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Numerical method for solving optimal control problem of the linear differential systems with inequality constraints

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Abstract In this paper, an efficient method for solving optimal control problems of the linear differential systems with inequality constraint is proposed. By using new adjustment of hat basis functions and their operational matrices of integration, optimal control problem is reduced to an optimization problem. Also, the error analysis of the proposed method is investigated and it is proved that the order of convergence is $O(h^4)$. Finally, numerical examples affirm the efficiency of the proposed method.

Keywords. Adjustment of hat basis functions, Operational matrices, Optimal control, Differential systems, Inequality constraint, Error analysis.

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1. Introduction

Finding analytic solution for optimal control problems with inequality constraints is difficult so numerical methods to get approximate solutions are important. In deterministic setting, there are many text books for analytic solutions of optimal control problems [1, 2, 5, 6, 16, 27, 28, 29]. Furthermore, numerical schemes for these problems have been provided in some articles [4, 8, 12, 13, 14, 15, 19, 22, 33].

Orthogonal functions, often used to solve various problems of dynamic systems. The aim of this technique is reducing these problems to a set of algebraic equations. Typical examples are the block-pulse functions [10], Legendre polynomials [3], Laguerre polynomials [11], Chebyshev polynomials [9] and Fourier series [30].

There are different basic functions for the solution of optimal control problems successfully solve the unconstrained problem such as block-pulse functions [10]. But often results in analytical and computational for solving the optimal control problems with inequality constraints are difficulties. In recent years, the development of computational techniques for

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solving problems such as hybrid of block-pulse functions and Legendre polynomials [20, 31], triangular orthogonal functions [7], B-spline functions [32].

The aim of this paper is developing a numerical scheme based on the adjustment of hat basis functions to solve the optimal control problem of the linear differential systems with inequality constraints. Operational matrices of the adjustment of hat basis functions reduce such problems to those that solve a system of algebraic equations which greatly simplify the problem. Consider the linear differential system

$$\dot{X}(t) = K(t)X(t) + F(t)U(t), \tag{1.1}$$

$$X(0) = Y, \tag{1.2}$$

with inequality constraints as

$$G(t)X(t) + H(t)U(t) \leq L(t), \tag{1.3}$$

where $X(t), U(t)$ are unknown functions and $L(t)$ is a known function. Also $K(t), F(t), G(t)$ and $H(t)$ are matrices of appropriate dimensions.
The aim of this paper is finding the numerical approximation of optimal control $U^*(t)$ and the corresponding optimal state $X^*(t)$, $0 \leq t \leq T$, satisfying Eqs. (1.1)-(1.3) while minimizing the quadratic cost functional

$$J = \int_0^T \left[X^T(t)Q(t)X(t) + U^T(t)R(t)U(t)\right] dt, \tag{1.4}$$

where $Q(t)$ and $R(t)$ are positive semi-definite and positive definite matrices, respectively.

Definitions of adjustment of hat basis functions and their properties are given in Section 2. In Section 3, the adjustment of hat functions are developed to approximate the solution of optimal control problem governed by linear differential systems. In Section 4, the error analysis is proved. In Section 5, the proposed method is used for solving some numerical examples. Finally, Section 6 affords some brief conclusion.

2. Definitions of adjustment of hat basis functions and their properties

A set of adjustment of hat functions are defined on $[0, T]$ as [24]

$$\phi_0(t) = \begin{cases} \frac{-1}{6h^3}(t-h)(t-2h)(t-3h) & 0 \leq t \leq 3h, \\ 0 & \text{otherwise}, \end{cases} \tag{2.1}$$

if $i = 3k-2$ and $1 \leq k \leq \frac{n}{3}$

$$\phi_i(t) = \begin{cases} \frac{1}{2h^3}(t-(i-1)h)(t-(i+1)h)(t-(i+2)h) & (i-1)h \leq t \leq (i+2)h, \\ 0 & \text{otherwise}, \end{cases} \tag{2.2}$$

if $i = 3k - 4$ and $2 \leq k \leq \frac{n}{3} + 1$

$$\phi_i(t) = \begin{cases} \frac{-1}{2h^3}(t-(i-2)h)(t-(i-1)h)(t-(i+1)h) & (i-2)h \leq t \leq (i+1)h, \\ 0 & \text{otherwise}, \end{cases} \tag{2.3}$$

if $i = 3k$ and $1 \leq k \leq \frac{n}{3} - 1$

$$\phi_i(t) = \begin{cases} \frac{1}{6h^3}(t-(i-3)h)(t-(i-2)h)(t-(i-1)h) & (i-3)h \leq t \leq ih, \\ \frac{-1}{6h^3}(t-(i+1)h)(t-(i+2)h)(t-(i+3)h) & ih \leq t \leq (i+3)h, \\ 0 & \text{otherwise}, \end{cases} \tag{2.4}$$

and

$$\phi_n(t) = \begin{cases} \frac{1}{6h^3}(t-(T-h))(t-(T-2h))(t-(T-3h)) & (T-3h) \leq t \leq T, \\ 0 & \text{otherwise}, \end{cases} \tag{2.5}$$

where $h = \frac{T}{n}$ is a sampling period and $n \geq 3$ is an integer of multiple three.

Let us divide interval $[0, T]$ into $\frac{n}{3}$ subintervals $[ih, (i+3)h]$ where $i = 0, 3, ..., n-3$, of equal lengths $3h$. By using the definition of adjustment of hat functions, we have

$$\phi_i(kh) = \begin{cases} 1 & i = k, \\ 0 & i \neq k, \end{cases} \tag{2.6}$$

and

$$\sum_{i=0}^{n} \phi_i(t) = 1.$$

An arbitrary real function $f(t)$ on $[0, T]$ can be expanded in an adjustment of hat series as follows

$$f(t) \simeq \sum_{i=0}^{n} f_i \phi_i(t) = F^T \Phi(t) = \Phi^T(t) F, \tag{2.7}$$

where

$$F = [f_0, f_1, ..., f_n]^T,$$

and

$$\Phi(t) = [\phi_0(t), \phi_1(t), ..., \phi_n(t)]^T, \tag{2.8}$$

with

$$f_i = f(ih),\ i = 0, 1, ..., n. \tag{2.9}$$

Also, expand $\int_0^t \phi_i(s)ds$ by relation (2.7) in terms of the adjustment of hat basis functions as

$$\int_0^t \phi_i(s)ds \simeq \sum_{j=0}^{n} a_{i,j}\phi_j(t),\ i = 0, 1, ...n. \tag{2.10}$$

By using relation (2.9), we can compute the coefficients $a_{i,j}$ as follows

$$a_{i,j} = \int_0^{jh} \phi_i(s)ds,\ i, j = 0, 1, ..., n. \tag{2.11}$$

Now, P is the $(n+1) \times (n+1)$ coefficients matrix with entries $a_{i,j}$, $i, j = 0, 1, ..., n$, we obtain

$$P = \frac{h}{24} \begin{pmatrix} 0 & 9 & 8 & 9 & 9 & \dots & 9 \\ \mathbf{0} & p_1 & p_2 & p_3 & p_3 & \dots & p_3 \\ \mathbf{0} & \mathbf{0} & p_1 & p_2 & p_3 & \dots & p_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & p_1 & p_2 & \dots & p_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & p_1 & \dots & p_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & p_1 \end{pmatrix}_{(n+1)\times(n+1)},$$

where

$$p_1 = \begin{pmatrix} 19 & 32 & 27 \\ -5 & 8 & 27 \\ 1 & 0 & 9 \end{pmatrix}_{3\times 3}, \qquad p_2 = \begin{pmatrix} 27 & 27 & 27 \\ 27 & 27 & 27 \\ 18 & 17 & 18 \end{pmatrix}_{3\times 3}, \qquad p_3 = \begin{pmatrix} 27 & 27 & 27 \\ 27 & 27 & 27 \\ 18 & 18 & 18 \end{pmatrix}_{3\times 3},$$

and $\mathbf{0}$ based on its location in the matrix, is the 3×3 zero matrix or 3-vector.

From relations (2.8) and (2.10), we obtain

$$\int_0^t \Phi(s)ds \simeq P\Phi(t). \tag{2.12}$$

Also, we have

$$\Phi_i(t)\Phi_j(t) = \begin{cases} 0 & i = 3k,\ 0 \le k \le \frac{n}{3} \text{ and } |i-j| \ge 4, \\ \\ 0 & \text{otherwise and } |i-j| \ge 3. \end{cases} \tag{2.13}$$

Now, from relations (2.8) and (2.13) we have

$$\Phi(t)\Phi^T(t) = \Lambda, \tag{2.14}$$

where matrix Λ is shown in page 236.

Therefore, from (2.14) we have

$$\int_0^T \Phi(s)\Phi^T(s)ds = Z, \tag{2.15}$$

where

$$Z = \frac{h}{560} \begin{pmatrix} 128 & 99 & -36 & 19 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 99 & 648 & -81 & -36 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -36 & -81 & 648 & 99 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 19 & -36 & 99 & 256 & 99 & -36 & 19 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 99 & 648 & -81 & -36 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ 0 & \cdots & 0 & 19 & -36 & 99 & 256 & 99 & -36 & 19 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 99 & 648 & -81 & -36 \\ 0 & \cdots & 0 & 0 & 0 & 0 & -36 & -81 & 648 & 99 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 19 & -36 & 99 & 128 \end{pmatrix}_{(n+1)\times(n+1)}. \tag{2.16}$$

Furthermore, by considering (2.6) and expanding entries $\Phi(t)\Phi^T(t)$ defined in (2.14) by the adjustment of hat functions, we obtain

$$\Phi(t)\Phi^T(t) = \begin{pmatrix} \phi_0(t) & 0 & \cdots & 0 \\ 0 & \phi_1(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_n(t) \end{pmatrix}_{(n+1)\times(n+1)}. \tag{2.17}$$

Definition 2.1. For two constant vectors $a^T = [a_0, a_1, \ldots, a_n]$ and $b^T = [b_0, b_1, \ldots, b_n]$, we define

$$a^T \odot b^T = [a_0 b_0, a_1 b_1, \ldots, a_n b_n],$$

where $\odot$ denotes the inner product.

$$
\Lambda = \begin{pmatrix}
\phi_0^2(t) & \phi_0(t)\phi_1(t) & \phi_0(t)\phi_2(t) & \phi_0(t)\phi_3(t) & 0 & 0 & 0 & 0 & \cdots & 0 \\
\phi_0(t)\phi_1(t) & \phi_1^2(t) & \phi_1(t)\phi_2(t) & \phi_1(t)\phi_3(t) & 0 & 0 & 0 & 0 & \cdots & 0 \\
\phi_0(t)\phi_2(t) & \phi_1(t)\phi_2(t) & \phi_2^2(t) & \phi_2(t)\phi_3(t) & 0 & 0 & 0 & 0 & \cdots & 0 \\
\phi_0(t)\phi_3(t) & \phi_1(t)\phi_3(t) & \phi_2(t)\phi_3(t) & \phi_3^2(t) & \phi_3(t)\phi_4(t) & \phi_3(t)\phi_5(t) & \phi_3(t)\phi_6(t) & 0 & \cdots & 0 \\
0 & 0 & 0 & \phi_3(t)\phi_4(t) & \phi_4^2(t) & \phi_4(t)\phi_5(t) & \phi_4(t)\phi_6(t) & 0 & \cdots & 0 \\
 & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & & & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \phi_{n-6}(t)\phi_{n-3}(t) & \phi_{n-5}(t)\phi_{n-3}(t) & \phi_{n-4}(t)\phi_{n-3}(t) & \phi_{n-3}^2(t) & \phi_{n-3}(t)\phi_{n-2}(t) & \phi_{n-3}(t)\phi_{n-1}(t) & \phi_{n-3}(t)\phi_n(t) \\
0 & \cdots & 0 & 0 & 0 & 0 & \phi_{n-3}(t)\phi_{n-2}(t) & \phi_{n-2}^2(t) & \phi_{n-2}(t)\phi_{n-1}(t) & \phi_{n-2}(t)\phi_n(t) \\
0 & \cdots & 0 & 0 & 0 & 0 & \phi_{n-3}(t)\phi_{n-1}(t) & \phi_{n-2}(t)\phi_{n-1}(t) & \phi_{n-1}^2(t) & \phi_{n-1}(t)\phi_n(t) \\
0 & \cdots & 0 & 0 & 0 & 0 & \phi_{n-3}(t)\phi_n(t) & \phi_{n-2}(t)\phi_n(t) & \phi_{n-1}(t)\phi_n(t) & \phi_n^2(t)
\end{pmatrix}.
$$

Theorem 2.1. *Let us approximate each of the functions of $a(t)$ and $b(t)$ by the adjustment of hat basis functions. That is,*

$$a(t) \simeq A^T\Phi(t) = \Phi^T(t)A,$$
$$b(t) \simeq B^T\Phi(t) = \Phi^T(t)B.$$

Then we have

$$a(t)b(t) \simeq (A^T \odot B^T)\Phi(t).$$

Proof. From (2.17), we have

$$a(t)b(t) \simeq A^T\Phi(t)\Phi^T(t)B \simeq A^T \begin{pmatrix} \phi_0(t) & 0 & \cdots & 0 \\ 0 & \phi_1(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_n(t) \end{pmatrix} B$$

$$= [a_0\phi_0(t), a_1\phi_1(t), ..., a_n\phi_n(t)]B = \Phi^T(t) \begin{pmatrix} a_0 & 0 & \cdots & 0 \\ 0 & a_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix} B$$

$$= a_0b_0\phi_0(t) + a_1b_1\phi_1(t) + ... + a_nb_n\phi_n(t) = [a_0b_0, a_1b_1, ..., a_nb_n]\Phi(t) = (A^T \odot B^T)\Phi(t).$$

Hence, this completes the proof. □

3. Basic idea

Firstly, we can rewrite relations (1.1) and (1.2) as

$$X(t) = Y + \int_0^t K(s)X(s)ds + \int_0^t F(s)U(s)ds. \tag{3.1}$$

Then, consider the ith equation of relation (3.1)

$$x_i(t) = y_i + \int_0^t \left(\sum_{j=0}^{n} k_{ij}(s)x_j(s) + \sum_{j=0}^{n} f_{ij}(s)u_j(s) \right) ds, \tag{3.2}$$

with ith inequality constraint of (1.3), we have

$$\sum_{j=0}^{n} g_{ij}(t)x_j(t) + \sum_{j=0}^{n} h_{ij}(t)u_j(t) \le l_i(t). \tag{3.3}$$

Now, we approximate functions $x_i, u_i, y_i, l_i, g_{ij}, h_{ij}, k_{ij}$ and f_{ij} by the adjustment of hat functions as follows

$$\begin{cases} x_i(t) \simeq \Phi^T(t)X_i = X_i^T\Phi(t), \\ u_i(t) \simeq \Phi^T(t)U_i = U_i^T\Phi(t), \\ y_i(t) \simeq \Phi^T(t)Y_i = Y_i^T\Phi(t), \\ l_i(t) \simeq \Phi^T(t)L_i = L_i^T\Phi(t), \end{cases} \tag{3.4}$$

$$\begin{cases} g_{ij}(t) \simeq \Phi^T(t)G_{ij} = G_{ij}^T\Phi(t), \\ h_{ij}(t) \simeq \Phi^T(t)H_{ij} = H_{ij}^T\Phi(t), \\ k_{ij}(t) \simeq \Phi^T(t)K_{ij} = K_{ij}^T\Phi(t), \\ f_{ij}(t) \simeq \Phi^T(t)F_{ij} = F_{ij}^T\Phi(t), \end{cases} \tag{3.5}$$

where matrices G_{ij}, H_{ij}, K_{ij} and F_{ij} and vectors X_i, U_i, Y_i and L_i are the adjustment of hat functions coefficients of $g_{ij}, h_{ij}, k_{ij}, f_{ij}, x_i, u_i, y_i$ and l_i, respectively.

Substituting (3.4) and (3.5) into (3.2) and (3.3), and using Theorem (1) we have

$$X_i^T\Phi(t) \simeq Y_i^T\Phi(t) + \sum_{j=0}^{n}(K_{ij}^T \odot X_j^T)\int_0^t \Phi(s)ds + \sum_{j=0}^{n}(F_{ij}^T \odot U_j^T)\int_0^t \Phi(s)ds,$$

subject to

$$\sum_{j=0}^{n}(G_{ij}^T \odot X_j^T)\Phi(t) + \sum_{j=0}^{n}(H_{ij}^T \odot U_j^T)\Phi(t) \leq L_i^T\Phi(t).$$

By using (2.12) and replacing $\simeq$ with $=$ and eliminating $\Phi(t)$, we get

$$X_i^T = Y_i^T + \sum_{j=0}^{n}(K_{ij}^T \odot X_j^T)P + \sum_{j=0}^{n}(F_{ij}^T \odot U_j^T)P,\ i = 0, 1, \dots, n, \tag{3.6}$$

subject to

$$\sum_{j=0}^{n}(G_{ij}^{T} \odot X_{j}^{T}) + \sum_{j=0}^{n}(H_{ij}^{T} \odot U_{j}^{T}) \leq L_{i}^{T}. \tag{3.7}$$

The system of linear equations (3.6) and (3.7), can be expressed in the following matrices form

$$\tilde{X}^{T} - V - W = \tilde{Y}^{T}, \tag{3.8}$$

subject to

$$\tilde{A} + \tilde{B} \leq \tilde{L}^{T}, \tag{3.9}$$

where

$$V^{T} = \begin{pmatrix} \sum_{j=0}^{n}(K_{0j}^{T} \odot X_{j}^{T})P \\ \sum_{j=0}^{n}(K_{1j}^{T} \odot X_{j}^{T})P \\ \vdots \\ \sum_{j=0}^{n}(K_{nj}^{T} \odot X_{j}^{T})P \end{pmatrix}, \; W^{T} = \begin{pmatrix} \sum_{j=0}^{n}(F_{0j}^{T} \odot U_{j}^{T})P \\ \sum_{j=0}^{n}(F_{1j}^{T} \odot U_{j}^{T})P \\ \vdots \\ \sum_{j=0}^{n}(F_{nj}^{T} \odot U_{j}^{T})P \end{pmatrix},$$

and

$$\tilde{A}^{T} = \begin{pmatrix} \sum_{j=0}^{n}(G_{0j}^{T} \odot X_{j}^{T}) \\ \sum_{j=0}^{n}(G_{1j}^{T} \odot X_{j}^{T}) \\ \vdots \\ \sum_{j=0}^{n}(G_{nj}^{T} \odot X_{j}^{T}) \end{pmatrix}, \; \tilde{B}^{T} = \begin{pmatrix} \sum_{j=0}^{n}(H_{0j}^{T} \odot U_{j}^{T}) \\ \sum_{j=0}^{n}(H_{1j}^{T} \odot U_{j}^{T}) \\ \vdots \\ \sum_{j=0}^{n}(H_{nj}^{T} \odot U_{j}^{T}) \end{pmatrix},$$

and

$$\begin{cases} \tilde{X} = [X_0, X_1, \ldots, X_n]^{T}, \\ \tilde{Y} = [Y_0, Y_1, \ldots, Y_n]^{T}, \\ \tilde{L} = [L_0, L_1, \ldots, L_n]^{T}, \end{cases}$$

where $\tilde{X}, \tilde{Y}$ and $\tilde{L}$ are $(n+1)^2$ dimensional vectors and

$$\begin{cases} X_i = [X_{i0}, X_{i1}, \dots, X_{in}]^T, \\ U_i = [U_{i0}, U_{i1}, \dots, U_{in}]^T, \\ K_{ij} = [K_{ij}^0, K_{ij}^1, \dots, K_{ij}^n]^T, \\ F_{ij} = [F_{ij}^0, F_{ij}^1, \dots, F_{ij}^n]^T, \end{cases}$$

$$\begin{cases} Y_i = [Y_{i0}, Y_{i1}, \dots, Y_{in}]^T, \\ L_i = [L_{i0}, L_{i1}, \dots, L_{in}]^T \\ G_{ij} = [G_{ij}^0, G_{ij}^1, \dots, G_{ij}^n]^T, \\ H_{ij} = [H_{ij}^0, H_{ij}^1, \dots, H_{ij}^n]^T. \end{cases}$$

By this method, system of Eq. (3.1) is reduced to system of $(n+1)^2$ algebraic equations.

Now, we have

$$J = \int_0^T \sum_{i=0}^{n} \left(\sum_{j=0}^{n} x_i(t) q_{ij}(t) x_j(t) \right) dt + \int_0^T \sum_{i=0}^{n} \left(\sum_{j=0}^{n} u_i(t) r_{ij}(t) u_j(t) \right) dt. \tag{3.10}$$

Let us approximate q_{ij} and r_{ij} by the adjustment of hat functions as follows

$$\begin{cases} q_{ij}(t) \simeq \Phi^T(t) Q_{ij} = Q_{ij}^T \Phi(t), \\ r_{ij}(t) \simeq \Phi^T(t) R_{ij} = R_{ij}^T \Phi(t), \end{cases} \tag{3.11}$$

where matrices Q_{ij} and R_{ij} are the adjustment of hat functions coefficient matrices of q_{ij} and r_{ij}, respectively.

Substituting (3.11) into (3.10), and using Theorem (1), we will have

$$J \simeq \sum_{i=0}^{n}\left(\sum_{j=0}^{n}(X_i^T \odot Q_{ij}^T)(\int_0^T \Phi(t)\Phi^T(t)dt)X_j\right)$$
$$+\sum_{i=0}^{n}\left(\sum_{j=0}^{n}(U_i^T \odot R_{ij}^T)(\int_0^T \Phi(t)\Phi^T(t)dt)U_j\right).$$

By using (2.15) and replacing $\simeq$ with $=$ and eliminating $\Phi(t)$, we have

$$J = \sum_{i=0}^{n}\left(\sum_{j=0}^{n}(X_i^T \odot Q_{ij}^T)ZX_j\right) + \sum_{i=0}^{n}\left(\sum_{j=0}^{n}(U_i^T \odot R_{ij}^T)ZU_j\right). \tag{3.12}$$

Now, we find X and U such that $J(X,U)$ in (3.12) is minimized subject to the constraints in (3.8) and (3.9). In this paper, the method used to solve the nonlinear constrained optimization problem is based on sequential quadratic programming (SQP) algorithm. SQP is an iterative method for nonlinear optimization. The idea of the SQP methods is to solving the nonlinearly constrained problem using a sequence of quadratic programming subproblems. Also, the approximated solution in each iteration need not be feasible points, since the computation of feasible points in case of the nonlinear constraints may be as difficult as the solution of the nonlinear programming itself [18].

4. Error analysis of the proposed method

In this section, we investigate that the rate of convergence of the mentioned approach is $O(h^4)$. We define

$$\|x(t)\| = \sup_{t\in[0,T]} |x(t)|. \tag{4.1}$$

Theorem 4.1. *Assume that* $X(t) = [X_0, X_1, \ldots, X_n] \in (C^4[0,T])^{n+1}$ *and*

$$X_m(t) = [X_{m0}(t), X_{m1}(t), \ldots, X_{mn}(t)] =$$
$$\left[\sum_{i=0}^{n} X_0(ih)\phi_i(t), \sum_{i=0}^{n} X_1(ih)\phi_i(t), \ldots, \sum_{i=0}^{n} X_n(ih)\phi_i(t)\right],$$

be the adjustment of hat functions expansion of $X(t)$.

Then we have

$$\begin{cases} (i)\ \forall j\ \|X_j(t) - X_{mj}(t)\| = O(h^4), \\ (ii)\ \forall j\ \|\int_o^t (X_j(s) - X_{mj}(s))ds\| = O(h^4). \end{cases}$$

Proof. (i) Let

$$E_k(t) = \begin{cases} X_j(t) - X_{mj}(t) & t \in I_k, \\ 0 & t \in [0,T] - I_k, \end{cases}$$

where $I_k = \{t | kh \le t \le (k+3)h\},\ k = 0, 3, ..., n-3$. Then, we obtain

$$E_k(t) = X_j(t) - \sum_{i=0}^{n} X_j(ih)\phi_i(t) = X_j(t) - (X_j(kh)\phi_k(t)$$

$$+X_j((k+1)h)\phi_{k+1}(t) + X_j((k+2)h)\phi_{k+2}(t) + X_j((k+3)h)\phi_{k+3}(t)).$$

By using third degree interpolation error, we obtain [25]

$$E_k(t) = \frac{(t-kh)(t-(k+1)h)(t-(k+2)h)(t-(k+3)h)}{24} . \frac{d^4 X_j(\eta_k)}{dt^4},$$

where $\eta_k \in (kh, (k+3)h)$.

Now consider $u(t) = (t-kh)(t-(k+1)h)(t-(k+2)h)(t-(k+3)h)$. Since, $u(t)$ is a continuous function and I_k is compacted, so $\sup_{t \in I_k} |u(t)| = \max_{t \in I_k} |u(t)| = 2.798h^4$.

Also, we have

$$|E_k(t)| \le \frac{1}{24} |u(t)| |\frac{d^4 X_j(\eta_k)}{dt^4}|.$$

Hence, we have

$$\|E(t)\| = \|X_j(t) - X_{mj}(t)\| = \max_{k=0,3,\dots,n-3} \sup_{t \in I_k} |E_k(t)| \le \max_{k=0,3,\dots,n-3} 0.0867h^4 |\frac{d^4 X_j(\eta_k)}{dt^4}|.$$

Then, there is a $l \in \{0, 3, ..., n-3\}$, where

$$\|E(t)\| \le \max_{k=0,3,\dots,n-3} 0.0867h^4 |\frac{d^4 X_j(\eta_k)}{dt^4}| = 0.0867h^4 |\frac{d^4 X_j(\eta_l)}{dt^4}|.$$

Finally, by using relation (4.1), we have

$$\|E(t)\| \le 0.0867h^4 |\frac{d^4 X_j(\eta_l)}{dt^4}| \le 0.0867h^4 \|\frac{d^4 X_j(t)}{dt^4}\|. \tag{4.2}$$

According to relation (4.2), we obtain

$$\|E(t)\| = O(h^4).$$

(ii) From case (i), we have

$$\| \int_o^t (X_j(s) - X_{mj}(s)) ds \| \le \int_o^t \|(X_j(s) - X_{mj}(s))\| ds$$

$$\leq 0.0867h^4 \| \frac{d^4 X_j(t)}{dt^4} \| \int_0^t ds = 0.0867h^4 t \| \frac{d^4 X_j(t)}{dt^4} \|,$$

since $t \in [kh, (k+3)h] \leq T$, then we have

$$\| \int_o^t (X_j(s) - X_{mj}(s))ds \| \leq 0.0867Th^4 \| \frac{d^4 X_j(t)}{dt^4} \|. \tag{4.3}$$

According to relation (4.3), we obtain

$$\| \int_o^t (X_j(s) - X_{mj}(s))ds \| = O(h^4).$$

Hence, this completes the proof. □

5. Numerical examples

In this section, we demonstrate the efficiency and accuracy of the proposed method by three examples and obtain the results for $n = 15, 63$. All computations were carried out using a program written in Matlab.

Example 5.1. Consider the minimization of functional [16]

$$J = \frac{1}{2} \int_0^1 \left(X^T(t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} X(t) + U^2(t) \right) dt,$$

subject to

$$\begin{cases} \dot{X}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} X(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} U(t), \\ \\ |U(t)| \leq 1, \\ \\ X(0) = \begin{pmatrix} 0 \\ 10 \end{pmatrix}. \end{cases}$$

where the optimal control of cost functional is $J = 8.07054$. A comparison between the cost functional obtained by the proposed method via the Rationalized Haar functions method [26] and Hybrid of block-pulse and Legendre method [20] is shown in Table 1.

Example 5.2. Consider the minimization of functional [17]

$$J = \int_0^1 \left(X^T(t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} X(t) + 0.005U^2(t) \right) dt,$$

subject to

$$\begin{cases} \dot{X}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} X(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} U(t), \\ X_2(t) \leq 8(t-0.5)^2 - 0.5, \\ X(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \end{cases}$$

A comparison between the cost functional obtained by the proposed method via the Rationalized Haar method [21] and Hybrid of block-pulse and Legendre method [20] is shown in Table 2. The computational results for $X_2(t)$ for $n = 15$ and $n = 63$ together with $r(t) = 8(t-0.5)^2 - 0.5$ are given in Figures 1 and 2.

Table 1. Estimated values and absolute errors of J for Example 5.1.

Methods	Estimated value	Absolute error	CPU time
Rationalized Haar functions [26]			
k=4	8.07473	$4.19e-04$	0.389
k=8	8.07065	$1.10e-04$	0.546
Hybrid of block-pulse and Legendre polynomials [20]			
k=4, m_1=3	8.07059	$4.99e-05$	1.592
k=4, m_1=4	8.07056	$2.00e-05$	4.304
Present method			
n=15	8.07243	$1.89e-04$	0.253
n=63	8.07055	$1.00e-05$	0.612

Example 5.3. Consider the minimization of functional [23]

$$J = \int_0^1 U^2(t)dt,$$

subject to

$$\begin{cases} \dot{X}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} X(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} U(t), \\ X_1(t) \leq 0.15, \\ X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ X(1) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \end{cases}$$

A comparison between the cost functional obtained by the proposed method via the Gradient-restoration method [23] is shown in Table 3.

Table 2. Estimated values of J for Example 5.2.

Methods	Estimated value	CPU time
Rationalized Haar functions [21]		
k=16, w=100	0.171973	-
k=32,w=100	0.170185	-
k=64, w=100	0.170115	-
k=128,w=100	0.170103	-
Hybrid of block-pulse and Legendre polynomials [20]		
k=4, m_1=3	0.17013645	0.951
k=4, m_1=4	0.17013640	1.545
Present method		
n=15	0.1700143	0.192
n=63	0.1698312	0.524

Figure 1. $r(t)$ and $X_2(t)$ obtained for $n = 15$ of Example 5.2.

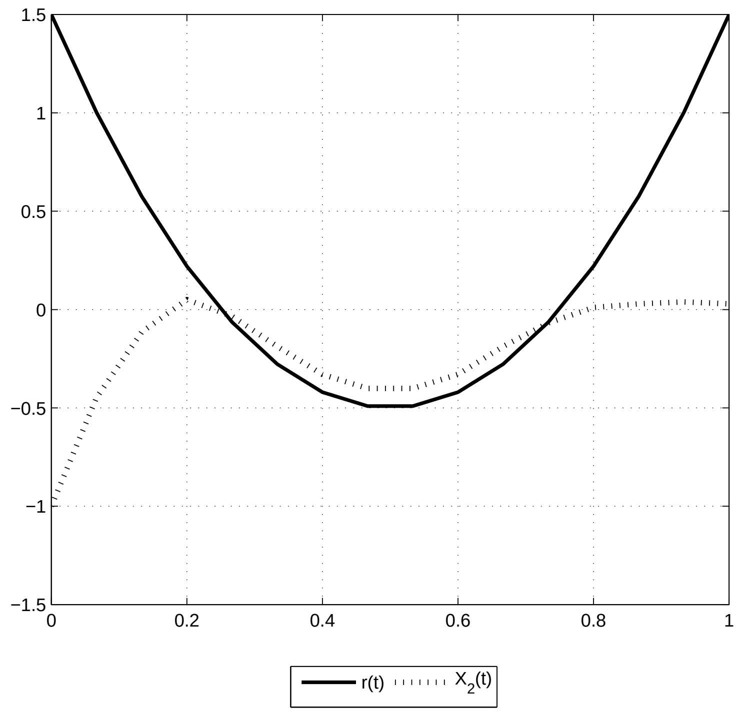


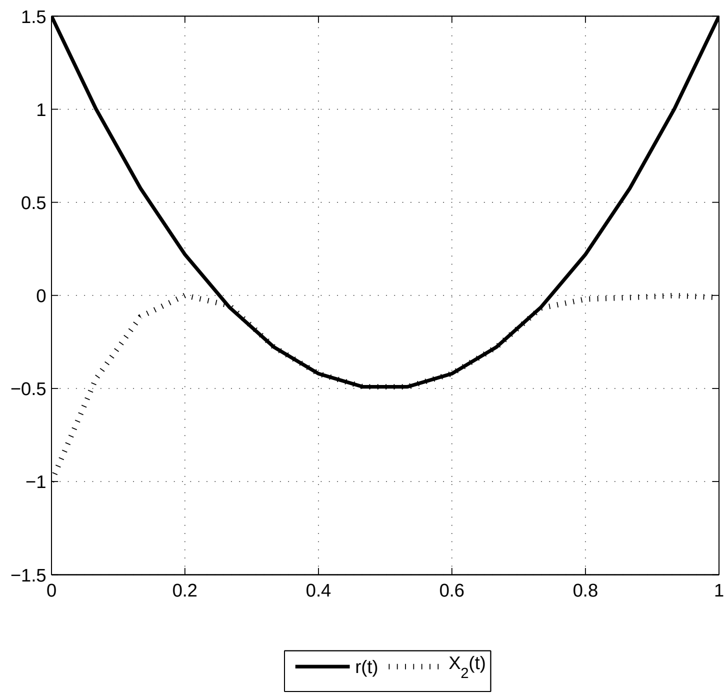
Table 3. Estimated values of J for Example 5.3.

Methods	Estimated value	CPU time
Gradient-restoration [23]		
N=16	5.927	-
Present method		
n=15	5.8451	1.025
n=63	5.7346	1.923

6. Conclusion

In the present work, the excellent properties of operational matrices of the adjustment of hat functions used to solve optimal control problem subject to linear differential systems with inequality constraint. The matrices P and Z in Eqs. (2.12) and (2.15) have large numbers of zero elements, hence the this method is very attractive and reduces the CPU time. Moreover, it is proved that method is convergent and the order of convergence of this method is $O(h^4)$. Illustrative examples are given to demonstrate the validity and applicability of the proposed method.

Figure 2. $r(t)$ and $X_2(t)$ obtained for $n = 63$ of Example 5.2.



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