Multiresolution analysis.** For functions $\phi^m \in L^2(R)$, $m = 0, \dots, r$, let a reference subspace or sample space V_0 be generated as the L^2-closure of the linear span of the integer translates of ϕ^m, namely:

$$V_0 = clos_{L^2} \langle \phi^m(. - k) : k \in Z \rangle, \qquad m = 0, \dots, r,$$

and consider other subspace

$$V_j = clos_{L^2} \left\langle \phi^m_{j,k} : k \in Z \right\rangle, \qquad j \in Z, m = 0, \dots, r,$$

where $\phi^m_{j,k} = \phi^m(2^j x - k)$, $j, k \in Z$, $m = 0, \dots, r$.

Definition 1. Functions $\phi^m \in L^2(R)$, is said to generate a multiresolution analysis (MRA) if they generate a nested sequence of closed subspaces V_j that satisfy

$$\begin{cases} i) & \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots, \\ ii) & clos_{L^2}\left(\bigcup_{j\in Z} V_j\right) = L^2(R), \\ iii) & \bigcap_{j\in Z} V_j = 0, \\ iv) & f(x) \in V_j \Longleftrightarrow f(x+2^{-j}) \in V_j \Longleftrightarrow f(2x) \in V_{j+1}, \\ v) & \{\phi^m(.-k)\}_{k\in Z}, \qquad \text{form a Riesz basis of } V_0. \end{cases} \tag{2.1}$$

If ϕ^m generate an MRA, then ϕ^m are called scaling functions. In case the different integer translate of ϕ^m are orthogonal (with respect to the standard inner product $< f, g >= \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$ for two functions in $L^2(R)$), denoted by $\phi^m(.-k)\perp\phi^{\tilde{m}}(.-\tilde{k})$ for $m \neq \tilde{m}, k \neq \tilde{k}$, the scaling functions are called an orthogonal scaling functions.

As the subspaces V_j are nested, there exist complementary orthogonal subspaces W_j such that

$$V_{j+1} = V_j \bigoplus W_j, \qquad j \in Z,$$

where $\bigoplus$ denotes orthogonal sums.

This give rise to an orthogonal decomposition of $L^2(R)$, namely:

$$L^2(R) = \bigoplus_{j\in Z} W_j.$$

Definition 2. Functions $\psi^m \in L^2(R)$ are called wavelets, if they generate the complementary orthogonal subspaces W_j of an MRA, i.e.,

$$W_j = clos_{L^2} < \psi^m_{j,k}, k \in Z >, \qquad j \in Z, m = 0, \ldots, r,$$

where $\psi^m_{j,k} = \psi^m(2^j x - k)$, $j, k \in Z$.

Obviously, $\psi^m_{j,k}\perp\psi^{\tilde{m}}_{\tilde{j},\tilde{k}}$ for $j \neq \tilde{j}$, $m \neq \tilde{m}$ and $k \neq \tilde{k}$ if $< 2^{j/2}\psi^m_{j,k}, 2^{\tilde{j}/2}\psi^{\tilde{m}}_{\tilde{j},\tilde{k}} >= \delta_{j,\tilde{j}}\delta_{k,\tilde{k}}\delta_{m,\tilde{m}}$ then ψ^m are called orthonormal wavelets.

Now we define Legendre scaling functions and its corresponding multiwavelets according to the above MRA.

2.2. **Construction of Scaling Functions.** Legendre multiwavelets system with multiplicity r consist of r scaling functions and r wavelets. The r-th order Legendre scaling functions are the set of $r+1$ functions $\phi^0(x), ..., \phi^r(x)$ where $\phi^i(x)$ is a polynomial of i-th order and all ϕ's form orthonormal basis, that is [2, 24], for $i = 0, 1, ..., r$,

$$\phi^i(x) = \sum_{k=0}^{i} a_{ik}x^k, \qquad \text{for } i = 0, 1, ..., r. \tag{2.2}$$

The coefficients a_{ik} are chosen so that

$$\int_0^1 \phi^i(x)\phi^k(x)dx = \delta_{i,k}, \qquad \text{for } i, k = 0, 1, ..., r, \tag{2.3}$$

where

$$\delta_{i,k} = \begin{cases} 1, & i = k, \\ 0, & i \neq k. \end{cases}$$

The scaling functions $\phi^i(x)$ for i is even, odd have symmetry, anti-symmetry properties, respectively. The two scale relations for Legendre scaling functions of order r, are in the form [2]:

$$\phi^i(x) = \sum_{j=0}^{r} p_{i,j}\phi^j(2x) + \sum_{j=0}^{r} p_{i,r+j+1}\phi^j(2x-1), \quad i = 0, 1, \ldots, r. \tag{2.4}$$

The coefficients $\{p\}$ are determined uniquely by substituting the equations (2.2) into (2.4). We would like to mention two remarks on the two-scale relations.

1. since $\phi^i(x)$ is a i-th order polynomial, the right hand side of (2.4) has at most i-th order scaling functions. Therefore, $p_{i,j} = p_{i,r+j+1} = 0$ for $i < j$.
2. The two-scale relations for the Legendre scaling function of the order n which is lower than r is a subset of the first n two-scale relations for ϕ^i for $i = 0, 1, ..., n$ form r-th order two scale relations.

2.3. **Construction of Wavelets.** The two-scale relations for the r-th order Legendre multiwavelets are in the form [2]:

$$\psi^i(x) = \sum_{j=0}^{r} q_{i,j}\phi^j(2x) + \sum_{j=0}^{r} q_{i,r+j+1}\phi^j(2x-1). \tag{2.5}$$

As we have $2(r+1)^2$ unknown coefficients $\{q\}$ in (2.5), use the following $2r(r+1)$ vanishing moment conditions (2.6) and $2(r+1)$ orthonormal conditions (2.7) to determine them.

1. Vanishing moments

$$\int_0^1 \psi^i(x)x^j = 0, \qquad \text{for } i = 0, 1, ..., r \ j = 0, 1, ..., i + r. \tag{2.6}$$

2. Orthonormality

$$\int_0^1 \psi^i(x)\psi^j(x) = \delta_{i,j}, \qquad for \ i, j = 0, 1, ..., r. \tag{2.7}$$

For example the cubic Legendre scaling functions consist of the four functions in (2.8).

$$\begin{cases} \phi^0(x) = 1, & 0 \le x < 1, \\ \phi^1(x) = \sqrt{3}(2x-1), & 0 \le x < 1, \\ \phi^2(x) = \sqrt{5}(6x^2 - 6x + 1), & 0 \le x < 1, \\ \phi^3(x) = \sqrt{7}(20x^3 - 30x^2 + 12x - 1), & 0 \le x < 1. \end{cases} \tag{2.8}$$

The closed form solution to the cubic Legendre multiwavelets $\psi^0(x)$, $\psi^1(x)$, $\psi^2(x)$ and $\psi^3(x)$ are in (2.9)-(2.12) which are determined using the condition (2.6) and (2.7). Figures 1 and 2 show the plots of cubic Legendre multiwavelets.

$$\psi^0(x) = \begin{cases} -\sqrt{\frac{15}{17}}(224x^3 - 216x^2 + 56x - 3), & 0 \le x < \frac{1}{2}, \\ \sqrt{\frac{15}{17}}(224x^3 - 456x^2 + 296x - 61), & \frac{1}{2} \le x < 1, \end{cases} \tag{2.9}$$

FIGURE 1. The plots of cubic Legendre multiwavelets ψ^0 (left), and ψ^1 (right).

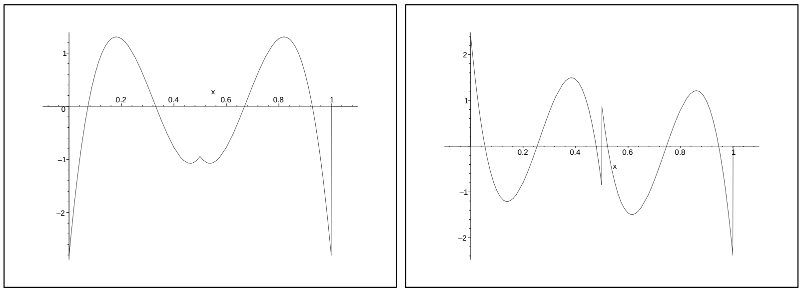
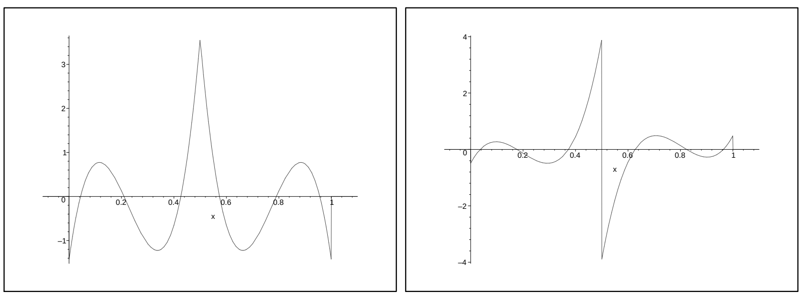


FIGURE 2. The plots of cubic Legendre multiwavelets ψ^2 (left), and ψ^3 (right).



$$\psi^1(x) = \begin{cases} \sqrt{\frac{1}{21}}(1680x^3 - 1320x^2 + 270x - 11), & 0 \le x < \frac{1}{2}, \\ \sqrt{\frac{1}{21}}(1680x^3 - 3720x^2 + 2670x - 619), & \frac{1}{2} \le x < 1, \end{cases} \tag{2.10}$$

$$\psi^2(x) = \begin{cases} -\sqrt{\frac{35}{17}}(256x^3 - 174x^2 + 30x - 1), & 0 \le x < \frac{1}{2}, \\ \sqrt{\frac{35}{17}}(256x^3 - 594x^2 + 450x - 111), & \frac{1}{2} \le x < 1, \end{cases} \tag{2.11}$$

$$\psi^3(x) = \begin{cases} \sqrt{\frac{5}{42}}(420x^3 - 246x^2 + 36x - 1), & 0 \le x < \frac{1}{2}, \\ \sqrt{\frac{5}{42}}(420x^3 - 1014x^2 + 804x - 209), & \frac{1}{2} \le x < 1. \end{cases} \tag{2.12}$$

2.4. **Function Approximation.** It can be verified that $V_j \oplus W_j = V_{j+1}$, thus we can write $V_j = V_0 \oplus (\oplus_{i=0}^{j-1} W_i)$ and we have two kind of basis sets for $J \in N$

$$\Phi_J(x) = \left[\phi^0_{J,0}(x), ..., \phi^r_{J,0}(x), | \cdots, \phi^0_{J,(2^J-1)}(x), ..., \phi^r_{J,(2^J-1)}(x)\right]^T, \tag{2.13}$$

$$\Psi_J(x) = \big[\phi^0_{0,0}(x), ..., \phi^r_{0,0}(x), |\psi^0_{0,0}(x), \ldots, \psi^r_{0,0}(x)|, \tag{2.14}$$

$$\ldots |\psi^0_{J-1,0}(x), \ldots, \psi^r_{J-1,0}(x)|, \ldots, \psi^0_{J-1,2^{J-1}-1}(x), \ldots, \psi^r_{J-1,2^{J-1}-1}(x)\Big]^T.$$

Now any function $f(x)$ on $[0,1]$ can be approximated using scaling functions as

$$f(x) = \sum_{k=0}^{2^J-1} \sum_{m=0}^{r} c_{J,k} \phi_{J,k}^{m}(x) = C^T \Phi_J(x), \tag{2.15}$$

and the corresponding wavelet functions as

$$f(x) = \sum_{m=0}^{r} \left\{ c_{0,0}^{m} \phi_{0,0}^{m}(x) + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} d_{j,k}^{m} \psi_{j,k}^{m}(x) \right\} = D^T \Psi_J(x), \tag{2.16}$$

where

$$c_{J,k}^{m} = \int_0^1 f(x) \phi_{J,k}^{m}(x) dx, \tag{2.17}$$

$$d_{j,k}^{m} = \int_0^1 f(x) \psi_{j,k}^{m}(x) dx, \tag{2.18}$$

and D and C are $(n \times 1)$ vectors with $n = (r+1)2^J$ given by

$$D = \left[c_{0,0}^{0}, ..., c_{0,0}^{r} | d_{0,0}^{0}, ..., d_{0,0}^{r} | ... | d_{J-1,0}^{0}, ..., d_{J-1,0}^{r} |, ..., d_{J-1,2^{J-1}-1}^{0}, ... d_{J-1,2^{J-1}-1}^{r} \right]^T, \tag{2.19}$$

$$C = \left[c_{J,0}^{0}, ..., c_{J,0}^{r} | ... | c_{J,2^J-1}^{0}, ..., c_{J,2^J-1}^{r} \right]^T. \tag{2.20}$$

2.5. **The Operational Matrix of Integral.** The integral of vectors $\Psi_J(x)$ and $\Phi_J(x)$ can be expressed as

$$\int_0^x \Psi_J(t) dt = I_\psi \Psi_J(x), \tag{2.21}$$

$$\int_0^x \Phi_J(t) dt = I_\phi \Phi_J(x), \tag{2.22}$$

where I_ψ and I_ϕ are $(n \times n)$ operational matrices of integral for Legendre scaling functions and multiwavelets. The matrix I_ψ can be obtained by the following process. Let

$$a_0 = 1, \qquad a_i = \sqrt{a_{i-1}^2 + 2}, \qquad \text{for } i = 1, 2, \ldots, r,$$

$$b_i = \frac{1}{a_{i-1} a_i}, \qquad \text{for } i = 1, 2, \ldots, r,$$

$$A_r = \begin{bmatrix} 0 & b_1 & 0 & & & & \\ & 0 & b_2 & 0 & & & \\ & & 0 & b_3 & 0 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 0 & b_{r-2} & 0 \\ & & & & & 0 & b_{r-1} \\ & & & & & & 0 \end{bmatrix}_{r \times r},$$

$$B_r = \frac{1}{2^J} \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}_{r \times r},$$

$$M_r = \frac{1}{2^{J+1}} \left(A_r - A_r^T + 2^J B_r \right).$$

Now it can be shown that

$$I_\phi = \begin{bmatrix} M_r & B_r & \cdots & B_r \\ & \ddots & \ddots & \vdots \\ & & M_r & B_r \\ 0 & & & M_r \end{bmatrix}_{n,n},$$

where I_ϕ is the operational matrix of Legendre scaling functions.

The matrix I_ψ can be obtained by considering

$$\Psi_J = G\Phi_{J+1}, \tag{2.23}$$

where G is a $(n \times n)$ matrix, which can be calculated as follows.

Equations (2.4) give

$$\Phi_j = P_j \Phi_{j+1}, \tag{2.24}$$

where $P_j,\ j = 1, 2, ..., J$ is a $(r2^{j-1}, r2^j)$ and members of P_j are the coefficient at (2.4).

From (2.5) we have

$$\Psi_j = Q_j \Phi_{j+1}, \tag{2.25}$$

where $Q_j,\ j = 1, 2, ..., J$ is a $(r2^{j-1}, r2^j)$ and members of Q_j are the coefficient at (2.5).

Using expressions (2.23), (2.24) and (2.25) we get

$$G = \begin{bmatrix} P_1 \times P_2 \times ... \times P_J \\ Q_1 \times P_2 \times ... \times P_J \\ \vdots \\ Q_{J-2} \times P_{J-1} \times \times P_J \\ Q_{J-1} \times P_J \\ Q_J \end{bmatrix}_{n \times n}. \tag{2.26}$$

Using expressions (2.21) (2.22) and (2.23) we have

$$\int_0^x \Psi_J(t)dt = G \int_0^x \Phi_{J+1}(t)dt = GI_\phi \Phi_{J+1}(x) = GI_\phi G^{-1} \Psi_J(x). \tag{2.27}$$

Comparing Eqs. (2.21) and (2.27) we get

$$I_\psi = GI_\phi G^{-1}. \tag{2.28}$$

3. Description of Numerical Method

In this section, we solve forcing Duffing equation of the form in (1.1), by using Legendre multiwavelets.

For this purpose, we first assume

$$z(t) = f(t, u(t)), . \tag{3.1}$$

Using Eq. (2.16) we get

$$z(t) = Z^T \Psi_J(t), \tag{3.2}$$

where Z is a $(n \times 1)$ unknown vector defined similarly to D in (2.19). Let

$$u''(t) = U^T \Psi_J(t), \tag{3.3}$$

by integrating from both sides of Eq.(3.3) and by using (2.21) we get

$$u'(t) - u'(0) = U^T \int_0^t \Psi_J(x)dx = U^T I_\psi \Psi_J(t), \tag{3.4}$$

now we put

$$u'(0) = \alpha,$$

thus

$$u'(t) = U^T I_\Psi \Psi_J(t) + \alpha. \tag{3.5}$$

Again by integrating from both sides of Eq. (3.5) we have

$$u(t) - u(0) = U^T I_\psi^2 \Psi_J(t) + \alpha t. \tag{3.6}$$

Suppose

$$u(0) = \beta,$$

so we get

$$u(t) = U^T I_\psi^2 \Psi_J(t) + \alpha t + \beta. \tag{3.7}$$

Using Eq. (2.19) we get

$$\alpha = \Lambda \Psi_J(t). \tag{3.8}$$

where Λ is a $(n \times 1)$ vector as

$$\Lambda = [\alpha, 0, \dots, 0]^T.$$

Using Eqs. (3.2) $-$ (3.8), in Eq. $(1,1)$, we get

$$U^T \Psi_J(t) + \sigma U^T I_\Psi \Psi_J(t) + \sigma \Lambda \Psi_J(t) + Z^T \Psi_J(t) = 0,$$

or

$$\left(U^T + \sigma U^T I_\Psi + \sigma \Lambda + Z^T\right) \Psi_J(t) = 0.$$

So we get

$$U^T + \sigma U^T I_\Psi + \sigma \Lambda + Z^T = 0. \tag{3.9}$$

Using Eqs.(3.2) and (3.7) in Eq. (3.1) we have

$$f(t, U^T I_\psi^2 \Psi_J(t) + \alpha t + \beta) = Z^T \Psi_J(t). \tag{3.10}$$

Collocating Eq. (3.10) in n points $t_i = i/(n-1)$, $i = 0, \cdots, n-1$ we get

$$f(t, U^T I_\psi^2 \Psi_J(t_i) + \alpha t_i + \beta) = Z^T \Psi_J(t_i). \tag{3.11}$$

The functions $h_1(s)$ and $h_2(s)$ in Eq. (1.2), using Eq. (2.16) may be approximated as

$$\begin{aligned} h_1(s) &= H_1^T \Psi_J(s), \\ h_2(s) &= H_2^T \Psi_J(s), \end{aligned} \tag{3.12}$$

where H_1 and H_2 are $(n \times 1)$ vectors with the entries as

$$\begin{aligned} (H_1)_i &= \int_0^1 h_1(s)(\Psi_J)_i(s)ds, \\ (H_2)_i &= \int_0^1 h_2(s)(\Psi_J)_i(s)ds. \end{aligned}$$

Applying Eqs. (3.5), (3.7) and (3.12) in Eq. (1.2) we get

$$\mu_1(U^T I_\psi^2 \Psi_J(0)+\beta)-\mu_2(U^T I_\Psi \Psi_J(0)+\alpha)-U^T I_\psi^2 \left(\int_0^1 \Psi_J(s)\Psi_J^T(s)ds\right) H_1^T$$

$$+\alpha H_1^T \int_0^1 s\Psi_J(s)ds + \beta H_1^T \int_0^1 \Psi_J(s)ds = 0, \tag{3.13}$$

and

$$\mu_3(U^T I_\psi^2 \Psi_J(1)+\alpha+\beta)-\mu_4(U^T I_\Psi \Psi_J(1)+\alpha)-U^T I_\psi^2 \left(\int_0^1 \Psi_J(s)\Psi_J^T(s)ds\right) H_2^T$$

$$+\alpha H_2^T \int_0^1 s\Psi_J(s)ds + \beta H_2^T \int_0^1 \Psi_J(s)ds = 0. \tag{3.14}$$

The second and the third integral terms in Eqs. (3.13) and (3.14), regarding Eq. (2.6) can be calculated as

$$\begin{aligned} V_1 &= \int_0^1 s\Psi(s)ds = [\tfrac{1}{2}, \tfrac{\sqrt{3}}{6}, 0, \ldots, 0]^T, \\ V_2 &= \int_0^1 \Psi(s)ds = [1, 0, \ldots, 0]^T. \end{aligned} \tag{3.15}$$

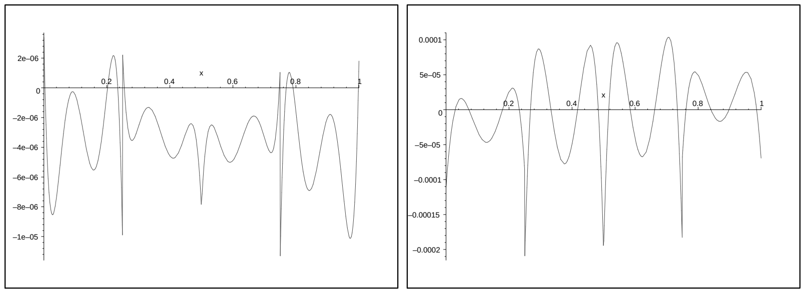
Using Eq.(2.7) in Eqs. (3.13) and (3.14) we get

$$\mu_1(U^T I_\psi^2 \Psi_J(0)+\beta)-\mu_2(U^T I_\Psi \Psi_J(0)+\alpha)-U^T I_\psi^2 H_1^T + \alpha H_1^T V_1 + \beta H_1^T V_2 = 0, \tag{3.16}$$

$$\mu_3(U^T I_\psi^2 \Psi_J(1)+\alpha+\beta)-\mu_4(U^T I_\Psi \Psi_J(1)+\alpha)-U^T I_\psi^2 H_2^T + \alpha H_2^T V_1 + \beta H_2^T V_2 = 0. \tag{3.17}$$

Equation (3.9), (3.11), (3.16) and (3.17) give a system of algebraic equations with $(2n+2)$ equations and unknowns, which can be solved to find U_k and $Z_k, k = 1, 2, \ldots, n$, α and β. So the unknown function $u(t)$ can be found using Eq. (3.7).

FIGURE 3. Absolute errors for $r = 4, J = 2$ (left), and $r = 3, J = 2$ (right).



4. EXAMPLE

In this section we give some computational results of numerical experiments with methods based on preceding section, to support our theoretical discussion. The non-linear systems obtained by the collocation method are solved by the Newton method.

Example 1. Consider the following forced Duffing equation [19]:

$$\begin{cases} u''(t) + u'(t) + t(1-t)u^3 = f(t), & 0 < t < 1, \\ u(0) - \frac{2}{\pi^2}u'(0) = -\int_0^1 u(s)ds, & u(1) + \frac{1}{\pi^2}u'(1) = -\int_0^1 su(s)ds, \end{cases}$$

where

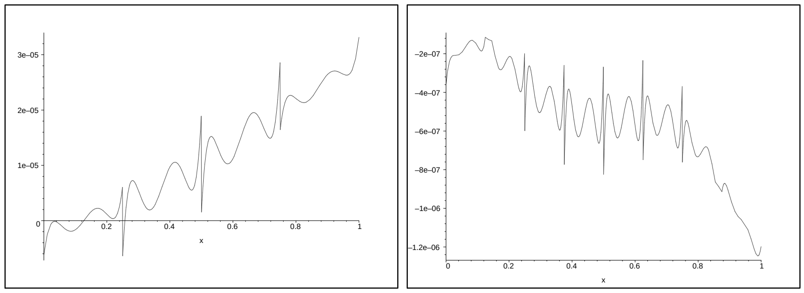
$$f(t) = \pi\cos(\pi t) - \sin(\pi t)\left(\pi^2 + (-1+t)\sin(\pi t)^2\right).$$

The exact solution is $u(x) = \sin(\pi t)$. Table 1 and Figure 3 represents the absolute values errors obtained in solving this test example with different values of r and J.

Table 1. Absolute errors for Example 1.

x	r=2 J=3	r=3 J=2	r=4 J=2
0.0	2.3×10^{-4}	1.1×10^{-4}	3.4×10^{-6}
0.1	1.1×10^{-4}	3.1×10^{-5}	4.9×10^{-7}
0.2	1.8×10^{-4}	2.6×10^{-5}	3.9×10^{-7}
0.3	3.9×10^{-4}	8.4×10^{-5}	2.6×10^{-6}
0.4	3.8×10^{-4}	4.9×10^{-5}	4.6×10^{-6}
0.5	3.8×10^{-4}	6.3×10^{-5}	4.8×10^{-6}
0.6	3.9×10^{-4}	4.1×10^{-5}	4.9×10^{-6}
0.7	4.2×10^{-4}	1.0×10^{-4}	3.4×10^{-6}
0.8	2.3×10^{-4}	5.0×10^{-5}	1.6×10^{-6}
0.9	1.7×10^{-4}	1.6×10^{-6}	2.0×10^{-6}
1.0	1.4×10^{-4}	6.9×10^{-5}	1.8×10^{-6}

FIGURE 4. Absolute errors for $r = 4, J = 2$ (left), and $r = 4, J = 3$ (right).



Example 2. Consider the following forced Duffing equation:

$$\begin{cases} u''(t) - u'(t) + tu^2 = f(t), & 0 < t < 1, \\ \frac{2}{3\pi}u(0) + u'(0) = -\int_0^1 \sin(\frac{\pi s}{2})u(s)ds, & \frac{2}{\pi^2}u(1) + u'(1) = \int_0^1 (s+2)u(s)ds, \end{cases}$$

where $f(t) = \pi\sin(\pi t) - \pi^2\cos(\pi t) + t\cos(\pi t)^2$. The exact solution is given by

$$u(x) = \cos(\pi t).$$

Table 2 and Figure 4 present the absolute errors for different values of r and J, using the present method.

Table 2. Absolute errors for Example 2.

x	r=2 J=3	r=3 J=2	r=4 J=2
0.0	3.7×10^{-5}	1.6×10^{-6}	6.1×10^{-6}
0.1	1.2×10^{-4}	2.5×10^{-5}	1.6×10^{-6}
0.2	2.0×10^{-4}	9.1×10^{-5}	1.2×10^{-6}
0.3	5.2×10^{-4}	2.2×10^{-5}	5.5×10^{-6}
0.4	7.1×10^{-4}	4.2×10^{-5}	9.7×10^{-6}
0.5	5.9×10^{-4}	1.4×10^{-5}	9.9×10^{-6}
0.6	6.6×10^{-4}	5.3×10^{-5}	1.1×10^{-5}
0.7	9.0×10^{-4}	1.3×10^{-4}	1.6×10^{-5}
0.8	1.3×10^{-3}	2.1×10^{-4}	2.2×10^{-5}
0.9	1.5×10^{-3}	1.0×10^{-4}	2.6×10^{-5}
1.0	1.9×10^{-3}	1.5×10^{-5}	3.3×10^{-5}

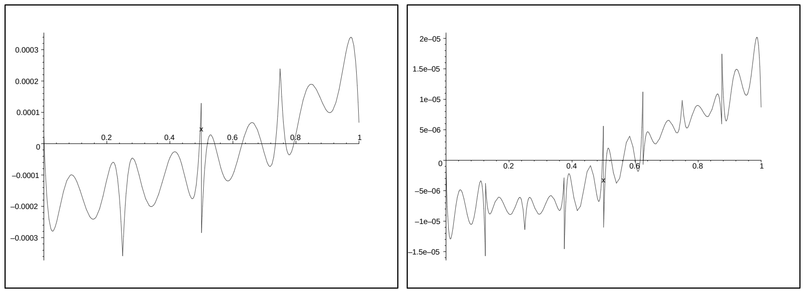
Example 3. Consider the following forced Duffing equation:

$$\begin{cases} u''(t) - 2u'(t) + t^2u^2 = f(t), & 0 < t < 1, \\ u(0) + \frac{4}{9\pi^3}u'(0) = -\int_0^1 s\sin(\pi s)u(s)ds, & u(1) - \frac{1}{2\pi^2}u'(1) = -\int_0^1 (2s-1)u(s)ds, \end{cases}$$

where

$$f(t) = 4\pi^2\sin(2\pi t) - 4\pi\cos(2\pi t) + t^2(1 - \cos(2\pi t)^2).$$

FIGURE 5. Absolute errors for $r = 4, J = 2$ (left), and $r = 4, J = 3$ (right).



The exact solution is $u(x) = \sin(2\pi t)$. The absolute errors are obtained in Table 3 and Figure 5, using the presented method, for different values of r and J.

Table 3. Absolute errors for Example 3.

x	r=2 J=3	r=3 J=3	r=4 J=2
0.0	7.8×10^{-5}	3.0×10^{-5}	2.3×10^{-5}
0.1	2.0×10^{-3}	1.4×10^{-4}	1.1×10^{-4}
0.2	2.3×10^{-3}	3.7×10^{-5}	1.1×10^{-4}
0.3	1.9×10^{-3}	1.9×10^{-5}	9.2×10^{-5}
0.4	5.4×10^{-4}	7.9×10^{-5}	4.4×10^{-5}
0.5	2.2×10^{-3}	3.1×10^{-5}	3.3×10^{-5}
0.6	1.5×10^{-3}	5.0×10^{-5}	9.9×10^{-5}
0.7	2.4×10^{-4}	9.2×10^{-6}	3.2×10^{-5}
0.8	1.5×10^{-3}	6.1×10^{-5}	3.0×10^{-5}
0.9	2.4×10^{-3}	2.1×10^{-4}	1.0×10^{-4}
1.0	2.1×10^{-3}	1.7×10^{-4}	6.8×10^{-5}

5. Conclusion

In this article, we presented a numerical scheme for solving the forced Duffing equation with integral boundary conditions. The Legendre multiwavelets [1–3] on interval [0, 1] are employed to solve this equation. The obtained results show that this approach can solve the problem effectively.

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