Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 4, No. 4, 2016, pp. 261-275

Solutions structure of integrable families of Riccati equations and their applications to the perturbed nonlinear fractional Schrodinger equation

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Abstract Some preliminaries about the integrable families of Riccati equations and solutions structure of these equations in several cases are presented in this paper, then by using of definitions for fractional derivative, we apply the new extended of tanh method to the perturbed nonlinear fractional Schrodinger equation with the kerr law nonlinearity. Finally by using of this method and solutions of Riccati equations, we obtain several analytical solutions for perturbed nonlinear fractional Schrodinger equation. The proposed technique enables a straightforward derivation of parameters of solitary solutions.

Keywords. Riccati equations, tanh method, analytical solution.

2010 Mathematics Subject Classification. 34B15, 47E05, 35G25.

1. Introduction

The Riccati equation (RE),

$$\phi' = \gamma(t)\,\phi^2 + \beta(t)\,\phi + \alpha(t)\,, \tag{1.1}$$

named after the Italian mathematician Jacopo Francesco Riccati [5], is a basic first-order nonlinear ordinary differential equation (ODE) that arises in different fields of mathematics and physics [29] is one of the most simple nonlinear differential equations because it is of first order and with quadratic nonlinearity. Obviously, this was the reason that as soon as Newton invented differential equations, RE was the first one to be investigated extensively since the end of the 17th century [30]. Which can be considered as the lowest order nonlinear approximation to the derivative of a function

Received: 19 September 2016 ; Accepted: 7 January 2017.
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in terms of the function itself. It is well known that solutions to the general Riccati equation are not available, and only special cases can be treated [19,26]. Even though the equation is nonlinear, similar to the second order inhomogeneous linear ODEs one needs only a particular solution to find the general solution.
Next by using these solutions we obtain the exact travelling wave solutions for perturbed nonlinear fractional Schrodinger equation. It is significant to looking for the exact traveling wave solutions for nonlinear partial differential equations (PDEs) in nonlinear sciences. As mathematical models of the phenomena, the investigation of exact solutions of these equations will help us to understand these phenomena better. Therefore, exact traveling wave solution methods of PDEs have become more and more important resulting in methods like the homogeneous balance method [39-40], the tanh-sech method [22-24,35,36], the extended tanh-coth method [7,8,43,33,1], the (G'/G)-expansion method [34,37,16,14,38], the modified simple equation method [15,41].
This article contains four sections. Section 2 introduces the preliminaries for Riccati equation and solution method, Section 3 presents an application and Section 4 brings a conclusion.

2. Preliminaries for integrable families of Riccati equations and solution method

In this section for obtaining a new integrability condition for the generalized Riccati equation (1.1), we consider the Riccati equation in the standard form which is given by the equation [6,9]

$$E(\xi,u)=u'(\xi)=u^2(\xi)+n(\xi). \tag{2.1}$$

For this aim, we need some definitions and theorems
Definition 2.1. The symmetries of (2.1) are given by the elements of a connected Lie group with parameter a

$$\begin{cases} \xi^*=\xi+af(\xi,u)+o\left(a^2\right), \\ u^*=u+ah(\xi,u)+o\left(a^2\right), \end{cases} \tag{2.2}$$

which transform solutions into solutions. Alternatively, the infinitesimal generators of the Lie algebra of (2.2), which are the components of the vector field associated to (2.2)

$$\partial=f(\xi,u)\,\partial_\xi+h(\xi,u)\,\partial_u, \tag{2.3}$$

where $\partial_\xi=\frac{\partial}{\partial\xi}$, and $\partial_u=\frac{\partial}{\partial u}$ are called symmetries of (2.1).
The symmetry variables$f(\xi,u)$, $h(\xi,u)$ can be found by solving the following equation, which is called determinant equation [6,25,28,13,9,10,11,12]

$$\begin{aligned} &h_\xi(\xi,u)+\left[h_u(\xi,u)-f_\xi(\xi,u)\right]E(\xi,u)-f_u(\xi,u)\,E^2(\xi,u)- \\ &f(\xi,u)\,E_\xi(\xi,u)-h(\xi,u)\,E_u(\xi,u)=0, \end{aligned} \tag{2.4}$$

where $E(\xi,u)$ is as in (2.1). Eq. (2.4) does not split into an overdetermined system, therefore it has an infinite set of solutions. That is why we can hope find solutions of certain forms only. On the other hand, we can see that (2.4) can be written as

$$\begin{aligned}
&-f_u(\xi,u)\,u^4(\xi) + (-2n(\xi)\,f_u(\xi,u) + h_u(\xi,u) - f_\xi(\xi,u))\,u^2(\xi) - \\
&2h(\xi,u)\,u(\xi) + (n(\xi)\,[h_u(\xi,u) - f_\xi(\xi,u) - f_u(\xi,u)\,n(\xi)] + \\
&h_\xi(\xi,u) - f(\xi,u)\,n'(\xi)) = 0.
\end{aligned} \tag{2.5}$$

Respect to this last equation, we have our first result which say us that f depend only on n and does not of $u(\xi)$

Proposition 2.2. [9,13,28] Let $f(\xi,u) = \sum f_i(\xi)\,u^i(\xi)\,, h(\xi,u) = h_0(\xi) + h_1(\xi)\,u(\xi)\,,$ a solution of (2.5). Then $f(\xi,u) = f(\xi)$.

Theorem 2.3. The standard Riccati equation (2.1) admits the vector fields

$$\partial = f(\xi)\,\partial_\xi - \left(f'(\xi)\,u(\xi) + \frac{f''(\xi)}{2}\right)\partial_u, \tag{2.6}$$

where $f(\xi)$ satisfies the third order ordinary differential equation

$$f'''(\xi) + 4f'(\xi)\,n(\xi) + 2n'(\xi)\,f(\xi) = 0. \tag{2.7}$$

Proof. In accordance with proposition 2.2, we seek solutions to (2.5) in the form

$$\left\{\begin{array}{l} f(\xi,u) = f(\xi)\,, \\ h(\xi,u) = k(\xi) + r(\xi)\,u(\xi)\,. \end{array}\right. \tag{2.8}$$

Substituting (2.8) into (2.5) and equaling the coefficients of this last equation to zero, we obtain the following system

$$\left\{\begin{array}{l} r(\xi) = -f'(\xi)\,, \\ k(\xi) = \frac{1}{2}r'(\xi)\,, \\ n(\xi)\,r(\xi) - [n(\xi)\,f(\xi)]' + k'(\xi) = 0. \end{array}\right. \tag{2.9}$$

From the firsts two equations in (2.9) we have

$$h(\xi,u) = -\frac{1}{2}f''(\xi) - f'(\xi)\,. \tag{2.10}$$

Substituting the expressions for $k(\xi)$ and $r(\xi)$ in the third equation that appear in (2.9), finally Eq. (2.8) is obtained and the proof is complete.

We cannot find solutions of (2.8) in the case that $n(\xi)$ is an arbitrary function. However, an analysis of (2.8) may be useful. In fact, if we consider (2.8) as an first order differential equation in the unknowns $n(\xi)$ and we solve it. We obtain

$$n(\xi) = \frac{\left(f'(\xi)^2\right) - 2f(\xi)\,f''(\xi) + 4k}{4f^2(\xi)}, \tag{2.11}$$

where K is an integration constant. According with the previous results, Eq. (2.11) say us that the following family of Riccati equations in standard form

$$u'(\xi) = u^2(\xi) + \frac{\left(f'(\xi)^2\right) - 2f(\xi)\,f''(\xi) + 4k}{4f^2(\xi)}, \tag{2.12}$$

is integrable by quadratures and the respective solutions are obtained using (2.10). We have the following new integrability condition to generalized Riccati equation (1.1)

Proposition 2.4. If in Eq. (1.1) the coefficients are defined in some interval $[a,b] \subset \Re$ and $p(\xi) \in C^2[a,b], q(\xi) \in C^1[a,b], r(\xi) \in C[a,b]$ are related as

$$\begin{cases} \frac{1}{4p^2(\xi)}, \\ \left\{(-p(\xi)\, q(\xi))^2 + 4p(\xi)^3\, r(\xi) - 2p(\xi)\, p'(\xi)\, q(\xi) - \right. \\ \left. 3p'^2(\xi) + 2p^{\ 2}(\xi)\, q'(\xi) + 2p(\xi)\, p''(\xi)\right\} = \frac{\left(f'(\xi)^2\right) - 2f(\xi)f''(\xi) + 4k}{4f^2(\xi)}, \end{cases} \tag{2.13}$$

with $f(\xi) \in C^2(a,b), f(\xi) \neq 0$ a properly chosen function and K an arbitrary constant, then a solution to (1.1) can be obtain using elementary integration.

Proof. With the change of variable

$$\phi(\xi) = \frac{1}{p(\xi)}\left[u(\xi) - \frac{q(\xi) + \frac{p'(\xi)}{p(\xi)}}{2}\right], \tag{2.14}$$

(1.1) reduces to (2.1) where

$$\begin{aligned} n(\xi) = \tfrac{1}{4p^2(\xi)} \Big[&- (p(\xi)\, q(\xi))^2 + 4p(\xi)^3\, r(\xi) - 2p(\xi)\, p'(\xi)\, q(\xi) - \\ &3p'^2(\xi) + 2p^2(\xi)\, q'(\xi) + 2p(\xi)\, p''(\xi)\Big]. \end{aligned} \tag{2.15}$$

Taking into account [6], we have the hypothesis given in the enunciate of the theorem. In accordance with the previous results, we can construct a variety of families of Riccati equations in standard form that are integrable by quadratures [10]. In fact, if we take an $f(\xi) \in C^2(a,b), f(\xi) \neq 0$ and K arbitrary constant, the family of Riccati equations

$$u'(\xi) = u^2(\xi) + \frac{\left(f'(\xi)^2\right) - 2f(\xi)\, f''(\xi) + 4k}{4f^2(\xi)}, \tag{2.16}$$

is integrable by quadratures. Moreover, $f(\xi, u) = f(\xi)$ and $h(\xi, u) = -\frac{1}{2}f''(\xi) - f'(\xi)$ are the components of the vector field associated to Lie group (2.2) admitted by this family. The following families may be considered as important examples
Case 1. We consider the Riccati equation [6,25,28,13,9]

$$\phi'(\xi) = \gamma(t)\,\phi^2(\xi) + \beta(t)\,\phi(\xi) + \alpha(t), \tag{2.17}$$

where $\alpha(t), \gamma(t) \neq 0, \beta(t)$ are functions that do not depend of ξ. By means of the substitution (2.14) then (2.15) reduces to following SRE

$$u'(\xi) = u^2(\xi) + \frac{4\alpha(t)\,\gamma(t) - \beta^2(t)}{4}. \tag{2.18}$$

The hypothesis in proposition 2.4 are satisfied if we take $K = \frac{4\alpha(t)\gamma(t) - \beta^2(t)}{4}$ and $f(\xi) = 1$. Therefore, by (2.8)-(2.12) with c = 0 and using (2.14) we get the following set of solutions to (2.17)

If $\alpha(t) \neq 0, \gamma(t) \neq 0$ and $\beta(t) \neq 0$

$$\phi(\xi) = \begin{cases} \frac{1}{\gamma(t)}\left(-\frac{1}{\xi} - \frac{\beta(t)}{2}\right), & \beta^2(t) = 4\gamma(t)\alpha(t), \\ \frac{1}{\gamma(t)}\left(\frac{\sqrt{4\gamma(t)\alpha(t)-\beta^2(t)}}{2}\tan\left[\frac{\sqrt{4\gamma(t)\alpha(t)-\beta^2(t)}}{2}\xi\right] - \frac{\beta(t)}{2}\right), & 4\gamma(t)\alpha(t) - \beta^2(t) > 0, \\ \frac{1}{\gamma(t)}\left(-\frac{\sqrt{4\gamma(t)\alpha(t)-\beta^2(t)}}{2}\cot\left[\frac{\sqrt{4\gamma(t)\alpha(t)-\beta^2(t)}}{2}\xi\right] - \frac{\beta(t)}{2}\right), & 4\gamma(t)\alpha(t) - \beta^2(t) > 0, \\ \frac{1}{\gamma(t)}\left(-\frac{\sqrt{4\gamma(t)\alpha(t)-\beta^2(t)}}{2}\tanh\left[\frac{\sqrt{4\gamma(t)\alpha(t)-\beta^2(t)}}{2}\xi\right] - \frac{\beta(t)}{2}\right), & 4\gamma(t)\alpha(t) - \beta^2(t) < 0, \\ \frac{1}{\gamma(t)}\left(-\frac{\sqrt{4\gamma(t)\alpha(t)-\beta^2(t)}}{2}\coth\left[\frac{\sqrt{4\gamma(t)\alpha(t)-\beta^2(t)}}{2}\xi\right] - \frac{\beta(t)}{2}\right), & 4\gamma(t)\alpha(t) - \beta^2(t) < 0. \end{cases} \tag{2.19}$$

If $\gamma(t) = 0$ and $\beta(t) \neq 0$

$$\phi(\xi) = \frac{-\alpha(t) + \beta(t) e^{\beta(t)\xi}}{\beta(t)}. \tag{2.20}$$

Case 2. To solve the Riccati equation [13]

$$\phi'(x) = \cos\xi - \sin(\xi - \phi(\xi))\, y(\xi), \tag{2.21}$$

we use (2.14) to obtain the SRE

$$u'(\xi) = u^2(\xi) + \frac{2\cos\xi - \sin^2\xi}{4}. \tag{2.22}$$

The conditions in Proposition 2.4 are satisfied if we take $K = 0$ and $f(\xi) = e^{\cos\xi}$. So that, we obtain the solution [26]

$$u(\xi) = \sin\xi - \frac{1}{e^{\cos\xi}\int e^{-\cos\xi} d\xi}. \tag{2.23}$$

Finally, using (2.14) with $p(\xi) = 1,\ q(\xi) = -\sin\xi$, a particular solution to initial equation is given by

$$\phi(\xi) = \sin\xi - \frac{1}{e^{\cos\xi}\int e^{-\cos\xi} d\xi} + \frac{\sin\xi}{2}. \tag{2.24}$$

Case 3. The Riccati equation [6]

$$\phi' = A\xi^m\left(\phi^2(\xi) + 1\right), \tag{2.25}$$

has the standard form

$$u'(\xi) = u^2(\xi) + \frac{-m^2 - 2m + 4A^2\xi^{2m+2}}{4\xi^2}. \tag{2.26}$$

The hypothesis in proposition 2.4 are satisfied if we take $K = A^2$ and $f(\xi) = \xi^{-m}$. Therefore, we have [26]

$$u(\xi) = \begin{cases} \frac{m+2|A|\xi^{m+1}\tan\left[|A|\left(\frac{\xi^{m+1}}{m+1}-c\right)\right]}{2\xi}, & m \neq -1, \\ \frac{|A|-\frac{1}{2}+\tan[|A|(\ln\xi-c)]}{x}, & m \neq -1. \end{cases} \tag{2.27}$$

Finally, by (2.14),

$$\phi(\xi) = \begin{cases} \pm\tan\left[|A|\left(c-\frac{\xi^{m+1}}{m+1}\right)\right], & m \neq -1, \\ \pm\tan\left[|A|\left(c-\ln\xi\right)\right], & m \neq -1, \end{cases} \tag{2.28}$$

are solutions to (2.25). In this case, c is an arbitrary constant.

Next for complete the structure of our method we consider some definitions about the fractional derivative.

Recently, a new modification of Riemann-Liouville derivative is proposed by Jumarie [35]

$$D_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x (x-\varepsilon)^{-\alpha}(f(\varepsilon)-f(0))\,d\varepsilon, \qquad 0<\alpha<1,$$

and gave some basic fractional calculus formulae, for example, formulae (4.12) and (4.13) in [35]

$$D_x^\alpha(u(x)v(x)) = v(x)D_x^\alpha(u(x)) + u(x)D_x^\alpha(v(x)), \tag{2.29}$$

$$D_x^\alpha(f(u(x))) = f_u'(u)D_x^\alpha(u(x)) = D_x^\alpha f(u)\left(u_x'\right)^\alpha, \tag{2.30}$$

The last formula (2.2) has been applied to solve the exact solutions to some nonlinear fractional order differential equations. If this formula were true, then we could take the transformation $\xi = x - \frac{kt^\alpha}{\Gamma(1+\alpha)}$ and reduce the partial derivative $\frac{\partial^\alpha U(x,t)}{\partial t^\alpha}$ to $U'(\xi)$. Therefore the corresponding fractional differential equations become the ordinary differential equations which are easy to study. But we must point out that Jumarie's basic formulae (2.1) and (2.2) are not correct, and therefore the corresponding results on differential equations are not true [35]. Now by using of most popular definitions and theorems as follows

Definition 2.5. Let $f^\alpha(t)$ stands for $T_\alpha(f)(t)$. Hence

$$f^\alpha(t) = \lim_{\xi\to 0}\frac{f\left(t+\xi t^{1-\alpha}\right)-f(t)}{\xi}.$$

If f is α-differentiable in some $(0,a)$, $a>0$, and $\lim_{t\to 0^+} f^\alpha(t)$ exists then by definition

$$f^\alpha(0) = \lim_{t\to 0^+} f^\alpha(t).$$

We should remark that $T_\alpha(t^\mu) = \mu t^{\mu-\alpha}$. Further, this definition coincides with the classical definitions of R-L and of Caputo on polynomials (up to a constant multiple). One can easily show that T_α satisfies all the properties in the theorem [2].

Theorem 2.6. Let $\alpha \in [0,1)$ and f, g be α-differentiable at a point t, Then

$(i)\, T_\alpha\,(af + bg) = aT_\alpha\,(f) + bT_\alpha\,(g)\,, for \quad all \quad a, b \in \mathrm{R},$
$(ii)\, T_\alpha\,(t^\mu) = \mu t^{\mu-\alpha}, for \quad all \quad \mu \in \mathrm{R},$
$(iii)\, T_\alpha\,(fg) = fT_\alpha\,(g) + gT_\alpha\,(f),$
$(i\nu)\, T_\alpha\left(\frac{f}{g}\right) = \frac{fT_\alpha(g) - gT_\alpha(f)}{g^2}.$

In addition, f is differentiable, then $T_\alpha\,(f)\,(t) = t^{1-\alpha}\frac{df}{dt}$.

Theorem 2.7. Let $f : [0, \infty) \to \mathrm{R}$ be a function such that f is differentiable and also differentiable. Let g be a function defined in the range of f and also differentiable; then, one has the following rule [1]

$T_\alpha\,(fog)\,(t) = t^{1-\alpha}g'\,(t)\, f'\,(g\,(t))\,.$

The above rule is referred to as Atangana beta-rule [35-43]. We will present new derivative for some special functions

$(i)\, T_\alpha\,(e^{cx}) = cx^{1-\alpha}e^{cx},\ c \in \mathrm{R},$
$(ii)\, T_\alpha\,(\sin bx) = bx^{1-\alpha}\cos bx,\ b \in \mathrm{R},$
$(iii)\, T_\alpha\,(\cos bx) = -bx^{1-\alpha}\sin bx,\ b \in \mathrm{R},$
$(i\nu)\, T_\alpha\left(\frac{1}{\alpha}x^\alpha\right) = 1.$

However, it is worth noting the following fractional derivatives of certain functions

$(i)\, T_\alpha\left(e^{\frac{1}{\alpha}t^\alpha}\right) = e^{\frac{1}{\alpha}t},$
$(ii)\, T_\alpha\left(\sin\frac{1}{\alpha}t\right) = \cos\frac{1}{\alpha}t,$
$(iii)\, T_\alpha\left(\cos\frac{1}{\alpha}t\right) = -\sin\frac{1}{\alpha}t.$

Definition 2.8. (Fractional Integral) Let $a \geq 0$ and $t \geq a$. Also, let f be a function defined on $(a, t]$ and $\alpha \in f$. Then the α fractional integral of f is defined by

$$I_\alpha^\alpha\,(f)\,(t) = \int_a^t \frac{f\,(x)}{x^{1-\alpha}}dx,$$

if the Riemann improper integral exists. It is interesting to observe that the α fractional derivative and the α fractional integral are inverse of each other as given in [2].

Theorem 2.9. (Inverse property). Let $a \geq 0$, and $\alpha \in (0, 1)$. Also, let f be a continuous function such that $I_a^\alpha f$ exists. Then
$T_\alpha\,(I_a^\alpha f)\,(t) = f\,(t)\,,$ for $t \geq a.$
Researchers with the aid of above contents and [35] introduce a new definition for wave transformations, that for fractional equations

$$G\left(u, D_t^\alpha u, D_x^\alpha u, D_y^\alpha u, D_t^{2\alpha} u, D_x^{2\alpha}, D_t^\alpha D_x^\alpha u, , ...\right) = 0, \qquad 0 < \alpha \leq 1.$$

As follow

$$u = u(\xi), \qquad \xi = a\frac{x^\alpha}{\alpha} + b\frac{t^\alpha}{\alpha}, \tag{2.31}$$

where k and c are real constants. This enables us to use the following changes

$$D_t^\alpha\,(.) = c\frac{d}{d\xi}, \quad D_x^\alpha\,(.) = k\frac{d}{d\xi}, \quad D_x^{2\alpha}\,(.) = k^2\frac{d^2}{d\xi^2},$$

where a and b are constants.

3. Method applied

Now with the aid of the solutions structure of Riccati equation and new definition for fractional derivative we consider the perturbed nonlinear fractional Schrodinger equation with the kerr law nonlinearity

$$\begin{aligned} & i\tfrac{\partial^\alpha u}{\partial t^\alpha} \ + \ u_{xx} \ + \ \gamma u\left|u\right|^2 \ + i\left[\gamma_1 u_{xxx} + \gamma_2 \left|u\right|^2 u_x \ + \ \gamma_3\left(\left|u\right|^2\right)_x u\right] \ = \ 0, \\ & \text{t} \ > \ 0, \ 0 \ < \alpha \leq 1, \end{aligned} \tag{3.1}$$

where γ_1 is third order dispersion, γ_2 is the nonlinear dispersion, while γ_3 is a also a version of nonlinear dispersion.
Substituting (2.31) into equation (3.2) we can show that equation (3.2) is reduced into an ordinary differential equation

$$ibu_\xi + a^2 u_{\xi\xi} + \ \gamma u\left|u\right|^2 + \ i\left[\gamma_1 a^3 u_{\xi\xi\xi} \ + \ \gamma_2 a\left|u\right|^2 u_\xi \ + \ \gamma_3 a\left(\left|u\right|^2\right)_\xi u\right] \ = \ 0. \tag{3.2}$$

Function u is a complex function so we can write

$$u\left(\xi\right) \ = \ e^{\,is\xi} w\left(\xi\right), \tag{3.3}$$

where s is a constant and $w\left(\xi\right)$ is a real function. Then (3.2) reduced to

$$\begin{aligned} & \left[\left(-bs \ - \ a^2 s^2 + \ \gamma_1 a^3 s^3\ \right) w + \left(\gamma \ - \ \gamma_2 as\right) w^3 + \left(a^2 - 3\gamma_1 a^3 s\right) w_{\xi\xi}\right] + \\ & i\left[\left(b + 2a^2 s - 3\gamma_1 a^3 s^2\ \right) w_\xi + \ \gamma_1 a^3 w_{\xi\xi\xi} + \ \left(\gamma_2 a \ + \ 2\gamma_3 a\right) w^2 w_\xi\right] \ = \ 0. \end{aligned} \tag{3.4}$$

Then we have two equations as follows

$$\begin{aligned} & \left[\left(-bs \ - \ a^2 s^2 + \ \gamma_1 a^3 s^3\ \right) w + \left(\gamma \ - \ \gamma_2 as\right) w^3 + \left(a^2 - 3\gamma_1 a^3 s\right) w_{\xi\xi}\right] = 0, \\ & \left(b + 2a^2 s - 3\gamma_1 a^3 s^2\ \right) w_\xi + \ \gamma_1 a^3 w_{\xi\xi\xi} + \ \left(\gamma_2 a \ + \ 2\gamma_3 a\right) w^2 w_\xi \ = \ 0. \end{aligned} \tag{3.5}$$

Integrating of second equation (3.5) and taking zero as the integration constant, we have

$$\left(b + 2a^2 s - 3\gamma_1 a^3 s^2\ \right) w + \ \gamma_1 a^3 w_{\xi\xi} + \ \frac{1}{3}\left(\gamma_2 a \ + \ 2\gamma_3 a\right) w^3 \ = \ 0. \tag{3.6}$$

By (3.6) and first equation (3.5) we have the same solutions. So, we have the following equation

$$\frac{-bs \ - \ a^2 s^2 + \ \gamma_1 a^3 s^3}{b + 2a^2 s - 3\gamma_1 a^3 s^2} \ = \ \frac{\gamma \ - \ \gamma_2 as}{\frac{1}{3}\left(\gamma_2 a \ + \ 2\gamma_3 a\right)} \ = \ \frac{a^2 - 3\gamma_1 a^3 s}{\gamma_1 a^3}, \tag{3.7}$$

from (3.7) we obtain

$$s \ = \ \frac{-b}{2a^2} + \ \frac{3}{2}\gamma_1 a s^2 \ + \ \frac{A}{2C}\gamma_1 a, \tag{3.8}$$

where we assume that

$$A = -bs \; - \; a^2s^2 + \; \gamma_1 a^3 s^3, \qquad B = a^2 - 3\gamma_1 a^3 s, \qquad C = \gamma \; - \; \gamma_2 as. \quad (3.9)$$

So first equation (3.5) is transformed into the following form

$$Aw + Bw_{\xi\xi} + Cw^3 = 0. \quad (3.10)$$

Now we introduce new extension of generalized tanh method

$$u\left(\xi\right) = \sum_{i=0}^{M} a_i \left(\phi\left(\xi\right) + h\right)^i + \sum_{i=M+1}^{2M} a_i \left(\phi\left(\xi\right) + h\right)^{M-i}, \quad (3.11)$$

Balancing w'' with w^3in Eq. (3.10) give

$$3M = M + 2 \Rightarrow M = 1. \quad (3.12)$$

We then assume that Eq. (3.11) has the following formal solution

$$w\left(\xi\right) = a_0 + a_1\left(\phi\left(\xi\right) + h\right) + a_2\left(\phi\left(\xi\right) + h\right)^{-1}, \quad (3.13)$$

and ϕ satisfied in following Riccati equation

$$\phi'\left(\xi\right) = \gamma\left(t\right)\phi^2\left(\xi\right) + \beta\left(t\right)\phi\left(\xi\right) + \alpha\left(t\right). \quad (3.14)$$

By considering the $\phi\left(\xi\right) + h = \Psi$ in equation (3.13) we have

$$w\left(\xi\right) = a_0 + a_1\Psi + a_2\Psi^{-1}, \quad (3.15)$$

and

$$\Psi' = \gamma\Psi^2 + (\beta - 2\gamma h)\Psi + \gamma h^2 - \beta h + \alpha. \quad (3.16)$$

By using equations (3.16) and (3.15) we obtain

$$\begin{aligned} w'' = {} & 2a_1\gamma^2\Psi^3 + \left[3a_1\gamma(\beta - 2\gamma h)\right]\Psi^2 + \left[2a_1\gamma(\gamma h^2 - \beta h + \alpha) + a_1(\beta - 2\gamma h)^2\right]\Psi + \\ & a_2(\beta - 2\gamma h)\gamma + a_1(\beta - 2\gamma h)(\gamma h^2 - \beta h + \alpha) + \\ & \left[a_2(\beta - 2\gamma h)^2 + 2a_2(\gamma h^2 - \beta h + \alpha)\gamma\right]\Psi^{-1} + \left[3a_2(\beta - 2\gamma h)(\gamma h^2 - \beta h + \alpha)\right]\Psi^{-2} + \\ & \left[2a_2(\gamma h^2 - \beta h + \alpha)^2\right]\Psi^{-3}. \end{aligned} \quad (3.17)$$

Substituting Eqs. (3.13)-(3.17) into (3.10) and collecting all terms with the same order of Ψ^j together, we convert the left-hand side of Eq. (3.10) into a polynomial in Ψ^j. Setting each coefficient of each polynomial to zero, we derive a set of algebraic equations for a_0, a_1, a_2 and h. By solving these algebraic equations we obtain several case of variables solutions for example we have

$$
\begin{aligned}
&a_1 = \frac{\gamma\sqrt{-2\beta C}}{C}, \\
&h = \frac{1}{6}\,\frac{\sqrt{\beta}(-3\beta^{\frac{3}{2}}B+3\sqrt{\beta}B^2-\sqrt{3}\sqrt{-\beta+B}\sqrt{B}\sqrt{-B\beta^2+4B\alpha\gamma+2A}}{B\gamma(-\beta+B)}, \\
&a_0 = \pm\frac{1}{6}\,\frac{\sqrt{-6C(-\beta+B)B(-B\beta^2+4B\alpha\gamma+2A)}}{C(-\beta+B)}, \\
&a_2 = -\frac{1}{6}\,\frac{2A\beta+4B\alpha\beta\gamma+12B\beta\gamma^2h^2-12B\beta^2\gamma h}{\gamma\sqrt{-2C\beta}\beta}+ \\
&\frac{2B\beta^3-3B^2\beta^2+12B^2\beta\gamma h-12B^2\gamma^2h^2}{\gamma\sqrt{-2C\beta}\beta}.
\end{aligned}
\tag{3.18}
$$

Now by substituting variables (3.18) into (3.13) along with (2.19) if $\beta^2(t) = 4\gamma(t)\alpha(t)$ we have

$$
\begin{aligned}
&w(x,t) = \pm\frac{1}{6}\,\frac{\sqrt{-6C(-\beta+B)B(-B\beta^2+4B\alpha\gamma+2A)}}{C(-\beta+B)}+ \\
&\frac{\gamma\sqrt{-2\beta C}}{C}\left[\frac{1}{\gamma(t)}\left(-\frac{1}{ax+\frac{bt^{\alpha}}{\alpha}}-\frac{\beta(t)}{2}\right)+\right. \\
&\left.\frac{1}{6}\,\frac{\sqrt{\beta}(-3\beta^{\frac{3}{2}}B+3\sqrt{\beta}B^2-\sqrt{3}\sqrt{-\beta+B}\sqrt{B}\sqrt{-B\beta^2+4B\alpha\gamma+2A}}{B\gamma(-\beta+B)}\right]- \\
&\frac{1}{6}\,\frac{2A\beta+4B\alpha\beta\gamma+12B\beta\gamma^2h^2-12B\beta^2\gamma h+2B\beta^3-3B^2\beta^2+12B^2\beta\gamma h-12B^2\gamma^2h^2}{\gamma\sqrt{-2C\beta}\beta}\times \\
&\left[\frac{1}{\gamma(t)}\left(-\frac{1}{ax+\frac{bt^{\alpha}}{\alpha}}-\frac{\beta(t)}{2}\right)+\right. \\
&\left.\frac{1}{6}\,\frac{\sqrt{\beta}(-3\beta^{\frac{3}{2}}B+3\sqrt{\beta}B^2-\sqrt{3}\sqrt{-\beta+B}\sqrt{B}\sqrt{-B\beta^2+4B\alpha\gamma+2A}}{B\gamma(-\beta+B)}\right]^{-1}.
\end{aligned}
$$

So in this case from (3.3) we obtain the solution of perturbed nonlinear fractional Schrodinger equation as follow

$$u_1(x,t) = \pm\frac{1}{6}\frac{e^{is\left(ax+\frac{bt^\alpha}{\alpha}\right)}\sqrt{-6C(-\beta+B)B(-B\beta^2+4B\alpha\gamma+2A)}}{C(-\beta+B)}+$$
$$e^{is\left(ax+\frac{bt^\alpha}{\alpha}\right)}\times\frac{\gamma\sqrt{-2\beta C}}{C}\left[\frac{1}{\gamma(t)}\left(-\frac{1}{ax+\frac{bt^\alpha}{\alpha}}-\frac{\beta(t)}{2}\right)+\right.$$
$$\left.\frac{1}{6}\,\frac{\sqrt{\beta}(-3\beta^{\frac{3}{2}}B+3\sqrt{\beta}B^2-\sqrt{3}\sqrt{-\beta+B}\sqrt{B}\sqrt{-B\beta^2+4B\alpha\gamma+2A}}{B\gamma(-\beta+B)}\right]-$$
$$\frac{1}{6}e^{is\left(ax+\frac{bt^\alpha}{\alpha}\right)}\times\left(\frac{2A\beta+4B\alpha\beta\gamma+12B\beta\gamma^2h^2-12B\beta^2\gamma h}{\gamma\sqrt{-2C\beta}\beta}+\right.$$
$$\left.\frac{2B\beta^3-3B^2\beta^2+12B^2\beta\gamma h-12B^2\gamma^2h^2}{\gamma\sqrt{-2C\beta}\beta}\right)\times\left[\frac{1}{\gamma(t)}\left(-\frac{1}{ax+\frac{bt^\alpha}{\alpha}}-\frac{\beta(t)}{2}\right)+\right.$$
$$\left.\frac{1}{6}\,\frac{\sqrt{\beta}(-3\beta^{\frac{3}{2}}B+3\sqrt{\beta}B^2-\sqrt{3}\sqrt{-\beta+B}\sqrt{B}\sqrt{-B\beta^2+4B\alpha\gamma+2A}}{B\gamma(-\beta+B)}\right]^{-1}.$$

If $4\gamma(t)\alpha(t)-\beta^2(t)>0$

$$u_2(x,t) = \pm\frac{1}{6}\frac{e^{is\left(ax+\frac{bt^\alpha}{\alpha}\right)}\sqrt{-6C(-\beta+B)B(-B\beta^2+4B\alpha\gamma+2A)}}{C(-\beta+B)}+$$
$$e^{is\left(ax+\frac{bt^\alpha}{\alpha}\right)}\times\frac{\gamma\sqrt{-2\beta C}}{C}\times$$
$$\left[\frac{1}{\gamma(t)}\left(\frac{\sqrt{4\gamma(t)\alpha(t)-\beta^2(t)}}{2}\tan\left[\frac{\sqrt{4\gamma(t)\alpha(t)-\beta^2(t)}}{2}\left(ax+\frac{bt^\alpha}{\alpha}\right)\right]-\frac{\beta(t)}{2}\right)+\right.$$
$$\left.\frac{1}{6}\,\frac{\sqrt{\beta}(-3\beta^{\frac{3}{2}}B+3\sqrt{\beta}B^2-\sqrt{3}\sqrt{-\beta+B}\sqrt{B}\sqrt{-B\beta^2+4B\alpha\gamma+2A}}{B\gamma(-\beta+B)}\right]-$$
$$\frac{1}{6}e^{is\left(ax+\frac{bt^\alpha}{\alpha}\right)}\times\left(\frac{2A\beta+4B\alpha\beta\gamma+12B\beta\gamma^2h^2-12B\beta^2\gamma h}{\gamma\sqrt{-2C\beta}\beta}+\right.$$
$$\left.\frac{2B\beta^3-3B^2\beta^2+12B^2\beta\gamma h-12B^2\gamma^2h^2}{\gamma\sqrt{-2C\beta}\beta}\right)\times$$
$$\left[\frac{1}{\gamma(t)}\left(\frac{\sqrt{4\gamma(t)\alpha(t)-\beta^2(t)}}{2}\tan\left[\frac{\sqrt{4\gamma(t)\alpha(t)-\beta^2(t)}}{2}\left(ax+\frac{bt^\alpha}{\alpha}\right)\right]-\frac{\beta(t)}{2}\right)+\right.$$
$$\left.\frac{1}{6}\,\frac{\sqrt{\beta}(-3\beta^{\frac{3}{2}}B+3\sqrt{\beta}B^2-\sqrt{3}\sqrt{-\beta+B}\sqrt{B}\sqrt{-B\beta^2+4B\alpha\gamma+2A}}{B\gamma(-\beta+B)}\right]^{-1}.$$

If $4\gamma(t)\alpha(t) - \beta^2(t) < 0$

$$u_3(x,t) = \pm\frac{1}{6}\frac{e^{\,is\left(ax\,+\frac{bt^\alpha}{\alpha}\right)}\sqrt{-6C(-\beta+B)B(-B\beta^2+4B\alpha\gamma+2A)}}{C(-\beta+B)} +$$

$$e^{\,is\left(ax\,+\frac{bt^\alpha}{\alpha}\right)} \times \frac{\gamma\sqrt{-2\beta C}}{C} \times$$

$$\left[\frac{1}{\gamma(t)}\left(-\frac{\sqrt{4\gamma(t)\alpha(t)-\beta^2(t)}}{2}\coth\left[\frac{\sqrt{4\gamma(t)\alpha(t)-\beta^2(t)}}{2}\left(ax\ +\ \frac{bt^\alpha}{\alpha}\right)\right] - \frac{\beta(t)}{2}\right) +\right.$$

$$\left.\frac{1}{6}\ \frac{\sqrt{\beta}(-3\beta^{\frac{3}{2}}B+3\sqrt{\beta}B^2-\sqrt{3}\sqrt{-\beta+B}\sqrt{B}\sqrt{-B\beta^2+4B\alpha\gamma+2A}}{B\gamma(-\beta+B)}\right] -$$

$$\frac{1}{6}e^{\,is\left(ax\,+\frac{bt^\alpha}{\alpha}\right)} \times \left(\frac{2A\beta+4B\alpha\beta\gamma+12B\beta\gamma^2h^2-12B\beta^2\gamma h}{\gamma\sqrt{-2C\beta}\beta}\ +\right.$$

$$\left.\frac{2B\beta^3-3B^2\beta^2+12B^2\beta\gamma h-12B^2\gamma^2h^2}{\gamma\sqrt{-2C\beta}\beta}\right) \times$$

$$\left[\frac{1}{\gamma(t)}\left(-\frac{\sqrt{4\gamma(t)\alpha(t)-\beta^2(t)}}{2}\coth\left[\frac{\sqrt{4\gamma(t)\alpha(t)-\beta^2(t)}}{2}\left(ax\ +\ \frac{bt^\alpha}{\alpha}\right)\right] - \frac{\beta(t)}{2}\right) +\right.$$

$$\left.\frac{1}{6}\ \frac{\sqrt{\beta}(-3\beta^{\frac{3}{2}}B+3\sqrt{\beta}B^2-\sqrt{3}\sqrt{-\beta+B}\sqrt{B}\sqrt{-B\beta^2+4B\alpha\gamma+2A}}{B\gamma(-\beta+B)}\right]^{-1}.$$

Also in these cases

$$s\ =\ \frac{-b}{2a^2} + \frac{3}{2}\gamma_1 a s^2\ +\ \frac{A}{2C}\gamma_1 a,$$

$$A = -bs\ -\ a^2s^2 +\ \gamma_1 a^3 s^3, \qquad B = a^2 - 3\gamma_1 a^3 s, \qquad C = \gamma\ -\ \gamma_2 a s.$$

If $\gamma(t) = 0$ and $\beta(t) \neq 0$

$$\phi(\xi) = \frac{-\alpha(t) + \beta(t)\,e^{\beta(t)\xi}}{\beta(t)},$$

$$u_4(x,t) = \pm\frac{1}{6}\,\frac{e^{is\left(ax+\frac{bt^{\alpha}}{\Gamma(\alpha+1)}\right)}\sqrt{-6C(-\beta+B)B(-B\beta^2+4B\alpha\gamma+2A)}}{C(-\beta+B)}+$$
$$\frac{\gamma\, e^{is\left(ax+\frac{bt^{\alpha}}{\Gamma(\alpha+1)}\right)}\sqrt{-2\beta C}}{C}\left(\frac{-\alpha(t)+\beta(t)e^{\beta(t)\left(ax+\frac{bt^{\alpha}}{\Gamma(\alpha+1)}\right)}}{\beta(t)}+\right.$$
$$\left.\frac{1}{6}\,\frac{\sqrt{\beta}(-3\beta^{\frac{3}{2}}B+3\sqrt{\beta}B^2-\sqrt{3}\sqrt{-\beta+B}\sqrt{B}\sqrt{-B\beta^2+4B\alpha\gamma+2A}}{B\gamma(-\beta+B)}\right)-$$
$$\left(\frac{1}{6}\,e^{is\left(ax+\frac{bt^{\alpha}}{\Gamma(\alpha+1)}\right)}\,\frac{2A\beta+4B\alpha\beta\gamma+12B\beta\gamma^2h^2-12B\beta^2\gamma h}{\gamma\sqrt{-2C\beta}\beta}+\right.$$
$$\left.\frac{2B\beta^3-3B^2\beta^2+12B^2\beta\gamma h-12B^2\gamma^2h^2}{\gamma\sqrt{-2C\beta}\beta}\right)\times$$
$$\left(\frac{-\alpha(t)+\beta(t)e^{\beta(t)\left(ax+\frac{bt^{\alpha}}{\Gamma(\alpha+1)}\right)}}{\beta(t)}+\right.$$
$$\left.\frac{1}{6}\,\frac{\sqrt{\beta}(-3\beta^{\frac{3}{2}}B+3\sqrt{\beta}B^2-\sqrt{3}\sqrt{-\beta+B}\sqrt{B}\sqrt{-B\beta^2+4B\alpha\gamma+2A}}{B\gamma(-\beta+B)}\right)^{-1}.$$

Also if we consider the Riccati equation in following form (case 2)

$$\phi'(x) = \cos\xi - \sin(\xi - \phi(\xi))\,y(\xi),$$

and by substituting this equation along with Eqs. (3.13)-(3.17) into equation (3.10) we obtain new solutions for parameters a_0, a_1, a_2 and h. Then by substituting these parameters into Eq. (3.13) along with general solution for Riccati equation we have new soliton solutions for perturbed nonlinear fractional Schrodinger equation with the kerr law nonlinearity.

4. Conclusion

A solitary wave is a wave, which propagates without any temporal evolution in shape or size when viewed in the reference frame moving with the group velocity of the wave. The envelope of the wave has one global peak and decays far away from the peak. Solitary waves arise in many contexts, including the elevation of the surface of water and the intensity of light in optical fibers. A soliton is a nonlinear solitary wave with the additional property that the wave retains its permanent structure, even after interacting with another soliton. For example, two solitons propagating in opposite directions effectively pass through each other without breaking. Solitons form a special class of solutions of model equations, including the Korteweg de-Vries (KdV) and the Nonlinear Schrodinger (NLS) equations. These model equations are approximations, which hold under a restrictive set of conditions. The soliton solutions obtained from the model equations provide important insight into the dynamics of solitary waves. In this work, we obtain new solutions for perturbed nonlinear fractional Schrodinger equation with the kerr law nonlinearity by using of several important Riccati equations solutions and application of new conformable fractional derivative. This extension of tanh method is new, reliable, efficient and gives new solutions.

5. Acknowledgements

The authors would like to express thanks to the editor and anonymous referees for their useful and valuable comments and suggestions.

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