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Positive solutions for discrete fractional intiail value problem

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Abstract In this paper, the existence and uniqueness of positive solutions for a class of nonlinear initial value problem for a finite fractional difference equation obtained by constructing the upper and lower control functions of nonlinear term without any monotone requirement .The solutions of fractional difference equation are the size of tumor in model tumor growth described by the Gompertz function. We use the method of upper and lower solutions and Schauder fixed point theorem to obtain the main results.

Keywords. discrete fractional calculus; existence of solutions; Positive solution; Fixed point theorem.

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1. Introduction

Fractional calculus, the continuous case, has a long history and there is a renewed interest in both the study of fractional calculus and fractional differential equations. In fact, rapid progress is being made in the study of problems for fractional differential equations. We refer the reader to [14]-[16] for accounts of the historical development; we refer the reader to [6] ,[7],[11],[17] for samples of recent studies in boundary value problems for fractional differential equations.

The kernel of the Riemann-Liouville fractional integral $\dfrac{(t-s)^{\nu-1}}{\Gamma(\nu)}$ is a clear analogue of the Cauchy function for ordinary differential equations. Hence, the authors are heavily influenced by the approach taken by Miller and Ross [12] who study the linear ν-th order fractional differential equation as an analogue of the linear n-th order ordinary differential equation.

To the authors knowledge, very little progress has been made to develop the theory of the fractional finite difference equation. Miller and Ross [13] produced an early paper; the authors [1] have developed and applied a transform method. The authors [2] also developed and applied a transform method for fractional q-calculus problems. An remarkable bibliography for the fractional q-calculus is provided in [2].

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We start with basic definitions and results. Let $\nu > 0$ and $\sigma(s) = s + 1$. The ν-th fractional sum of f is defined by

$$\Delta^{-\nu} f(t; a) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t - \sigma(s))^{(\nu-1)} f(s). \tag{1.1}$$

Note that f is defined for $s = a \bmod (1)$ and $\Delta^{-\nu} f$ is defined for $t = a + \nu \bmod (1)$; in particular, $\Delta^{-\nu} f$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+\nu}$, where $\mathbb{N}_t = \{t, t + 1, t + 2, ...\}$. We point out that we employ throughout the notation, $\sigma(s)$, because eventually progress will be made to develop the theory of the fractional calculus on time scales [9]. We remind the reader that $t^{(\nu)} = \dfrac{\Gamma(t + 1)}{\Gamma(t + \nu + 1)}$, we shall suppress the dependence on a in $\Delta^{-\nu} f(t; a)$ since domains will be clear by the context, and finally we point out that Miller and Ross [13] have argued that $\lim_{\nu\to 0^+} \Delta_a^{-\nu} f(t) = f(t)$.

The μ-th fractional difference is defined as

$$\Delta^{\mu} u(t) = \Delta^{m-\nu} u(t) = \Delta^{m}(\Delta^{-\nu})u(t) \tag{1.2}$$

where $\mu > 0$ and $m - 1 < \mu < m$, m denotes a positive integer and $-\nu = \mu - m$.

Tumor growth, a special relationship between tumor size and time, is of special interest since growth estimation is very critical in clinical practice. Three different terms are typically used to model growth behavior in biology: exponential, logistic and sigmoidal. Tumor growth, however, is best described by sigmoidal functions. In 1825, Benjamin Gompertz introduced the Gompertz function, a sigmoid function, which is found to be applicable to various growth phenomena, in particular tumor growth (see [10])The Gompertz difference equation describes the growth models and these models can be studied on the basis of the parameters a (growth rate) and b (exponential rate of growth deceleration) in the recursive formulation of the Gompertz law of growth [8].

The Gompertz difference equation in [8] is given by

$$\ln G(t + 1) = a + b \ln G(t).$$

Next we introduce Gompertz fractional difference equation

$$\Delta^{\nu} \ln G(t - \nu + 1) = (b - 1) \ln G(t) + a.$$

For simplicity if we replace $\ln \ln G(t) = y(t)$, we obtain

$$\Delta_0^{\nu} y(t - \nu + 1) = (b - 1)y(t) + a. \tag{1.3}$$

Next we are concerned with the following optimization problem

$$J[y] = \min \sum_{t=0}^{T-1} U(y(t-\nu+1)),$$

with a constraint

$$\Delta_0^{\nu} y(t-\nu+1) = (b-1)y(t) + a, \quad y(0) = c,$$

or

$$\Delta_{\nu-1}^{\nu} y(t) = (b-1)y(t-\nu+1) + a,$$

where y(t) is the size of tumor and U is a function with continuous partial derivatives.
We have

$$J[y] = \min \sum_{t=0}^{T-1} \Big\{ U(y(t-\nu+1)) + \lambda(t-\nu+1)(\Delta_{\nu-1}^{\nu} y(t) - (b-1)y(t-\nu+1) - a) \Big\}.$$

It follows from (Theorem 4.1 in [5]), Euler-Lagrange equations with respect to λ and y are

$$U_y\Big(y(t-\nu+1)) + \lambda(t-\nu+1)(b-1) + {}_{T+\nu-1}\Delta^{\nu}\nu_t \lambda^{\rho}(t+2(\nu-1)\Big) = 0, \tag{1.4}$$

or

$$\mathrm{U}_y\Big(y(t-\nu+1)) + \lambda(t-\nu+1)(b-1)$$
$$+ \frac{1}{\Gamma(1-\nu)}(-\Delta) \textstyle\sum_{s=t+\nu-1}^{T+\nu-1} (s - \sigma(t+(2\nu-1)))^{(-\nu)} \lambda(\rho(s)\Big),$$

and

$$\Delta_{\nu-1}^{\alpha} y(t) - (b-1)y(t-\nu+1) - a = 0, \tag{1.5}$$

where $t \in [1, T-1]$.

At this point, to the best of the authors knowledge, there is no known or published numerical methods of solving the above systems of Eqs. (1.4)-(1.5). Therefore we shall focus on proving the existence and uniqueness result for the Gompertz fractional difference equation (1.3) with an initial condition $y(0) = c$.

Consider the following fractional difference equation with an initial condition

$$\Delta_0^{\nu} y(t-\nu+1) = f(t, y(t)), \quad t = 0, 1, 2, ..., \qquad y(0) = c, \tag{1.6}$$

where $\alpha \in (0, 1]$, f is a real-valued function, and c is a real number. With $t + \alpha - 1$ shift at the same time, we obtain

$$\Delta_0^{\nu} y(t) = f(t+\nu-1, y(t+\nu-1)). \tag{1.7}$$

In this article, we shall consider the following nonlinear fractional difference equation with an initial condition (fractional initial value problem, FIVP)

$$\begin{cases} \Delta^{\nu} y(t) = f(t+\nu-1, y(t+\nu-1)), \quad t = 0,1,2,...,b, \\ y(\nu-1) = 0, \end{cases} \tag{1.8}$$

where $\nu \in (0,1]$, $f : [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}} \times \mathbb{R} \to \mathbb{R}$ is continuous, $b \geq 2$ is an integer and $[\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}} = \{0,1,...,b+2\}$.

In Section 2, together with the definitions of discrete fractional sum operator $\Delta^{-\nu}$ and discrete fractional difference operator Δ^{ν}, we shall list some basic theorems in discrete fractional calculus. For these general results, we shall remove the constraint $0 < \nu \leq 1$. In Section 3, we obtain sufficient conditions for the existence of solutions of the nonlinear FIVP (1.8). In Section 4, we obtain sufficient conditions for the uniqueness of positive solutions of the nonlinear FIVP (1.8)

2. Discrete fractional difference operators

Througout this section, $\nu > 0$. The first two results (the commutative property of the fractional sum operator and the power rule) and their proofs can be found in [1].

Theorem 2.1. *Let f be a real-valued function defined on $\mathbb{N}_a$ and let $\mu; \nu > 0$. Then the following equalities hold*

$$\Delta^{-\nu}[\Delta^{-\mu} f(t)] = \Delta^{-(\mu+\nu)} = \Delta^{-\mu}[\Delta^{-\nu} f(t)].$$

Lemma 2.2. *Let $\mu \neq 1$ and assume $\mu+\nu+1$ is not a non-positive integer. Then,*

$$\Delta^{-\nu} t^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} t^{(\mu+\nu)}.$$

The next result, proved earlier by the authors in [3], addresses the commutativity of fractional sums and fractional differences of arbitrary order.

Theorem 2.3. *For any real number ν and any positive integer p, the following equality holds*

$$\Delta^{-\nu}\Delta^{p} g(t) = \Delta^{p}\Delta^{-\nu} g(t) - \sum_{k=0}^{p-1} \frac{(t-a)^{(\nu-p+k)}}{\Gamma(\nu+k-p+1)} \Delta^{k} g(a),$$

where g is defined on $\mathbb{N}_a$.

Assume $0 \leq N-1 < \nu \leq N$. Using power rule Lemma 2.2, implies that there are N linearly independent solutions to $\Delta^{\nu} y(t) = 0$. To see this, note that

$$\begin{aligned} \Delta^{\nu} t^{(\mu)} &= \Delta^{N-(N-\nu)} t^{(\mu)} = \Delta^{N} \Delta^{-(N-\nu)} t^{(\mu)} \\ &= \Delta^{N} \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+N-\nu)} t^{(\mu+N-\nu)}. \end{aligned}$$

Replace μ by $\nu - m$ to obtain

$$\Delta^{\nu} t^{(\nu-m)} = \frac{\Gamma(\nu-m+1)}{\Gamma(N-m+1)} \Delta^{N} t^{(N-m)}.$$

Since $\Delta^N t^{(N-m)} = 0$, if $m = 1; ...; N$,

$$\Delta^\nu t^{(\nu-m)} = 0, \quad m = 1; ...; N.$$

We state this observation in the form of a lemma.

Lemma 2.4. *Let $0 \le N-1 < \nu < N$. Let $C_i \in \mathbb{R}, i = 1; 2; ...; N$. Then,*

$$y(t) = C_1 t^{(\nu-1)} + C_2 t^{(\nu-2)} + ... + C_N t^{(\nu-N)},$$

satisfie $\Delta^\nu y(t) = 0$.

By Theorems 2.1 and 2.3, it follows that

$$\Delta^{-\nu}\Delta^\nu y(t) = y(t) + C_1 t^{(\nu-1)} + C_2 t^{(\nu-2)} + ... + C_N t^{(\nu-N)}.$$

Moreover,

$$\Delta^{-\nu}\Delta^\nu y(t) = \Delta^{-\nu}\Delta^{N-(N-\nu)} y(t) = \Delta^{-\nu}\Delta^N \Delta^{-(N-\nu)} y(t),$$

and

$$\begin{aligned}
\Delta^{-\nu}\Delta^N\Delta^{-(N-\nu)}y(t) &= \Delta^N[\Delta^{-\nu}\Delta^{-(N-\nu)}]y(t) - \sum_{k=0}^{N-1} \frac{t^{(\nu-N+k)}}{\Gamma(\nu+k-1)}\Delta^k[\Delta^{\nu-N}y(a)] \\
&= \Delta^N[\Delta^{-N}]y(t) + C_1 t^{(\nu-1)} + C_2 t^{(\nu-2)} + ... + C_N t^{(\nu-N)} \\
&= y(t) + C_1 t^{(\nu-1)} + C_2 t^{(\nu-2)} + ... + C_N t^{(\nu-N)}
\end{aligned}$$

This proves the following lemma.

Lemma 2.5. *Let $0 \le N-1 < n \le N$. Then*

$$\Delta^{-\nu}\Delta^\nu y(t) = y(t) + C_1 t^{(\nu-1)} + C_2 t^{(\nu-2)} + ... + C_N t^{(\nu-N)},$$

for some $C_i \in \mathbb{R}, i = 1; 2; ...; N$.

3. Existence of Positive Solution

For our purposes, define the Banach space $\mathcal{B}$ by

$$\mathcal{B} = \{y : [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}} \to \mathbb{R} : y(\nu-1) = 0\},$$

with norm

$$\|y(t)\| = \max|y(t)| \quad t \in [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}}.$$

and define cones

$$K = \{y \in \mathcal{B}, y(t) \ge 0 \;\; for \;\; t \in [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}}\}.$$

The positive solution which we consider in this article is such that $u(\nu-1) = 0, u(t) > 0, t \in (\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}}, u(t) \in \mathcal{B}$.

Eq. (1.8) is equivalent to the following equation

$$y(t)=\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{(\nu-1)}f(s+\nu-1,y(s+\nu-1)), \tag{3.1}$$

where Γ denotes the Gamma function. The equation (3.1) is equivalent to the fixed point equation $Ay(t)=y(t), y(t)\in\mathcal{B}$, where operator $A:K\to K$ is defined as

$$A(y(t))=\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{(\nu-1)}f(s+\nu-1,y(s+\nu-1)). \tag{3.2}$$

We shall obtain sufficient conditions for the existence of a fixed point of A. First, note that A is a summation operator on a discrete finite set. Hence, A is completely continuous.

Lemma 3.1. *The operator $A:K\to K$ is compact.*

Proof. The operator $A:K\to K$ is continuous in view of the assumption of nonnegativeness and continuity of $f(t,y)$.

Let $M\subset K$ be bounded, that is, there exists a positive constant l such that $\|y\|\le l$ for any $y\in M$, and let $L=\max_{t\in[\nu-1,\nu+b+1]_{\mathbb{N}_{\nu-1}},0\le y\le 1} f(t,y(t))+1$, then, for any $y\in M$, we have

$$\begin{aligned}
|A(y(t))| &= \Big|\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{(\nu-1)}f(s+\nu-1,y(s+\nu-1))\Big|\\
&\le \frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{(\nu-1)}\Big|f(s+\nu-1,y(s+\nu-1))\Big|\\
&\le \frac{L}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{(\nu-1)}\le\frac{L}{\Gamma(\nu)\Gamma(\nu+1)}t^{\nu}.
\end{aligned}$$

Thus,

$$\|Ay\|\le\frac{L(b+2)^{\nu}}{\Gamma(\nu)\Gamma(\nu+1)},$$

Hence, $A(M)$ is uniformly bounded.

Now, we will prove that the operator A is equicontinuous. For each $y\in M$, any $\varepsilon>0$, $t_1,t_2\in[\nu-1,\nu+b+1]_{\mathbb{N}_{\nu-1}}$, $t_1<t_2$., let $\delta=\dfrac{\varepsilon\Gamma(\nu)\Gamma(\nu+1)}{L}$, then, when

$|t_2 - t_1| < \delta$ we have

$$\begin{aligned} |A(y(t_2)) - A(y(t_1))| &= \Big|\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t_2-\nu}(t_2-\sigma(s))^{(\nu-1)}f(s+\nu-1,y(s+\nu-1)) \\ &- \frac{1}{\Gamma(\nu)}\sum_{s=0}^{t_1-\nu}(t_1-\sigma(s))^{(\nu-1)}f(s+\nu-1,y(s+\nu-1))\Big| \\ &\le \frac{L}{\Gamma(\nu)}\Big[\sum_{s=0}^{t_2-\nu}(t_2-\sigma(s))^{(\nu-1)}\sum_{s=0}^{t_1-\nu}(t_1-\sigma(s))^{(\nu-1)}\Big] \\ &\le \frac{L}{\Gamma(\nu)\Gamma(\nu+1)}|t_2^{\nu}-t_1^{\nu}| < \frac{L}{\Gamma(\nu)\Gamma(\nu+1)}|t_2-t_1| \\ &< \frac{L}{\Gamma(\nu)\Gamma(\nu+1)}\delta = \varepsilon. \end{aligned}$$

Therefore, $A(M)$ is equicontinuous. The Arzela-Ascoli Theorem implies that A is compact. □

Let $f(t,y) : [\nu-1,\nu+b+1]_{\mathbb{N}_{\nu-1}} \times [0,+\infty) \to [0,+\infty)$ be a given continuous function. Take $c, d \in \mathbb{R}^+$, and $d > c$. For any $y \in [c,d]$ we define the upper-control function $H(t,u) = \sup_{c\le\eta\le y} f(t,\eta)$ and lower-control function $h(t,y) = \inf_{y\le\eta\le d} f(t,\eta)$. Obviously, $H(t,y), h(t,y)$ are monotonous non-decreasing on y and $h(t,y) \le f(t,y) \le H(t,y)$.

Suppose $\tilde{y}(t), \hat{y}(t) \in K$,$d \ge \tilde{y}(t) \ge \hat{y}(t) \ge c$, satisfy

$$\begin{aligned} \Delta^{\nu}\tilde{y}(t) \ge H(t+\nu-1,\tilde{y}(t+\nu-1))\Big(&or\ \tilde{y}(t) \\ &\ge \frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{(\nu-1)}H(s+\nu-1,\tilde{y}(s+\nu-1))\Big), \end{aligned}$$

and

$$\begin{aligned} \Delta^{\nu}\hat{y}(t) \le h(t+\nu-1,\hat{y}(t+\nu-1))\Big(&or\ \hat{y}(t) \\ &\le \frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{(\nu-1)}h(s+\nu-1,\hat{y}(s+\nu-1))\Big). \end{aligned}$$

Then the function $\tilde{y}(t), \hat{y}(t)$ are called a pair of upper and lower solution for Eq. (1.8), respectively.

Now, we give the main results of this article.

Theorem 3.2. *Assume that $f(t,y) : [\nu-1,\nu+b+1]_{\mathbb{N}_{\nu-1}} \times [0,+\infty) \to [0,+\infty)$ is continuous and $\tilde{y}(t)$ and $\hat{y}(t)$ are a pair of upper and lower solutions of Eq. (1.8), then the initial value problem(1.8) has at least one solution $y(t) \in \mathcal{B}$. Moreover,*

$$\tilde{y}(t) \ge y(t) \ge \hat{y}(t), \quad t \in [\nu-1,\nu+b+1]_{\mathbb{N}_{\nu-1}}$$

Proof. Let

$$S = \{y(t)|y(t) \in K, \tilde{y}(t) \geq y(t) \geq \hat{y}(t), \quad t \in [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}}\}$$

be a set endowed with the norm $\|v(t)\| = max|v(t)|$. Then we have $\|v(t)\| \leq d$, and S is a convex, bounded, and closed subset of the Banach space $\mathcal{B}$. The operator $A : K \to K$ is compact. Then, we need only to prove A maps S into S.

For any $v(t) \in S$, we have $\tilde{y}(t) \geq v(t) \geq \hat{y}(t)$, then,

$$\begin{aligned} A(v(t)) &= \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-\sigma(s))^{(\nu-1)} f(s+\nu-1, v(s+\nu-1)) \\ &\leq \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-\sigma(s))^{(\nu-1)} H(s+\nu-1, v(s+\nu-1)) \\ &\leq \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-\sigma(s))^{(\nu-1)} H(s+\nu-1, \tilde{y}(s+\nu-1)) \leq \tilde{y}(t), \end{aligned}$$

and

$$\begin{aligned} A(v(t)) &= \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-\sigma(s))^{(\nu-1)} f(s+\nu-1, v(s+\nu-1)) \\ &\geq \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-\sigma(s))^{(\nu-1)} h(s+\nu-1, v(s+\nu-1)) \\ &\geq \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-\sigma(s))^{(\nu-1)} h(s+\nu-1, \hat{y}(s+\nu-1)) \geq \hat{y}(t). \end{aligned}$$

Hence, $\tilde{y}(t) \geq A(v(t)) \geq \hat{y}(t), t \in [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}}$, that is, $A : S \to S$. According to Schauder fixed point theorem, the operator A has at least one fixed point $y(t) \in S, t \in [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}}$. Therefore, the initial value problem (1.8) has at least one solution $y(t) \in \mathcal{B}$ and $\tilde{y}(t) \geq v(t) \geq \hat{y}(t), t \in [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}}$.

□

Lemma 3.3. *Assume that $f(t,u) : [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}} \times [0, +\infty) \to [0, +\infty)$ is continuous and there exists $k_2 \geq k_1 > 0$, such that*

$$k_1 \leq f(t,l) \leq k_2, \quad (t,l) \in [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}} \times [0, +\infty). \tag{3.3}$$

Then, the initial value problem (1.8) has at least one positive solution $y(t) \in \mathcal{B}$. Moreover

$$\frac{k_1}{\Gamma(\nu)\Gamma(\nu+1)} t^{\nu} \leq y(t) \leq \frac{k_2}{\Gamma(\nu)\Gamma(\nu+1)} t^{\nu}.$$

Proof. By assumption (3.3) and the definition of control function, we have

$$k_1 \leq h(t,l) \leq H(t,l) \leq k_2, \quad (t,l) \in [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}} \times [c,d].$$

Now, we consider the equation

$$\Delta^{\nu} w(t) = k_2, \quad w(\nu-1) = 0. \tag{3.4}$$

Obviously, equation (3.4) has a positive solution

$$w(t) = \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-\sigma(s))^{(\nu-1)} k_2 = \frac{k_2}{\Gamma(\nu)\Gamma(\nu+1)} t^{\nu},$$

and $w(t) = \Delta^{-\nu} k_2 \geq \Delta^{-\nu} H(t, w(t))$, namely, $w(t)$ is a upper solution of Eq. (1.8). In a similar way, we verify that $v(t) = \Delta^{-\nu} k_1$ is the lower solution of Eq. (1.8). An application of Theorem 3.2 now yields that the initial value problem (1.8) has at least one solution $y(t) \in \mathcal{B}$ and satisfies

$$\frac{k_1}{\Gamma(\nu)\Gamma(\nu+1)} t^{\nu} \leq y(t) \leq \frac{k_2}{\Gamma(\nu)\Gamma(\nu+1)} t^{\nu}.$$

□

Lemma 3.4. *Assume that* $f(t,u) : [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}} \times [0,+\infty) \to [0,+\infty)$ *is continuous, where* a *is a positive constant, moreover,*

$$a < \lim_{y \to +\infty} f(t,y) < +\infty, \;\; t \in [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}}. \tag{3.5}$$

Then, the initial value problem (1.8) has at least one positive solution $y(t) \in \mathcal{B}$.

Proof. By assumption (3.5), there are positive constants N, R, such that when $y > R$, we have $f(t,y) \leq N$. Let

$$C = \max_{t \in [\nu-1,\nu+b+1]_{\mathbb{N}_{\nu-1}}, 0 \leq y \leq R} f(t,y),$$

then, $a \leq f(t,y) \leq N + C, 0 < y < +\infty$.
By Lemma 3.3, the initial problem (1.8) has at least one positive solution $y(t) \in \mathcal{B}$ and this solution satisfies

$$\frac{a}{\Gamma(\nu)\Gamma(\nu+1)} t^{\nu} \leq y(t) \leq \frac{N+C}{\Gamma(\nu)\Gamma(\nu+1)} t^{\nu}.$$

□

Lemma 3.5. *Assume that* $f(t,u) : [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}} \times [0,+\infty) \to [0,+\infty)$ *is continuous, where* a *is a positive constant, moreover,*

$$a < \lim_{y \to +\infty} \max_{t \in [\nu-1,\nu+b+1]_{\mathbb{N}_{\nu-1}}} \frac{f(t,y)}{y} < +\infty. \tag{3.6}$$

Then, the initial value problem (1.8) has at least one positive solution $y(t) \in \mathcal{B}'$ *, where*

$$\mathcal{B}' = \{y : [\nu-1, \delta]_{\mathbb{N}_{\nu-1}} \to \mathbb{R} : y(\nu-1) = 0\}$$

and $\nu - 1 < \delta < \nu + b + 1$.

Proof. According to

$$a < \lim_{y\to+\infty} \max_{t\in[\nu-1,\nu+b+1]_{\mathbb{N}_{\nu-1}}} \frac{f(t,y)}{y} < +\infty,$$

there exist $M > 0$ and $c > 0$, such that for any $y(t) \in \mathcal{B}$, we have

$$f(t, y(t)) \leq My(t) + c.$$

By the definition of control function, we have

$$H(t, y(t)) \leq My(t) + c. \tag{3.7}$$

Next, we consider the equation

$$\Delta^{\nu} y(t) = My(t+\nu-1) + c, \quad t \in [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}}. \tag{3.8}$$

Eq.(3.8) is equivalent to following equation

$$y(t) = \Delta^{-\nu}(My(t+\nu-1) + c) \tag{3.9}$$

$$= \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-\sigma(s))^{(\nu-1)} (My(s+\nu-1) + c). \tag{3.10}$$

Let $T : K \to K$ be an operator defined as follow

$$Ty(t) = \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-\sigma(s))^{(\nu-1)} (My(s+\nu-1) + c), \quad t \in [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}}.$$

Then by Lemma 3.1 the operator T is compact.

Let

$$\mathcal{B}_R = \{y(t) \in K| \ \ \|y - \frac{c}{\Gamma(\nu)\Gamma(\nu+1)} t^s\| \leq R < +\infty\},$$

then, $\mathcal{B}_R$ is convex, bounded, and closed subset of the Banach space $\mathcal{B}'$, where $\nu - 1 < \delta < \nu + b + 1$.

For any $y \in \mathcal{B}_R$, we have

$$\begin{aligned}
\|y\| &\leq \frac{c}{\Gamma(\nu)\Gamma(\nu+1)} t^{\nu} + R \\
&\leq \frac{c}{\Gamma(\nu)\Gamma(\nu+1)} \delta^{\nu} + R \\
&\leq \frac{c}{\Gamma(\nu)\Gamma(\nu+1)} (b+\nu+1)^{\nu} + R,
\end{aligned}$$

then,

$$\begin{aligned}
\|A(y) - \frac{c}{\Gamma(\nu)\Gamma(\nu+1)} t^{\nu}\| \ &\leq \ \frac{M}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-\sigma(s))^{(\nu-1)} y(s+\nu-1) \\
&\leq \ \frac{M}{\Gamma(\nu)} \|y\| \sum_{s=0}^{t-\nu} (t-\sigma(s))^{(\nu-1)} \leq \frac{M}{\Gamma(\nu)\Gamma(\nu+1)} \|y\| t^{\nu} \\
&\leq \ \frac{M}{\Gamma(\nu)\Gamma(\nu+1)} (\frac{c}{\Gamma(\nu)\Gamma(\nu+1)} (b+\nu+1)^{\nu} + R) \delta^{\nu}.
\end{aligned}$$

Take

$$\delta < \min\{[\frac{\Gamma(\nu)\Gamma(\nu+1)}{2M}]^{\frac{1}{\nu}}, [\frac{\Gamma(\nu)^2\Gamma(\nu+1)^2 R}{2Mc}]^{\frac{1}{\nu}}, (b+\nu+1)\},$$

then

$$\|A(y) - \frac{c}{\Gamma(\nu)\Gamma(\nu+1)} t^{\nu}\| \leq R.$$

Hence, the Schauder fixed point theorem assures operator T has at least one fixed point, and then, Eq.(3.8) has at least one positive solution $w^*(t)$, where $t \in [\nu - 1, \delta]_{\mathbb{N}_{\nu-1}}$. Therefore, we have

$$w^*(t) = \Delta^{-\nu}(Mw^*(t+\nu-1)+c) = \frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{(\nu-1)}(w^*(s+\nu-1)+c).$$

Combining with condition (10), we have

$$w^*(t) \geq \frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{(\nu-1)}H(s, w^*(s+\nu-1)).$$

Obviously,$w^*(t)$ is the upper solution of initial value problem (1.8), and $v^*(t) = \Delta^{-\nu}(a) > 0$ is the lower solution of Eq.(1.8). By Theorem 3.2, the system (1.8) has at least one positive solution $y(t) \in \mathcal{B}' = \{y : [\nu-1, \delta]_{\mathbb{N}_{\nu-1}} \to \mathbb{R} : y(\nu-1) = 0\}$, where $\nu - 1 < \delta < \nu + b + 1$ and $v^*(t) \leq y(t) \leq w^*(t)$. □

Lemma 3.6. *Assume that $f(t,u) : [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}} \times [0,+\infty) \to [0,+\infty)$ is continuous, where a is a positive constant, and there exist $d > 0, c > 0$, such that*

$$\max\{f(t,l) : (t,l) \in [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}} \times [0,d]\} \leq c\Gamma(\nu)\Gamma(\nu+1). \tag{3.11}$$

Then, the initial value problem (1.8) has at least one positive solution $y(t) \in \mathcal{B}$ and satisfies

$$0 < \|y(t)\| \leq c.$$

Proof. By the definition of control function, we have

$$a \leq H(t,l) \leq c\Gamma(\nu)\Gamma(\nu+1), (t,l) \in [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}} \times [0,d].$$

By Lemma 3.3, the initial problem (1.8) has at least one positive solution $y(t) \in \mathcal{B}$, and satisfies $0 < y(t) \leq c$. Hence, $0 < \|y(t)\| \leq c$. □

4. Uniqueness of Positive Solution

In this section, we shall prove the uniqueness of the positive solution using Banach contraction principle.

Lemma 4.1. *If the operator $A : X \to X$ is a contraction mapping, where X is the Banach space, then A has a unique fixed point in X.*

Theorem 4.2. *If Eq. (1.8) has a pair of positive upper and lower solutions and for any* $u(t), v(t) \in \mathcal{B}, t \in [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}}$, *there exists* $l > 0$, *such that*

$$\|f(t,u) - f(t,v)\| \leq l\|u - v\|, \tag{4.1}$$

then for $\dfrac{l}{\Gamma(\nu)\Gamma(\nu+1)} < 1$, *the initial problem (1.8) has a unique positive solution* $y(t) \in \mathcal{B}$.

Proof. From Theorem 3.2, if the condition in Theorem 4.2 holds, it follows that the initial value problem (1.8) has at least one positive solution in S. Hence, we need only to prove that the operator A defined in (3.1) is a contractiont in $\mathcal{B}$. In fact, for any $y_1(t), y_2(t) \in \mathcal{B}$, by assumption (4.1), we have

$$\begin{aligned}
\|A(y_1)(t) - A(y_2)(t)\| &\leq \frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{(\nu-1)}\big\|f(s+\nu+1, y_1(s+\nu+1)) \\
&- f(s+\nu+1, y_2(s+\nu+1))\big\| \\
&\leq \frac{l}{\Gamma(\nu)}\|y_1(s+\nu+1) - y_2(s+\nu+1)\|\sum_{s=0}^{t-\nu}(t-\sigma(s))^{(\nu-1)} \\
&\leq \frac{l}{\Gamma(\nu)\Gamma(\nu+1)}\|y_1(s+\nu+1) - y_2(s+\nu+1)\|.
\end{aligned}$$

Thus, for $\dfrac{l}{\Gamma(\nu)\Gamma(\nu+1)} < 1$, the operator A is a contraction mapping. Therefore, the initial value problem (1.8) has a unique positive solution $y(t) \in \mathcal{B}$. □

Finally, we give an example to illustrate our results.

Example 4.3. We consider the fractional equation

$$\Delta^{\frac{2}{3}}y(t) = 1 + \frac{y(t+\nu+1)}{y(t+\nu+1) + \sin(y(t+\nu+1)+1)}, \quad y(-\frac{1}{3}) = 0, \tag{4.2}$$

where $f(t,y) = 1 + \dfrac{y}{y + \sin(y+1)}$. Due to $\lim_{y\to+\infty} f(t,y) = 2$ and $f(t,y) \geq 1, y \in [0,+\infty)$, by Lemma 3.4, the equation (4.2) has a positive solution.

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