



On asymptotic stability of Prabhakar fractional differential systems

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Abstract In this article, we survey the asymptotic stability analysis of fractional differential systems with the Prabhakar fractional derivatives. We present the stability regions for these types of fractional differential systems. A brief comparison with the stability aspects of fractional differential systems in the sense of Riemann-Liouville fractional derivatives is also given.

Keywords. Asymptotically stable, Prabhakar fractional derivative, Generalized Mittag-Leffler function, Riemann-Liouville fractional derivative.

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1. INTRODUCTION

With the extension of theory of fractional calculus (the integral and derivative of arbitrary order), the stability analysis of differential system

$$x'(t) = A x(t), \quad x(0) = x_0, \quad A \in \mathbb{R}^{n \times n}, \quad (1.1)$$

was developed in the last decades for the following fractional differential system

$$\mathcal{D}_t^\alpha x(t) = A x(t), \quad x(0) = x_0, \quad 0 < \alpha \leq 1, \quad (1.2)$$

where \mathcal{D}_t^α is a fractional differential operator. For the first time in 1996, the stability of the above system with the Caputo fractional derivatives was surveyed by Matignon [11]. Later other researchers extended some similar results for the stability

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of fractional differential systems containing the Riemann-Liouville fractional derivatives [15, 16, 19], the Hilfer fractional derivatives [17], distributed order fractional derivatives [1, 2, 18] and fractional differential equations with time delays [3, 4].

In this paper, we intend to survey the stability analysis of linear differential systems containing the Prabhakar fractional derivatives. This type of fractional derivative was introduced by Garra et al. [7] in terms of the generalized Mittag-Leffler function and can be considered as a generalization of the most popular definitions of fractional derivatives. See papers [5, 6],[9, 13].

For this purpose, in Section 2 we recall some definitions and lemmas in generalized fractional calculus. In Section 3, we introduce the linear differential system containing Prabhakar fractional derivative and discuss about the asymptotic stability analysis of these types of fractional differential systems. In Section 4, we compare the stability aspects of Prabhakar fractional differential systems with the Riemann-Liouville fractional differential systems.

2. PRELIMINARIES

In this section, we recall some definitions and lemmas which are used in the next sections.

Definition 2.1. [10, 12]. For $0 < \alpha < 1$ and $f \in L^1[0, b]$, $0 < t < b \leq \infty$, the Riemann-Liouville fractional integral and derivative of the order α are defined as

$${}_0+I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau)(t - \tau)^{\alpha-1} d\tau, \quad t > 0, \quad 0 < \alpha < 1, \quad (2.1)$$

$${}_0+D_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau)(t - \tau)^{-\alpha} d\tau, \quad t > 0, \quad 0 < \alpha < 1. \quad (2.2)$$

Also, for the absolutely continuous function f , the Caputo fractional derivatives of order α is defined as follows

$${}_0^C D_t^\alpha f(t) = {}_0+I_t^{1-\alpha} \frac{d}{dt} f(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - \tau)^{-\alpha} \frac{d}{d\tau} f(\tau) d\tau. \quad (2.3)$$

Definition 2.2. For $m - 1 < \Re(\mu) < m$ and function $f \in L^1[0, b]$, $0 < t < b \leq \infty$, the Prabhakar fractional integral is defined as follows [7]

$$(\mathbf{E}_{\rho, \mu, \omega, 0+}^\gamma f)(t) = \int_0^t (t - \tau)^{\mu-1} E_{\rho, \mu}^\gamma(\omega(t - \tau)^\rho) f(\tau) d\tau, \quad (2.4)$$

where $E_{\rho, \mu}^\gamma$ is the generalized Mittag-Leffler function introduced by Prabhakar in 1971 [14]

$$E_{\rho, \mu}^\gamma(z) = \frac{1}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma + n)}{n! \Gamma(\rho n + \mu)} z^n, \quad \Re(\rho), \Re(\mu) > 0. \quad (2.5)$$

Definition 2.3. [7]. For the function $f \in L^1[0, b]$, the Prabhakar fractional derivative is defined as

$$(D_{\rho, \mu, \omega, 0+}^\gamma f)(t) = \frac{d^n}{dt^n} \mathbf{E}_{\rho, n-\mu, \omega, 0+}^{-\gamma} f(t), \quad t > 0, \quad (2.6)$$



where $m - 1 < \Re(\mu) < m$ and $\rho, \mu, \omega, \gamma \in \mathbb{C}$.

Lemma 2.4. [7]. *The Laplace transform of Prabhakar fractional derivative for $m - 1 < \Re(\mu) < m$ is given by*

$$\mathcal{L}(D_{\rho, \mu, \omega, 0^+}^\gamma f(t)) = s^\mu (1 - \omega s^{-\rho})^\gamma F(s) - \sum_{k=0}^{n-1} s^k (D_{\rho, \mu-k, \omega, 0^+}^\gamma f)(0), \quad (2.7)$$

where $F(s)$ is the Laplace transform of $f(t)$

$$F(s) = \int_0^\infty e^{-st} f(t) dt. \quad (2.8)$$

Lemma 2.5. *The Laplace transform of generalized Mittag-Leffler function $t^{\mu-1} E_{\rho, \mu}^\gamma(\omega t^\rho)$ is given by [7]*

$$\mathcal{L}\{t^{\mu-1} E_{\rho, \mu}^\gamma(\omega t^\rho); s\} = \frac{s^{\gamma\rho-\mu}}{(s^\rho - \omega)^\gamma}, \quad \left| \frac{\omega}{s^\rho} \right| < 1. \quad (2.9)$$

Also, for $\rho, \mu, \gamma, a > 0$ the following asymptotic behavior holds [8]

$$\Gamma(\rho) E_{\rho, \mu}^\gamma(a(\rho t)^\gamma) \cong \frac{1}{(1 + a(\frac{t}{\rho})^\gamma)^\mu}, \quad t \rightarrow \infty. \quad (2.10)$$

3. ASYMPTOTIC STABILITY ANALYSIS OF LINEAR AUTONOMOUS PRABHAKAR FRACTIONAL DIFFERENTIAL SYSTEMS

3.1. Main theorem. In this section, we introduce the linear autonomous Prabhakar fractional differential systems and discuss about the asymptotic stability of systems. We consider the following fractional system

$$\begin{aligned} D_{\rho, \mu, \omega, 0^+}^\gamma x(t) &= A x(t), \quad t > 0, \quad 0 < \gamma < 1, \quad 0 < \rho < 1, \quad 0 < \mu < 1, \\ D_{\rho, \mu, \omega, 0^+}^\gamma x(0^+) &= x_0, \end{aligned} \quad (3.1)$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$ is matrix, $x_0 = (x_{10}, x_{20}, \dots, x_{n0})$ and $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$ such that $0 < \gamma < 1$.

Remark 3.1. The system (3.1) is called commensurate order system if $\gamma_1 = \gamma_2 = \dots = \gamma_n$.

Definition 3.2. The Prabhakar fractional derivatives system (3.1)

- i): is stable if for any initial value x_0 and $t > 0$, there exists an $\epsilon > 0$ such that $\|x(t)\| < \epsilon$.
- ii): is asymptotically stable if at first it is stable and $\lim_{t \rightarrow \infty} \|x(t)\| = 0$.

Theorem 3.3. *The solution of the linear commensurate order system (3.1) is given by*

$$x(t) = \sum_{n=0}^\infty t^{\mu+\mu n-1} E_{\rho, \mu+\mu n}^{1+\gamma n+\gamma}(\omega t^\rho) A^n x_0. \quad (3.2)$$



Proof. By applying the Laplace transform on the equation (3.1) and using the relation (2.7), we have

$$s^\mu(1 - \omega s^{-\rho})^\gamma X(s) - x_0 = AX(s), \tag{3.3}$$

which for $\|\frac{A}{s^\mu(1-\omega s^{-\rho})^\gamma}\| < 1$ leads to

$$\begin{aligned} X(s) &= \frac{x_0}{s^\mu(1 - \omega s^{-\rho})^\gamma - A} \\ &= \frac{x_0}{s^\mu(1 - \omega s^{-\rho})^\gamma} (I - s^{-\mu}(1 - \omega s^{-\rho})^{-\gamma} A)^{-1}. \end{aligned} \tag{3.4}$$

Using the inverse Laplace transform and considering relation (2.9), we get the result

$$x(t) = \sum_{n=0}^{\infty} t^{\mu+\mu n-1} E_{\rho, \mu+\mu n}^{1+\gamma n+\gamma}(\omega t^\rho) A^n x_0, \tag{3.5}$$

and proof is completed. □

Theorem 3.4. *If all the eigenvalues of A (set $\lambda(A)$) satisfy the following condition for any $r > 0$*

$$|\arg(\lambda(A))| > \frac{\mu\pi}{2} - \gamma(\tan^{-1} \frac{\omega r^{-\rho} \sin(\frac{\rho\pi}{2})}{1 - \omega r^{-\rho} \cos(\frac{\rho\pi}{2})}), \tag{3.6}$$

then the solution of system (3.1) is asymptotically stable.

Proof. We begin with inequality $\|\frac{A}{s^\mu(1-\omega s^{-\rho})^\gamma}\| < 1$ as a necessary condition of the solution (3.2). If we set $s = r e^{i\theta}$ as an arbitrary point in the complex plane, then for holding this inequality, the argument of absolute value $|\frac{1}{r^\mu e^{i\mu\theta} (1-\omega r^{-\rho} \cos(\frac{\rho\pi}{2}))^\gamma}|$ should be less than $\frac{\pi}{2}$, which implies that

$$\theta = |\arg(\lambda(A))| > \frac{\mu\pi}{2} - \gamma(\tan^{-1} \frac{\omega r^{-\rho} \sin(\frac{\rho\pi}{2})}{1 - \omega r^{-\rho} \cos(\frac{\rho\pi}{2})}). \tag{3.7}$$

Now, we consider the solution (3.2) and analyze the asymptotic stability of system in the two cases.

Case 1: Suppose that the matrix A is diagonalizable and J is the Jordan canonical form of the matrix A such that $J = P^{-1}AP = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ where P is an invertible matrix. In this case, we have

$$A^n = (PJP^{-1})^n = PJ^nP^{-1} = P(\text{diag}(\lambda_1^n, \lambda_2^n, \dots, \lambda_n^n))P^{-1}. \tag{3.8}$$

At this point, if we apply the asymptotic relation (2.10) for solution (3.2) and consider $\omega_n = \frac{t^{\mu+\mu n-1}}{(1 + \frac{\omega t^\rho \sin(\frac{\rho\pi}{2})}{\rho^2(\gamma n + \gamma)})^{\mu+\mu n}}$, then we have

$$x(t) = \sum_{n=0}^{\infty} \omega_n \lambda_i^n x_0 \rightarrow 0, \quad t \rightarrow \infty, \quad 1 \leq i \leq n, \tag{3.9}$$



hence

$$\lim_{t \rightarrow \infty} \left\| \sum_{n=0}^{\infty} \omega_n \text{diag}(\lambda_1^n, \lambda_2^n, \dots, \lambda_n^n) x_0 \right\| = 0, \tag{3.10}$$

and consequently

$$\lim_{t \rightarrow \infty} \|x(t)\| = \lim_{t \rightarrow \infty} \left\| \sum_{n=0}^{\infty} \omega_n A^n x_0 \right\| = \lim_{t \rightarrow \infty} \left\| P \left(\sum_{n=0}^{\infty} \omega_n J^n x_0 \right) P^{-1} \right\| = 0. \tag{3.11}$$

Case 2: Suppose that the matrix A has a Jordan canonical form $J = (J_1, J_2, \dots, J_s)$, where $J_i, 1 \leq i \leq s$, is shown by

$$J_i = \begin{pmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{pmatrix}_{n_i \times n_i}, \lambda_i \in \mathbb{C}, \tag{3.12}$$

and $\sum_{i=1}^s n_i = n$. In this case, we have

$$A^n = (PJP^{-1})^n = PJ^nP^{-1} = P(\text{diag}(J_1^n, J_2^n, \dots, J_h^n))P^{-1}, \tag{3.13}$$

and for the solution (3.2), we obtain

$$\sum_{n=0}^{\infty} \omega_n J_i^n x_0 = \sum_{n=0}^{\infty} \omega_n \begin{pmatrix} \lambda_i^n & D_n^1 \lambda_i^{n-1} & \dots & D_n^{n_i-1} \lambda_i^{n-n_i+1} \\ 0 & \lambda_i^n & \ddots & \vdots \\ \vdots & \ddots & \ddots & D_n^1 \lambda_i^{n-1} \\ 0 & \dots & 0 & \lambda_i^n \end{pmatrix} x_0, \tag{3.14}$$

the matrix (3.14), can be written as

$$\sum_{n=0}^{\infty} \omega_n J_i^n x_0 = \sum_{n=0}^{\infty} \omega_n (a_{st})_{n_i \times n_i} x_0, \tag{3.15}$$

where $(a_{st})_{n_i \times n_i}$, has the following form

$$a_{st} = \sum_{m=0}^{n_i-s} D_n^m \lambda_i^{n-m}, s = 1, 2, \dots, n_i, t = s, s + 1, \dots, n_i, \tag{3.16}$$

Substituting (3.16) into (3.15), we obtain

$$\sum_{n=0}^{\infty} \omega_n J_i^n x_0 = \sum_{n=0}^{\infty} \sum_{m=0}^{n_i-s} \omega_n \left(\frac{1}{m!} \left(\frac{\partial}{\partial \lambda_i} \right)^m \lambda_i^n \right)_{n_i \times n_i} x_0. \tag{3.17}$$

where $D_k^j, 1 \leq j \leq n_i$, are the binomial coefficients such that

$$D_k^j = \binom{k}{j} = \begin{cases} \frac{k!}{j!(k-j)!} & 0 \leq j \leq k, \\ 0 & \text{etc.} \end{cases}$$



If we consider the non-zero element of the above matrix

$$\frac{1}{(j-1)!} \left\{ \left(\frac{\partial}{\partial \lambda} \right)^{j-1} \sum_{n=0}^{\infty} \omega_n \lambda_i^n \right\}, \quad 1 \leq j \leq n_i, \quad 1 \leq i \leq s,$$

and use the asymptotic behavior (2.10) once again, then we have

$$\frac{1}{(j-1)!} \left(\frac{\partial}{\partial \lambda_i} \right)^{j-1} \sum_{n=0}^{\infty} \omega_n \lambda_i^n = \sum_{n=0}^{\infty} \frac{(n-j+2) \cdots (n-1)n}{(j-1)!} \omega_n \lambda_i^{n-j+1}, \quad (3.18)$$

which implies that

$$\left| \frac{1}{(j-1)!} \left(\frac{\partial}{\partial \lambda} \right)^{j-1} \sum_{n=0}^{\infty} \omega_n \lambda_i^n \right| \rightarrow 0, \quad 1 \leq j \leq n_i, \quad t \rightarrow \infty, \quad (3.19)$$

and consequently

$$\lim_{t \rightarrow \infty} \|x(t)\| = \lim_{t \rightarrow \infty} \left\| \sum_{n=0}^{\infty} \omega_n A^n x_0 \right\| = \lim_{t \rightarrow \infty} \left\| P \left(\sum_{n=0}^{\infty} \omega_n J^n x_0 \right) P^{-1} \right\| = 0. \quad (3.20)$$

□

3.2. Asymptotic stability region. The asymptotic stability regions of system (3.1) for different values of parameters ρ, μ, γ can be plotted by the relation (4.1). We plot these regions in blue for parameters $\omega = 1, \mu = 0.5, 0.75, 0.95, 1$ and $\rho = \gamma = 0.5$. See Figure 1.

4. COMPARISON WITH RIEMANN-LIOUVILLE FRACTIONAL DIFFERENTIAL SYSTEMS

It is obvious that for $\gamma = 0$, the Prabhakar fractional integral (2.4) coincides with the Riemann-Liouville fractional integral of order μ (2.1). In this case, the stability region of fractional systems with the Riemann-Liouville fractional derivative can be plotted by the following condition

$$|\arg(\lambda(A))| > \frac{\mu\pi}{2}. \quad (4.1)$$

This region has been shown for $0 < \mu < 1$ in Figure 2. The difference between two regions of the Prabhakar and Riemann-Liouville fractional derivatives is considered as term $\gamma \left(\tan^{-1} \frac{\omega r^{-\rho} \sin(\frac{\rho\pi}{2})}{1 - \omega r^{-\rho} \cos(\frac{\rho\pi}{2})} \right)$. The graph of this difference has been plotted in Figure 3.

Remark 4.1. The shaded region in Figure 3 shows that the fractional differential systems with the Riemann-Liouville and Prabhakar derivatives have not the same stability status for order $0 < \mu < 1$. It means that, for a determined parameter μ , the fractional differential system with the Riemann-Liouville derivative is asymptotically stable, but the associated fractional differential system with the Prabhakar derivative is unstable.



FIGURE 1. The asymptotic stability regions of system (3.1) for parameters $\mu = 0.5, 0.75, 0.95, 1$ and $\rho = \gamma = 0.5$.

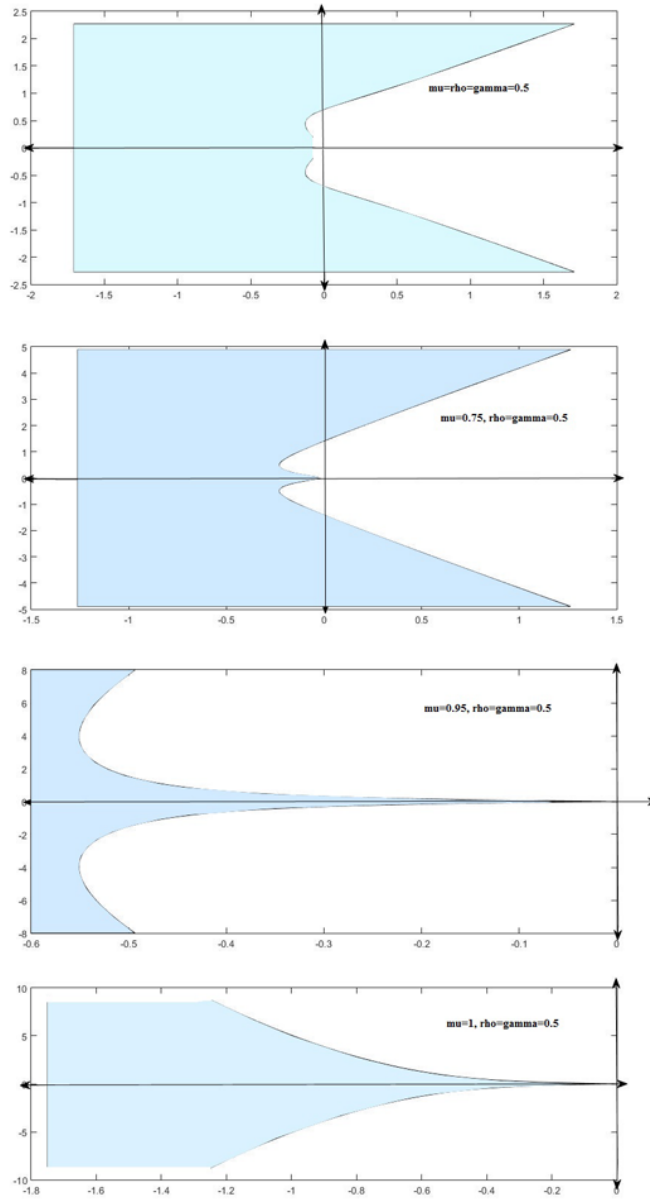


FIGURE 2. The Stability region of the fractional differential system with Riemann-Liouville derivative of order $0 < \mu < 1$.

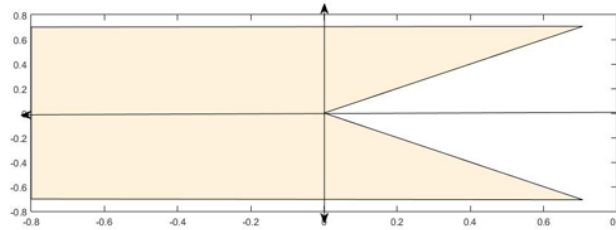
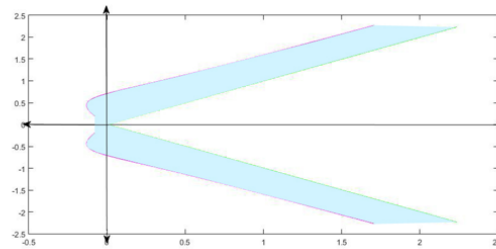


FIGURE 3. The difference region between the fractional differential systems with the Riemann-Liouville and Prabhakar derivatives of order $0 < \mu < 1$.



REFERENCES

- [1] H. Aminikhah, A. R. Sheikhan and H. Rezazadeh, *Stability analysis of linear distributed order fractional system with multiple time delays*, U.P.B.Sci. Bull. Ser. A., 77 (2015), 207-218.
- [2] H. Aminikhah, A. R. Sheikhan and H. Rezazadeh, *Stability analysis of distributed order fractional Chen system*, The Sci. World J., 2013. DOI 10.1155/2013/645080.
- [3] Y. Chen and K. L. Moore, *Analytical stability bound for a class of delayed fractional-order dynamic systems*, Nonlinear Dyn., 29 (2002), 191-200.
- [4] W. Deng, C. Li and J. Lu, *Stability analysis of linear fractional differential system with multiple time delays*, Nonlinear Dynam., 48 (2007), 409-416.
- [5] S. Eshaghi and A. Ansari, *Lyapunov inequality for fractional differential equations with Prabhakar derivative*, Mathematical Inequalities and Applications, 19 (2016), 349-358.
- [6] S. Eshaghi and A. Ansari, *Finite fractional Sturm-Liouville transforms for generalized fractional derivatives*, Iranian Journal of Science and Technology, Accepted, 2016.
- [7] R. Garra, R. Gorenflo, F. Polito and Z. Tomovski, *Hilfer-Prabhakar derivatives and some applications*, Applied Mathematics and Computation, 242 (2014), 576-589.
- [8] R. Gorenflo, F. Mainardi and S. Rogosin, *Mittag-Leffler Functions: Related Topics*, Springer Monographs in Mathematics, New York, 2010.
- [9] G. Farid, J. Pecaric and Z. Tomovski, *Opial-type inequalities for fractional integral operator involving Mittag-Leffler function*, 5 (2015), 93-106.
- [10] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.



- [11] D. Matignon, *Stability results for fractional differential equations with applications to control processing*, Comput. Engrg. Syst. Appl. *2* (1996), 963-968.
- [12] I. Podlubny, *Fractional differential equations*, Academic Press, New York, 1999.
- [13] F. Polito and Z. Tomovski, *Some properties of Prabhakar-type fractional calculus operators*, arXiv:1508.03224.
- [14] T. R. Prabhakar, *A singular integral equation with a generalized Mittag-Leffler function in the kernel*, Yokohama Journal of Mathematics, *19* (1971), 7–15.
- [15] D. Qian, C. Li, R. P. Agarwal and P. J. Wong, *Stability analysis of fractional differential system with Riemann-Liouville derivative*, Math. Comput. Model. *52* (2010), 862-874.
- [16] Z. Qin, R. Wu and Y. Lu, *Stability analysis of fractional-order systems with the Riemann-Liouville derivative*, Systems Sci. Control Eng., *2* (2014), 727-731.
- [17] H. Rezaadeh, H. Aminikhah and A. R. Sheikhan, *Stability analysis of Hilfer fractional differential systems*, Math. Commun., *21* (2016), 1–20.
- [18] H. Saberi Najafi, A. Refahi Sheikhan and A. Ansari, *Stability analysis of distributed order fractional differential equations*, Abstr. Appl. Anal., 2011. DOI 10.1155/2011/175323.
- [19] Q. S. Zeng, G. Y. Cao and X. J. Zhu, *The asymptotic stability on sequential fractional-Order systems*, J. Shanghai JiaoTong Univ., *39* (2005), 346–348.

