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# Polynomial and non-polynomial solutions set for wave equation using Lie point symmetries

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**Abstract** This paper obtains the exact solutions of the wave equation as a second-order partial differential equation (PDE). We are going to calculate polynomial and non-polynomial exact solutions by using Lie point symmetry. We demonstrate the generation of such polynomial through the medium of the group theoretical properties of the equation. A generalized procedure for polynomial solution is presented and this extended to the construction of related polynomials.

**Keywords.** Wave equation, Symmetry, Similarity solution.

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## 1. Introduction

The wave equation is an important hyperbolic second-order non-linear PDE

$$u_{tt} - (f(u)u_x)_x - (g(u)u_y)_y = 0, \tag{1.1}$$

where $f$ and $g$ are arbirary smooth functions of $u$ and $u$ is the dependent variable of $(t, x, y)$. This equation uses for the description of waves as they occur in physics such as sound waves, light waves and water waves. It arises in another fields like acoustics, electromagnetics, and fluid dynamics.

This paper considers the Eq. (1.1) for $f(u) = g(u) = 1$. Thus, we will focus on the revised form of the equation in the following form

$$u_{tt} - u_{xx} - u_{yy} = 0. \tag{1.2}$$

Then, the wide range of solution with Lie symmetry's method is given. This method is based on finding some differential operators (vector fields) called symmetries, in order to find the exact solutions of differential equations. These operators are the largest local group of transformations acting on the independent and dependent variables of the system with the property that they transform solutions of the system to other

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solutions. When one is confronted with a complicated system of PDEs arising from some physically important problem, the discovery of any explicit solutions whatsoever is of great interest. Explicit solutions can be used as a models for physical experiments, as benchmarks for testing numerical methods, etc., and often reflect the asymptotic or dominant behaviour of more general types of solutions.

One of the most important application of symmetry's method is the reducing systems of differential equations, i.e., finding equivalent systems of differential equations of simpler form, that is called reduction. This method provides a systematic computational algorithm for determining a large classes of special solutions. The solutions of the obtained equivalent system will correspond to solutions of the original system. There is a lot of papers in the literature for this process and one can find the classical reduction method in [1, 2, 8, 9, 10, 13].

Solutions of the wave equation describe propagation of disturbances out from the region at a fixed speed in one or in all spatial directions, as doing physical waves from plane or localized sources. The solutions of the Eq. (1.2) have $10 + 1 + \infty$ Lie point symmetries. The classical solutions are recovered with the use of the non-generic symmetries to construct similarity solutions. Further solutions, both polynomial and non-polynomial, are constructed by using the invariants of the Lie point symmetries as seed solutions and the property of mapping solutions into solutions. These solutions are analogous to the well-known wave polynomials.

The presented paper is organized as follows: the second section is devoted to introduce the important concept of Lie point symmetry, in the third chapter the reduction forms of the Eq. (1.2) including the invariants for finding the similarity solutions are given [3]. Then, we applied the Lie bracket of the symmetries to find some new solutions from the old solutions. In section four, some new non-polynomial solutions are given from the old non-polynomial solutions. Finally some special solutions are plotted at the end of this section.

## 2. Lie Symmetries of the Equation

Symmetry plays a very important role in various fields of nature. As is known to all, Lie method is an effective method and a large number of equations [13] are solved with the aid of this method. There are still many authors who use this method to find the exact solutions [2, 4, 6, 7, 13] of non-linear differential equations. Since this method has powerful tools to find exact solutions of non-linear problems [13, 14]. For example, as it said in the introduction, when we are confronted with a complicated system of PDEs or ODEs, it is intersting to find a vast set of exact solutions for the given system via a systematic methods with no limitation, this would be done by using Lie's symmetry as an analytic applicable method. The general procedure to obtain Lie symmetries of differential equations, and their applications to find analytic solutions of the equations are described in detail in several monographs on the subject (e.g. [3, 4, 8]) and in numerous papers in the literature (e.g. [5, 6, 11, 12, 13]).

A PDE with $p-$independent and $q-$dependent variables has a Lie point transformations

$$\tilde{x}_i = x_i + \xi_i(x,u) + \mathcal{O}(\epsilon^2), \qquad \tilde{u}_\alpha = u_\alpha + \phi_\alpha(x,u) + \mathcal{O}(\epsilon^2). \tag{2.1}$$

where

$$\xi_i = \frac{\partial \tilde{x}_i}{\partial \epsilon}|_{\epsilon=0} \quad \text{for} \quad i = 1, ..., p, \qquad \phi_\alpha = \frac{\partial \tilde{u}_\alpha}{\partial \epsilon}|_{\epsilon=0} \quad \text{for} \quad \alpha = 1, ..., q. \tag{2.2}$$

The action of the Lie group can be considered by its associated infinitesimal generator

$$X = \sum_{i=1}^{p} \xi_i(x, u)\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \phi_\alpha(x, u)\frac{\partial}{\partial u_\alpha}, \tag{2.3}$$

on the total space of PDE (the space containing independent and dependent variables). Furthermore, the characteristic of the vector field (2.3) is given by

$$Q^\alpha(x, u^{(1)}) = \phi_\alpha(x, u) - \sum_{i=1}^{p} \xi_i(x, u)\frac{\partial u^\alpha}{\partial x_i},$$

and its $n-$th prolongation is determined by

$$X^{(n)} = \sum_{i=1}^{p} \xi_i(x, u)\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \sum_{J=j=0}^{n} \phi_\alpha^J(x, u^{(j)})\frac{\partial}{\partial u_J^\alpha}, \tag{2.4}$$

where $\phi_\alpha^J = D_J Q^\alpha + \sum_{i=1}^{p} \xi_i u_{J,i}^\alpha$ are the prolong coefficients.

The aim is to analyse the point symmetry structure of the wave equation, where $u$ is a smooth function of $(x, y, t)$. Let us consider a one-parameter Lie group of infinitesimal transformations given by

$$\begin{array}{ll} \tilde{t} = t + \varepsilon\xi_1(t, x, y, u) + \mathcal{O}(\epsilon^2), & \tilde{x} = x + \varepsilon\xi_2(t, x, y, u) + \mathcal{O}(\epsilon^2), \\ \tilde{y} = y + \varepsilon\xi_3(t, x, y, u) + \mathcal{O}(\epsilon^2), & \tilde{u} = u + \varepsilon\phi(t, x, y, u) + \mathcal{O}(\epsilon^2), \end{array} \tag{2.5}$$

where $\epsilon$ is the group parameter. Then, one requires that this transformations leaves invariant the set of solutions of the Eq. (1.2) . This yields to the linear system of equations for the infinitesimals $\xi_1, \xi_2, \xi_3, \phi$ . The Lie algebra of infinitesimal symmetries is the set of vector fields in the form of

$$X = \xi_1\frac{\partial}{\partial t} + \xi_2\frac{\partial}{\partial x} + \xi_3\frac{\partial}{\partial y} + \phi\frac{\partial}{\partial u}.$$

This vector field has the second prolongation

$$\begin{aligned} X^{(2)} = X &+ \phi^t\frac{\partial}{\partial u_t} + \phi^x\frac{\partial}{\partial u_x} + \phi^y\frac{\partial}{\partial u_y} + \phi^{tt}\frac{\partial}{\partial u_{tt}} \\ &+\phi^{tx}\frac{\partial}{\partial u_{tx}} + \phi^{ty}\frac{\partial}{\partial u_{ty}} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{xy}\frac{\partial}{\partial u_{xy}} + \phi^{yy}\frac{\partial}{\partial u_{yy}}. \end{aligned} \tag{2.6}$$

Acting (2.6) on the Eq. (1.2) and using the invariance condition, yields the full symmetry group of the equation spanned by the following eleven vector fields:

$$\begin{aligned}
&X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = \frac{\partial}{\partial t},\\
&X_4 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \qquad X_5 = t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t},\\
&X_6 = t\frac{\partial}{\partial y} + y\frac{\partial}{\partial t} \qquad X_7 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t},\\
&X_8 = (t^2 + x^2 - y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} + 2xt\frac{\partial}{\partial t} - xu\frac{\partial}{\partial u},\\
&X_9 = 2xy\frac{\partial}{\partial x} + (t^2 - x^2 + y^2)\frac{\partial}{\partial y} + 2yt\frac{\partial}{\partial t} - yu\frac{\partial}{\partial u},\\
&X_{10} = 2xt\frac{\partial}{\partial x} + 2yt\frac{\partial}{\partial y} + (t^2 + x^2 + y^2)\frac{\partial}{\partial t} - tu\frac{\partial}{\partial u},\\
&X_{11} = u\frac{\partial}{\partial u}, \qquad X_{12} = f(t,x,y)\frac{\partial}{\partial u}.
\end{aligned}$$

where $f\,(t,x,y)$ is a solution of Eq (1.2).

## 3. Calculation of Solutions

The number of Lie point symmetries is written in the form given to high light their different provenances. The eleventh symmetry, the infinity, is a feature of linear evolution equation and those non-linear evolution equation which can be linearized by means of a point transformation. For a linear evolution equation the function, $f(t,x,y)$, is a solution of the equation itself as is the case of the wave equation (1.2).

Determination of similarity solutions of the wave (and other) equation is a standard procedure to be found in many texts. We remind the reader, $X_{11}$ and $X_{12}$ do not provide similaity solutions. The associated Lagrange's system for $X_1$ is

$$\frac{dx}{d\varepsilon} = 1, \quad \frac{dt}{d\varepsilon} = 0, \quad \frac{dy}{d\varepsilon} = 0, \quad \frac{du}{d\varepsilon} = 0, \tag{3.1}$$

where $\varepsilon$ is the group parameter in (2.5). The solution of the system (3.1) yields invariants are $y$ , $t$ and $u$. Thus, we can write $u = g(t,y)$ and substitue this into Eq. (1.2) that reduced to

$$g_{tt} - g_{yy} = 0. \tag{3.2}$$

This equation have two solution sets; polynomial and non-polynomial solution,

$$g(t,y) = t^2 + y^2, \tag{3.3}$$

is polynomial solution and

$$g(t,y) = \ln\frac{y-t}{y+t}, \tag{3.4}$$

TABLE 1. Invariants and the solution set for the symmetry $X_i, i = 1, ..., 10$

| Symmetry | Invariant transformations | Reduced equations |
|---|---|---|
| $X_1$ | $q = t, r = y, g = u$ | $g_{qq} - g_{rr} = 0$ |
| $X_2$ | $q = t, r = x, g = u$ | $g_{qq} - g_{rr} = 0$ |
| $X_3$ | $q = x, r = y, g = u$ | $g_{qq} + g_{rr} = 0$ |
| $X_4$ | $q = t, r = x^2 + y^2, g = u$ | $g_{qq} - 4g_r - 4rg_{rr} = 0$ |
| $X_5$ | $q = y, r = t^2 - x^2, g = u$ | $g_{qq} + 4g_r + 4rg_{rr} = 0$ |
| $X_6$ | $q = x, r = y^2 - t^2, g = u$ | $g_{qq} + 4g_r + 4rg_{rr} = 0$ |
| $X_7$ | $q = \frac{t}{x}, r = \frac{y}{x}, g = u$ | $r^2 g_{rr} + q^2 g_{qq} + 2qrg_{rq} + 2qg_q$ $+2rg_r - g_{rr} + g_{qq} = 0$ |
| $X_8$ | $q = \frac{x^2+y^2-t^2}{y}, r = \frac{t}{y}, g = u\sqrt{y}$ | $4r^2 q_{rr} + 8grq_{gr} + 4g^2 q_{gg} + 12gq_g$ $+12rq_r - 4q_{gg} + 3q = 0$ |
| $X_9$ | $q = \frac{x^2+y^2-t^2}{x}, r = \frac{t}{x}, g = u\sqrt{x}$ | $4r^2 q_{rr} + 8grq_{gr} + 4g^2 q_{gg} + 12gq_g$ $+12rq_r - 4q_{gg} + 3q = 0$ |
| $X_{10}$ | $q = \frac{t^2-y^2-x^2}{x}, r = \frac{y}{x}, g = u\sqrt{x}$ | $4r^2 q_{rr} + 8grq_{gr} + 4g^2 q_{gg} + 12gq_g$ $+12rq_r + 4q_{gg} + 3q = 0$ |

is non-polynomial solution for the reduced equation (3.2). The associated Lagrange's system for $X_5$ is

$$\frac{dt}{d\varepsilon} = x, \quad \frac{dx}{d\varepsilon} = t, \quad \frac{dy}{d\varepsilon} = 0, \quad \frac{du}{d\varepsilon} = 0. \tag{3.5}$$

So the solutions of the system (3.5) yields the invariants $y, t^2 - x^2$ and $u$. We write $u = g(y, r)$ such that $r = t^2 - x^2$ and substitue this into Eq. (1.2) that reduced to

$$4rg_{rr} + 4g_r + g_{yy} = 0. \tag{3.6}$$

The equation (3.6) have two solution sets too, polynomial and non-polynomial solutions,

$$g(r, y) = r + 2y^2, \tag{3.7}$$

is polynomial solution and

$$g(r, y) = \operatorname{arctanh}\sqrt{\frac{y^2 + r}{y^2}}, \tag{3.8}$$

is non-polynomial solution for this reduced equation (3.6). The procedure is the same for other symmetries. We summarize the results in Table 1 and 2.

From these solutions we may construct further solutions by the property that symmetries map solutions to solutions. Linear PDEs have an infinite number of solutions and under quite general conditions an admitted symmetry must be fiber preserving. To construct the solution, one uses the property that the Lie bracket of $X_i, i = 1, ..., 10$ with $X_{12}$ produces another member of the class of symmetries of the form of $X_{12}$. This provide a route to the generation of new and non-trivial solutions from trivial similarity solutions that are associated with $X_i, i = 1, ..., 10$. The structure of the new solutions from the property of the Lie bracket with the solution symmetry summarized in Table 3. For example we can obtain other solutions from

Table 2. Invariants and the solution set for the symmetry $X_i$, $i = 1, 6$

| Symmetry | Invariants | Polynomial solutions | Non-polynomial solutions |
|---|---|---|---|
| $X_1$ | $t, y, u$ | $t^2 + y^2$ | $\ln \frac{y-t}{y+t}$ |
| $X_2$ | $t, x, u$ | $t^2 + x^2$ | $\ln \frac{x-t}{x+t}$ |
| $X_3$ | $x, y, u$ | $x^2 - y^2$ | $0$ |
| $X_4$ | $t, x^2 + y^2, u$ | $2t^2 + x^2 + y^2$ | $\arctan\sqrt{\frac{x^2+y^2-t^2}{t^2}}$ |
| $X_5$ | $t^2 - x^2, y, u$ | $t^2 - x^2 + 2y^2$ | $\operatorname{arctanh}\sqrt{\frac{y^2+t^2-x^2}{y^2}}$ |
| $X_6$ | $x, y^2 - t^2, u$ | $y^2 - t^2 - 2x^2$ | $\operatorname{arctanh}\sqrt{\frac{y^2+x^2-t^2}{x^2}}$ |

Table 3. Structure of the new solutions generated by the Lie bracket

| $[X_i, X_{12}]$ | New Symmetry | New Solutions |
|---|---|---|
| $[X_1, X_{12}]$ | $\frac{\partial f_{\text{old}}}{\partial x}\frac{\partial}{\partial u}$ | $f_{\text{new}} = \frac{\partial f_{\text{old}}}{\partial x}$ |
| $[X_2, X_{12}]$ | $\frac{\partial f_{\text{old}}}{\partial y}\frac{\partial}{\partial u}$ | $f_{\text{new}} = \frac{\partial f_{\text{old}}}{\partial y}$ |
| $[X_3, X_{12}]$ | $\frac{\partial f_{\text{old}}}{\partial t}\frac{\partial}{\partial u}$ | $f_{\text{new}} = \frac{\partial f_{\text{old}}}{\partial t}$ |
| $[X_4, X_{12}]$ | $\left(-y\frac{\partial f_{\text{old}}}{\partial x} + x\frac{\partial f_{\text{old}}}{\partial y}\right)\frac{\partial}{\partial u}$ | $f_{\text{new}} = -y\frac{\partial f_{\text{old}}}{\partial x} + x\frac{\partial f_{\text{old}}}{\partial y}$ |
| $[X_5, X_{12}]$ | $\left(t\frac{\partial f_{\text{old}}}{\partial x} + x\frac{\partial f_{\text{old}}}{\partial t}\right)\frac{\partial}{\partial u}$ | $f_{\text{new}} = t\frac{\partial f_{\text{old}}}{\partial x} + x\frac{\partial f_{\text{old}}}{\partial t}$ |
| $[X_6, X_{12}]$ | $\left(t\frac{\partial f}{\partial y} + y\frac{\partial f}{\partial t}\right)\frac{\partial}{\partial u}$ | $f_{\text{new}} = t\frac{\partial f_{\text{old}}}{\partial y} + y\frac{\partial f_{\text{old}}}{\partial t}$ |
| $[X7, X_{12}]$ | $\left(x\frac{\partial f_{\text{old}}}{\partial x} + y\frac{\partial f_{\text{old}}}{\partial y} + t\frac{\partial f_{\text{old}}}{\partial t}\right)\frac{\partial}{\partial u}$ | $f_{\text{new}} = x\frac{\partial f_{\text{old}}}{\partial x} + y\frac{\partial f_{\text{old}}}{\partial y} + t\frac{\partial f_{\text{old}}}{\partial t}$ |
| $[X_8, X_{12}]$ | $\left(xf_{\text{old}} + (t^2 + x^2 - y^2)\frac{\partial f_{\text{old}}}{\partial x}\right.$ | $f_{\text{new}} = xf_{\text{old}} + (t^2 + x^2 - y^2)\frac{\partial f_{\text{old}}}{\partial x}$ |
| | $\left.+2xy\frac{\partial f_{\text{old}}}{\partial y} + 2xt\frac{\partial f_{\text{old}}}{\partial t}\right)\frac{\partial}{\partial u}$ | $+2xy\frac{\partial f_{\text{old}}}{\partial y} + 2xt\frac{\partial f_{\text{old}}}{\partial t}$ |
| $[X_9, X_{12}]$ | $\left(yf_{\text{old}} + 2xy\frac{\partial f_{\text{old}}}{\partial x} + (t^2 - x^2 + y^2)\frac{\partial f_{\text{old}}}{\partial y}\right.$ | $f_{\text{new}} = yf_{\text{old}} + 2xy\frac{\partial f_{\text{old}}}{\partial x}$ |
| | $\left.+2yt\frac{\partial f_{\text{old}}}{\partial t}\right)\frac{\partial}{\partial u}$ | $+(t^2 - x^2 + y^2)\frac{\partial f_{\text{old}}}{\partial y} + 2yt\frac{\partial f_{\text{old}}}{\partial t}$ |
| $[X_{10}, X_{12}]$ | $\left(tf_{\text{old}} + 2xt\frac{\partial f_{\text{old}}}{\partial x} + 2yt\frac{\partial f_{\text{old}}}{\partial y}\right.$ | $f_{\text{new}} = tf_{\text{old}} + 2xt\frac{\partial f_{\text{old}}}{\partial x} + 2yt\frac{\partial f_{\text{old}}}{\partial y}$ |
| | $\left.+(t^2 + x^2 + y^2)\frac{\partial f_{\text{old}}}{\partial t}\right)\frac{\partial}{\partial u}$ | $+(t^2 + x^2 + y^2)\frac{\partial f_{\text{old}}}{\partial t}$ |

the seed solution $f(x, t, y) = y^2 + t^2$ by $X_1$ that is 0, or by $X_2$ that are $2y$, 2 and 0, by $X_3$ that are $2xy$, $2(x^2 - y^2)$, $-8xy$, $-8(x^2 - y^2)$ and etc. These results summarized in Table 4 and 5.

## 4. Non-polynomial Solutions

The seed solutions of $X_i$ provide a source for non-polynomial solutions of wave equation. For example for non-polynomial solution (3.4) $X_1$ gives 0 as a trivial solution. But, $X_2$ gives

$$f^1(t, x, y) = -2\frac{t}{t^2 - y^2}, \qquad f^2(t, x, y) = -2\frac{t(y+t)}{(y-t)^2} + 2\frac{t}{y-t},$$
$$f^3(t, x, y) = 4\frac{t(y+t)}{(y-t)^3} - 4\frac{t}{(y-t)^2},$$

TABLE 4. Classification of exact polynomial solutions for wave equation

| $f_{\text{old}} = y^2 + t^2$ | $f_{\text{old}} = x^2 + t^2$ | $f_{\text{old}} = x^2 - y^2$ |
|---|---|---|
| $f_{\text{new}} = 2y$ | $f_{\text{new}} = 2x$ | $f_{\text{new}} = 2x$ |
| $f_{\text{new}} = 2t$ | $f_{\text{new}} = 2t$ | $f_{\text{new}} = 2y$ |
| $f_{\text{new}} = c$ | $f_{\text{new}} = c$ | $f_{\text{new}} = c$ |
| $f^n_{\text{new}} = (-1)^n(2^{2n+1})xy$ | $f^n_{\text{new}} = (-1)^{n+1}(2^{2n+1})xy$ | $f^n_{\text{new}} = (-1)^n(2^{2n})xy$ |
| $f^n_{\text{new}} = (-1)^n(2^{2n+1})xy$ | $f^n_{\text{new}} = (-1)^{n+1}(2^{2n+1})xy$ | $f^n_{\text{new}} = (-1)^n(2^{2n})xy$ |
| $f^n_{\text{new}} = (2^{2n+1})xt$ | $f^n_{\text{new}} = (2^{2n})xt$ | $f^n_{\text{new}} = (2^{2n+1})xt$ |
| $f^n_{\text{new}} = (2^{2n+1})(x^2 + t^2)$ | $f^n_{\text{new}} = (2^{2n})(x^2 + t^2)$ | $f^n_{\text{new}} = (2^{2n+1})(x^2 + t^2)$ |
| $f_{\text{new}} = 5xt^2 + 5xy^2$ | $f_{\text{new}} = 7xt^2 + 3x^3 - 2xy^2$ | $f_{\text{new}} = 2xt^2 + 3x^3 - 7xy^2$ |
| $f_{\text{new}} = 7yt^2 - 2yx^2 + 3y^3$ | $f_{\text{new}} = 5yt^2 + 5yx^2$ | $f_{\text{new}} = 7yx^2 - 2yt^2 - 3y^3$ |
| $f_{\text{new}} = 2tx^2 + 7ty^2 + 3t^3$ | $f_{\text{new}} = 7tx^2 + 2ty^2 + 3t^3$ | $f_{\text{new}} = 5tx^2 - 5ty^2$ |

TABLE 5. Classification of exact polynomial solutions for wave equation

| $f_{\text{old}} = y^2 + 2t^2 + x^2$ | $f_{\text{old}} = 2y^2 + t^2 - x^2$ | $f_{\text{old}} = y^2 - t^2 - 2x^2$ |
|---|---|---|
| $f_{\text{new}} = 2x$ | $f_{\text{new}} = 2x$ | $f_{\text{new}} = 2x$ |
| $f_{\text{new}} = 2t$ | $f_{\text{new}} = 2t$ | $f_{\text{new}} = 2y$ |
| $f_{\text{new}} = c$ | $f_{\text{new}} = 2y$ | $f_{\text{new}} = 2t$ |
| $f^n_{\text{new}} = 3(-1)^n(2^{2n+1})(x^2 - y^2)$ | $f^n_{\text{new}} = (-1)^n(2^{2n+1})(x^2 - y^2)$ | $f^n_{\text{new}} = (-1)^{n+1}(2^{2n+1})(x^2 - y^2)$ |
| $f^n_{\text{new}} = (-1)^n(2^{2n})(x^2 + t^2)$ | $f^n_{\text{new}} = 3(-1)^n(2^{2n+1})xy$ | $f^n_{\text{new}} = 3(-1)^n(2^{2n+1})(x^2 - y^2$ |
| $f^n_{\text{new}} = (2^{2n+1})yt$ | $0$ | $f^n_{\text{new}} = (-3)(2^{2n+1})xt$ |
| $f^n_{\text{new}} = 3(2^{2n+1})(y^2 + t^2)$ | $f^n_{\text{new}} = 3(2^{2n+1})yt$ | $f^n_{\text{new}} = (-3)(2^{2n+1})(x^2 + t^2)$ |
| $f_{\text{new}} = 12xt^2 + 3xy^2 + 3x^3$ | $f_{\text{new}} = 3xt^2 + 12xy^2 - 3x^3$ | $f_{\text{new}} = 9xy^2 - 9xt^2 - 6x^3$ |
| $f_{\text{new}} = 12yt^2 + 3yx^2 + 3y^3$ | $f_{\text{new}} = 9yt^2 - 9yx^2 + 6y^3$ | $f_{\text{new}} = 3y^3 - 12yx^2 - 3yt^2$ |
| $f_{\text{new}} = 9ty^2 + 9tx^2 + 6t^3$ | $f_{\text{new}} = 12ty^2 - 3tx^2 + 3t^3$ | $f_{\text{new}} = 3ty^2 - 12tx^2 - 3t^3$ |

as non-polynomial solutions. A straightforward calculation shows that

$$f^n(t,x,y) = 2tn!\left[\left(\frac{(-1)^n t(y+t)}{(y-t)^{(n+1)}}\right) + \left(\frac{(-1)^{n+1}}{(y-t)^n}\right)\right],$$

is a general solution for the Eq. (1.2).

Similarly $X_3$ provides the solutions:

$$f^1(t,x,y) = 2\frac{y}{t^2-y^2}, \qquad f^2(t,x,y) = -2\frac{y(y+t)}{(y-t)^2} - 2\frac{y}{y-t},$$

$$f^3(t,x,y) = -12\frac{y(y+t)}{(y-t)^4} - 12\frac{y}{(y-t)^3}.$$

Observation outcomes is expressed in Table 6. We can also run this process for other non-polynpmial solutions in the last column of the Table 1 to obtain a number of solutions for wave equation. These results are comming in Tables 7, 8, 9 and 10. Also some special cases in polynomial and non-polynomial form are plotted in Figure (1).

TABLE 6. Classification of exact non-polynomial solutions for wave equation

| $X_i$ | $f_{\text{old}} = \ln \frac{y-t}{y+t}$ |
|---|---|
| $X_1$ | $f_{\text{new}} = 0$ |
| $X_2$ | $f_{\text{new}} = 2tn![\frac{(-1)^n t(y+t)}{(y-t)^{(n+1)}}) + (\frac{(-1)^{n+1}}{(y-t)^n})]$ |
| $X_3$ | $f_{\text{new}} = 2yn![\frac{(-1)^n t(y+t)}{(y-t)^{(n+1)}}) + (\frac{(-1)^n}{(y-t)^n})]$ |
| $X_4$ | $f_{\text{new}} = -2\frac{tx}{t^2-y^2}, 2\frac{t(yt^2-2tx^2-y^3)}{(-y+t)^2}, 2\,\frac{tx(t^3+5yt^2-4tx^3-7ty^2+y^3)}{(-y+t)^3}$, etc |
| $X_5$ | $f_{\text{new}} = 2\,\frac{yx}{t^2-y^2}, 2\,\frac{yt\left(t^2-2\,x^2-y^2\right)}{(t^2-y^2)^2}, -2\,\frac{yx\left(5\,t^4-6\,t^2x^2-4\,t^2y^2-2\,x^2y^2-y^4\right)}{(t^2-y^2)^3}$, etc |
| $X_6$ | $f_{\text{new}} = 0$ |
| $X_7$ | $f_{\text{new}} = 0$ |

TABLE 7. Classification of exact non-polynomial solutions for wave equation

| $X_i$ | $f_{\text{old}} = \ln \frac{x-t}{x+t}$ |
|---|---|
| $X_1$ | $f_{\text{new}} = -2\,\frac{t}{t^2-x^2}, -4\,\frac{xt}{(t^2-x^2)^2}, -4\,\frac{t\left(t^2+3\,x^2\right)}{(t^2-x^2)^3}$ ,etc |
| $X_2$ | $f_{\text{new}} = 0$ |
| $X_3$ | $f_{\text{new}} = 2\,\frac{x}{t^2-x^2}, -4\,\frac{xt}{(t^2-x^2)^2}, 4\,\frac{x\left(3\,t^2+x^2\right)}{(t^2-x^2)^3}$ ,etc |
| $X_4$ | $f_{\text{new}} = 2\,\frac{ty}{t^2-x^2}, \frac{-2xt(t^2-x^2-2y^2)}{(t^2-x^2)^2}, -4\,\frac{yt(2t^2x^2-t^2y^2-2x^4-x^2y^2)}{(t^2-x^2)^2}$, etc |
| $X_5$ | $f_{\text{new}} = -2, 0$ |
| $X_6$ | $f_{\text{new}} = 0$ |
| $X_7$ | $f_{\text{new}} = -2\frac{t(t^2-x^2-y^2)}{(t^2-x^2)}$ ,etc |

TABLE 8. Classification of exact non-polynomial solutions for wave equation

| $X_i$ | $f_{\text{old}} = \arctan \sqrt{\frac{x^2+y^2-t^2}{t^2}}$ |
|---|---|
| $X_1$ | $f_{\text{new}} = \frac{xt^2}{\sqrt{-t^2+x^2+y^2}(t^4-t^2+x^2+y^2)}$ ,etc |
| $X_2$ | $f_{\text{new}} = \frac{t^2y}{\sqrt{-t^2+x^2+y^2}(t^4-t^2+x^2+y^2)}$ ,etc |
| $X_3$ | $f_{\text{new}} = \frac{\left(t^2-2\,x^2-2\,y^2\right)t}{\sqrt{-t^2+x^2+y^2}(t^4-t^2+x^2+y^2)}$ ,etc |
| $X_4$ | $f_{\text{new}} = 0$ |
| $X_5$ | $f_{\text{new}} = -2\,\frac{xt\sqrt{-t^2+x^2+y^2}}{t^4-t^2+x^2+y^2}$ |
| $X_6$ | $f_{\text{new}} = -\frac{t^2\sqrt{-t^2+x^2+y^2}}{t^4-t^2+x^2+y^2}$, etc |
| $X_7$ | $f_{\text{new}} = -3\frac{xt^2\sqrt{-t^2+x^2+y^2}}{t^4-t^2+x^2+y^2}$, etc |

Table 9. Classification of exact non-polynomial solutions for wave equation

| $X_i$ | $f_{\text{old}} = \operatorname{arctanh}\sqrt{\frac{y^2+t^2-x^2}{y^2}}$ |
|---|---|
| $X_1$ | $f_{\text{new}} = \frac{xy^2}{\sqrt{t^2-x^2+y^2}\,(-y^4+t^2-x^2+y^2)}$,etc |
| $X_2$ | $f_{\text{new}} = \frac{(2t^2-2x^2+y^2)y}{\sqrt{t^2-x^2+y^2}\,(-y^4+t^2-x^2+y^2)}$,etc |
| $X_3$ | $f_{\text{new}} = -\frac{ty^2}{\sqrt{t^2-x^2+y^2}\,(-y^4+t^2-x^2+y^2)}$ ,etc |
| $X_4$ | $f_{\text{new}} = 2\,\frac{xy(t^2-x^2)}{\sqrt{t^2-x^2+y^2}\,(-y^4+t^2-x^2+y^2)}$, etc |
| $X_5$ | $f_{\text{new}} = 0$ |
| $X_6$ | $f_{\text{new}} = \frac{\sqrt{t^2-x^2+y^2}}{-y^4+t^2-x^2+y^2}$, etc |
| $X_7$ | $f_{\text{new}} = \frac{xy^2(3t^2-3x^2+y^2)}{\sqrt{t^2-x^2+y^2}\,(-y^4+t^2-x^2+y^2)}$, etc |

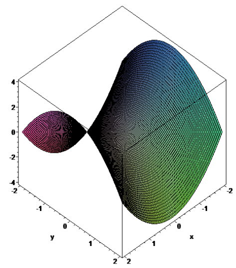
Table 10. Classification of exact non-polynomial solutions for wave equation

| $X_i$ | $f_{\text{old}} = \operatorname{arctanh}\sqrt{\frac{y^2+x^2-t^2}{x^2}}$ |
|---|---|
| $X_1$ | $f_{\text{new}} = \frac{y^2x}{\sqrt{t^2-x^2+y^2}\,(-y^4+t^2-x^2+y^2)}$,etc |
| $X_2$ | $f_{\text{new}} = \frac{yx^2}{\sqrt{-t^2+x^2+y^2}\,(x^4+t^2-x^2-y^2)}$,etc |
| $X_3$ | $f_{\text{new}} = \frac{-tx^2}{\sqrt{-t^2+x^2+y^2}\,(x^4+t^2-x^2-y^2)}$ ,etc |
| $X_4$ | $f_{\text{new}} = \frac{2yx\sqrt{-t^2+x^2+y^2}}{(x^4+t^2-x^2-y^2)}$ ,etc |
| $X_5$ | $f_{\text{new}} = \frac{-2tx\sqrt{-t^2+x^2+y^2}}{(x^4+t^2-x^2-y^2)}$ ,etc |
| $X_6$ | $f_{\text{new}} = \frac{-x^2\sqrt{-t^2+x^2+y^2}}{(x^4+t^2-x^2-y^2)}$ ,etc |
| $X_7$ | $f_{\text{new}} = \frac{xy^2(3t^2-3x^2+y^2)}{\sqrt{t^2-x^2+y^2}\,(-y^4+t^2-x^2+y^2)}$, etc |

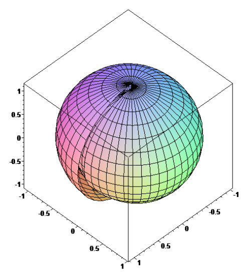
## 5. Conclusion

In this paper, by using the Lie symmetry groups, we studied the symmetry properties and similarity reduction forms of the (2+1)-dimensional linear wave equation (1.2). Moreover, we also derived the polynomial and non-polynomial solutions of Eq. (1.2), by virtue of this fact, that symmetries and their Lie brackets map solutions to solutions. The method is applicable for any other differential equations which admitts a symmetries such as $X_{12}$.

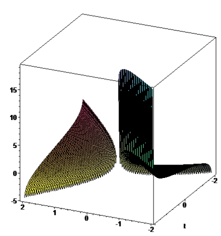
FIGURE 1. Graphs of solutions



(A) $u = x^2 - y^2$



(B) $u = \arctan\left(\sqrt{\frac{x^2+y^2-t^2}{t^2}}\right)$



(C) $u = \ln\left(\frac{y-t}{y+t}\right)$

## REFERENCES

[1] GOERGE W. BLUMAN, ALEXEI F. CHEVIAKOV AND STEPHEN C. ANCO, *Construction of conservation laws: How the direct method Generalizes Noether's theorem*, Proceeding of 4th Workshop "Group analysis of differential equations & integribility", (2009), 1-23.

[2] GOERGE W. BLUMAN, ALEXEI F. CHEVIAKOV AND STEPHEN C. ANCO, *Application of symmetry methods to partial differential equations*, Springer-Verlage, New York, 2000.

[3] Goerge W. Bluman and Julian D. Cole, *The general similarity solution of the heat equation*, J. Math. Mech. *18* (1969) 1025-1042.

[4] Wilhelm I. Fushchych and Roman O. Popovych, *Symmetry reduction and exact solutions of the NavierStokes equations*, J. Non-linear Math. Phys., *1* (1994), 75-113, 156-188.

[5] S. Reza Hejazi, *Lie group analaysis, Hamiltonian equations and conservation laws of Born-Infeld equation*, Asian-European Journal of Mathematics, *7(3)* (2014), 1450040(19 pages).

[6] Peter E. Hydon, *Symmetry method for differential equations*, Cambridge University Press, Cambridge, UK, 2000.

[7] Nail H. Ibragimov, *Transformation group applied to mathematical physics*, Riedel, Dordrecht 1985.

[8] Nail H. Ibragimov, Alexandre V. Aksenov, Valentin A. Baikov, Vladimir A. Chugunov, Roman K. Gazizov and Alexadre G. Meshkov, CRC handbook of Lie group analysis of differential equations, Applications in engineering and physical sciences, *2*, Boca Raton: CRC Press; 1995.

[9] Nail H. Ibragimov, *Non-linear self-adjointness in constructing conservation laws*, Arch ALGA 20102011, 7/8:199 (2011) 1104.

[10] Nail H. Ibragimov and Robert L. Anderson, *Lie theory of differential equations*, In: Ibragimov NH, editor. Lie group analysis of differential equations, *1*, Symmetries, exact solutions and conservation laws. Boca Raton: CRC Press; 1994. 714.

[11] Mehdi Nadjafikhah and S. Reza Hejazi, *Symmetry analysis of cylindrical Laplace equation*, Balkan journal of geometry and applications, *14(2)*, (2009) 63-74.

[12] Peter J. Olver, *Equivalence, Invariant and Symmetry*, Cambridge University Press, Cambridge University Press, Cambridge 1995.

[13] Peter J. Olver, *Applications of Lie groups to differential equations*, Second Edition, GTM, *107*, Springer-Verlage, New York, 1993.

[14] Lyev V. Ovsiannikov, *Group analysis of differential equations*, Academic Press, New York, 1982.