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Numerical solution of two-dimensional integral equations of the first kind by multi-step methods

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Abstract In this paper, we develop multi-step methods to solve a class of two-dimensional nonlinear Volterra integral equations (2D-NVIEs) of the first kind. Here, we convert a 2D-NVIE of the first kind to a one-dimensional linear VIE of the first kind and then we solve the resulted equation numerically by multi-step methods. We also verify convergence and error analysis of the method. At the end, we give some illustrative examples to show the efficiency and accuracy of the presented method.

Keywords. Two-dimensional nonlinear Volterra integral equations, Integral equations of the first kind, Multi-step methods.

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1. Introduction

In general, the integral equations of the first kind are ill-posed problems, that is, a small perturbation in the given data makes a large perturbation in the solution [5]. These equations in two-dimensional case have many interesting applications, for example in mechanic, physics and other applied sciences [11]. But many of these equations can not be usually solved analytically and they should be solved by numerical methods. Therefore, giving suitable numerical methods for these equations is very worthwhile. Recently, many researchers have studied two-dimensional integral equations. For example, homotopy analysis method, the method based on the piecewise approximation by Chebyshev polynomials and wavelet method have been presented in [1], [6] and [13], respectively. In [4], the nonlinear Volterra-Fredholm integral equations have been solved by collocation methods based on polynomials of spline spaces. In [15], an Euler-type method has been presented for 2D-VIEs of the first kind. In [12], an adaptive multi-scale moment method has been proposed for solving two-dimensional Fredholm integral equations (2D-FIEs) of the first kind. The authors of [17, 18] have developed the well-known Tau method to solve linear and

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nonlinear Volterra integral equations of the first kind. Also in [3] a Runge-Kutta method and in [14] a method based on application of two-dimensional block-pulse functions (2D-BPFs) have been applied to solve 2D-VIEs of the first kind. On the other hand, multi-step methods have been applied to solve one-dimensional integral equations. For example, in [2, 9, 10] multi-step methods and their stability and convergence have been investigated for one-dimensional linear Volterra integral equations of the first kind.

As mentioned above, in this paper, we develop multi-step methods to solve 2D-NVIEs of the first kind. Here, we consider 2D-NVIEs of the first kind of the form:

$$\int_0^t \int_0^x K(x,t,z,s)G(u(z,s))dzds = f(x,t), \quad (x,t) \in D := [0,X] \times [0,T], \tag{1.1}$$

where f, K and G are continuous functions on their corresponding domains.

By setting $y(z,s) = G(u(z,s))$ the equation (1.1) is converted to

$$\int_0^t \int_0^x K(x,t,z,s)y(z,s)dzds = f(x,t), \quad (x,t) \in D, \tag{1.2}$$

which is linear.

Now, we solve this equation numerically and finally we obtain the numerical solution of the main unknown function by $u(z,s) = G^{-1}(y(z,s))$.

It is clear that for solvability of the equation (1.2) , it is necessary that $f(x,0) = f(0,t) = 0$. Also, we assume that $K(x,t,x,t) \neq 0$.

To discuss the existence and uniqueness of the solution of (1.2), by differentiating of (1.2) with respect to x, we obtain

$$\int_0^t \left[K(x,t,x,s)y(x,s) + \int_0^x \frac{\partial K}{\partial x}(x,t,z,s)y(z,s)dz \right] ds = \frac{\partial f}{\partial x}(x,t). \tag{1.3}$$

Now, by differentiating of (1.3) with respect to t, we have

$$\begin{aligned} K(x,t,x,t)y(x,t) + \int_0^x \frac{\partial K}{\partial x}(x,t,z,t)y(z,t)dz \\ + \int_0^t \left[\frac{\partial K}{\partial t}(x,t,x,s)y(x,s) + \int_0^x \frac{\partial^2 K}{\partial t \partial x}(x,t,z,s)y(z,s)dz \right] ds \\ = \frac{\partial^2 f}{\partial t \partial x}(x,t). \end{aligned} \tag{1.4}$$

Under the suitable conditions, (1.2) and (1.4) are equivalent. Existence and uniqueness results for equation (1.4) were obtained by Gronwall in [8] and Goursat in [7], who used techniques based on Picard's method of successive approximations [15]. Also, a different approach to prove the existence and uniqueness of the solution of (1.2) and (1.4) by applying the Picard fixed point iterative procedure to (1.4) can be found in [15].

For more details about the existence and uniqueness of the solution of two-dimensional Volterra integral equations, see [16].

2. MULTI-STEP METHODS

In this section, we introduce multi-step methods for one-dimensional Volterra integral equations of the first kind, then we extend them to the two-dimensional case. Consider the equation:

$$\int_0^t K(t,s)y(s)ds = f(t), \quad t \in [0,T], \tag{2.1}$$

and set $t = t_i := ih,\ i = 0,1,\ldots,N$ for $h = T/N$ and given positive integer N, then

$$\int_0^{t_i} K(t_i,s)y(s)ds = f(t_i),\ i = 1,\ldots,n. \tag{2.2}$$

Assume that, we have obtained $y(t_j) \simeq y_j$, for $j = 0,\ldots,i-1$. To obtain an approximation of the unknown function at the point t_i, we use a quadrature rule to approximate the integral part of (2.2), depending on the points $t_j,\ j = 0,\ldots,i$. Therefore, by knowing the values $y_j,\ j = 0,\ldots,i-1$, we obtain the approximation y_i from an algebraic equation, at the ith step.
To explain an r-step method, we define [2]:

$$[\varphi_h y]_i = \begin{cases} h(y_i - \widehat{y}_i), & i = 0,1,\ldots,r-1 \\ h\sum_{j=0}^{i} w_{ij}K_{ij}y_j - f_i, & i = r,\ldots,N. \end{cases} \tag{2.3}$$

where, for $i = 0,1,\ldots,r-1$, $\widehat{y}_i$s are the starting values and for $i = r,\ldots,N$, f_i and K_{ij} denote $f(t_i)$ and $K(t_i,t_j)$, respectively. Also w_{ij}s are the weights of the quadrature rules and for $i = 0,\ldots,N$, y_i denotes approximation of $y(t_i)$.
Using a matrix-vector multiplication, Eq.(2.2) can be reduced to the linear system of algebraic equations:

$$\varphi_h y = hU_h y - g$$

where

$$U_h = (U_{ij}), \quad U_{ij} = \begin{cases} \delta_{ij}, & 0 \le i,j \le r-1 \\ w_{ij}K_{ij}, & r \le j \le i \\ 0, & i < j, \end{cases} \tag{2.4}$$

$$y = (y_0,\ldots,y_{r-1},y_r,\ldots,y_N)^T$$

and

$$g = (h\widehat{y}_0,\ldots,h\widehat{y}_{r-1},f_r,\ldots,f_N)^T. \tag{2.5}$$

For example, for $N = 8$, $r = 3$ and 3/8-Simpson rule, as the main repeated rule and trapezoidal and Simpson's rules as the end formulae, the method has the following

representation (for $K \equiv 1$):

$$h\begin{bmatrix} 1 & & & & & & & & \\ 0 & 1 & & & & & & & \\ 0 & 0 & 1 & & & & & & \\ 3/8 & 9/8 & 9/8 & 3/8 & & & & & \\ 3/8 & 9/8 & 9/8 & 3/8+1/2 & 1/2 & & & & \\ 3/8 & 9/8 & 9/8 & 3/8+1/3 & 4/3 & 1/3 & & & \\ 3/8 & 9/8 & 9/8 & 3/8+3/8 & 9/8 & 9/8 & 3/8 & & \\ 3/8 & 9/8 & 9/8 & 3/8+3/8 & 9/8 & 9/8 & 3/8+1/2 & 1/2 & \\ 3/8 & 9/8 & 9/8 & 3/8+3/8 & 9/8 & 9/8 & 3/8+1/3 & 4/3 & 1/3 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \end{bmatrix} = \begin{bmatrix} h\widehat{y}_0 \\ h\widehat{y}_1 \\ h\widehat{y}_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \end{bmatrix}$$

To obtain the first exact starting value, we assume $K(t,t) \neq 0$ for all $t \in [0,T]$ and differentiate (2.1) to get:

$$y(t) + \int_0^t (\frac{\partial K}{\partial t}(t,s)/K(t,t))ds = \frac{f'(t)}{K(t,t)}. \tag{2.6}$$

Then setting $t = 0$ in (2.6) yields:

$$\widehat{y}_0 = y(0) = \frac{f'(0)}{K(0,0)}. \tag{2.7}$$

Now, one can use any one-step method to produce other required starting values.
The following algorithm can be given to the presented method.

Algorithm of the method:
Step 1: For a given positive integer N, set $h = T/N$ and $t_i = ih$ for $i = 0, \ldots, N$.
Step 2: Select main repeated formula and end formula(e) and form the matrix U_h by (2.4).
Step 3: Obtain $y_0 = \widehat{y}_0$ exactly by (2.7).
Step 4: For $i = 1, \ldots, r-1$ obtain $\widehat{y}_i$ by a one-step method.
Step 5: Form the vector g by (2.5).
Step 6: Solve the lower triangular system $hU_h y = g$.

Here, we will use 3/8-Simpson's rule as the main repeated formula and 4-step and 5-step rules as the end formulae.

3. MAIN RESULTS

In this section, we describe the method of this paper. As we mentioned previously, it is sufficient to consider the equation (1.2).
First, we set $t = 0$ and $x = 0$ separately in equation (1.4) and obtain,

$$K(x,0,x,0)y(x,0) + \int_0^x \frac{\partial K}{\partial x}(x,0,z,0)y(z,0)dz = \frac{\partial^2 f}{\partial t \partial x}(x,0), \tag{3.1}$$

and

$$K(0,t,0,t)y(0,t) + \int_0^t \frac{\partial K}{\partial t}(0,t,0,s)y(0,s)ds = \frac{\partial^2 f}{\partial t \partial x}(0,t). \tag{3.2}$$

By solving these equations by multi-step methods, the values of the unknown function are obtained on the boundary of the first quarter of the xt-plane, that is, on the lines $x = 0$ and $t = 0$.
Then, we use a quadrature rule to convert Eq. (1.2) to a one-dimensional case and solve it by the method described in previous section.

Let $t_i = ik$, $i = 0, 1, \ldots, M$ and $x_i = ih$, $i = 0, 1, \ldots, N$ with $k = T/M$ and $h = X/N$ for some given positive integers M and N. Setting $x = x_i$ in (1.2) implies:

$$\int_0^t \int_0^{x_i} K(x_i, t, z, s) y(z, s) dz ds = f(x_i, t), \quad i = 1, \ldots, N. \tag{3.3}$$

Now, we substitute the inner integral by a quadrature rule and get by (3.3):

$$\int_0^t \sum_{j=0}^{i} w_{ij} K(x_i, t, x_j, s) y(x_j, s) ds = f(x_i, t), \quad i = 1, \ldots, N, \tag{3.4}$$

where w_{ij}s are the weights of given quadrature rule.
Letting $K_{ij}(t, s) := K(x_i, t, x_j, s)$, $y_j(s) := y(x_j, s)$ and $f_i(t) := f(x_i, t)$, the Eq. (3.4) can be written as:

$$\sum_{j=0}^{i} w_{ij} \int_0^t K_{ij}(t, s) y_j(s) ds = f_i(t), \qquad i = 1, \ldots, N, \tag{3.5}$$

Now, for $i = 1, \ldots, N$ we use the method described in section 2 to obtain approximate values of $y(x_i, t_j)$ by using previous approximate values. Finally, to obtain the main solution of the equation (1.1) at the mesh points we set $u(x_i, t_j) = G^{-1}(y(x_i, t_j))$.
At the first step, that is for $x = x_1$, we use the trapezoidal rule for inner integral because we just have two points x_0, x_1, that is for $i = 1$ the equation (3.5) is converted to:

$$w_{10} \int_0^t K_{10}(t, s) y_0(s) ds + w_{11} \int_0^t K_{11}(t, s) y_1(s) ds = f_1(t), \tag{3.6}$$

where, $w_{10} = w_{11} = h/2$ are the weights of trapezoidal rule.
Now, we set $t = t_j$ for $j = 1, \ldots, M$ in the equation (3.6). The values of $y_0(t_j) = y(0, t_j)$ are known from the previous step. Therefore, by substituting the integrals of the equation (3.6) by appropriate quadrature rule, for various values of t_j, we obtain $y_1(t_j) = y(x_1, t_j)$ for $j = 1, \ldots, M$.
For $x = x_2$, we use Simpson's rule, that is for $i = 2$, we have:

$$w_{20} \int_0^t K_{20}(t, s) y_0(s) ds + w_{21} \int_0^t K_{21}(t, s) y_1(s) ds \tag{3.7}$$

$$+ w_{22} \int_0^t K_{22}(t, s) y_2(s) ds = f_2(t),$$

where, $w_{20} = w_{22} = h/3$, $w_{21} = 4h/3$ are the Simpson's weights.
Again, we set $t = t_j$ for $j = 1, \ldots, M$ in the equation (3.7) and we use appropriate quadrature rules depending on t_j, for the integrals of the equation (3.7). The values of $y_0(t_j) = y(0, t_j)$ and $y_1(t_j) = y(x_1, t_j)$ are known from previous steps. Therefore, in this stage we obtain $y_2(t_j) = y(x_2, t_j)$ for $j = 1, \ldots, M$. Similarly for $x = x_3, x_4, \ldots, x_N$ we use the 3/8-Simpson's rule as the main repeated rule with the 4-step and 5-step rules as the end formulae. To apply the described method, we need the initial values $y(x_1, t_1)$, $y(x_1, t_2)$, $y(x_2, t_1)$ and $y(x_2, t_2)$, which we use the trapezoidal and Simpson's rules at two directions, x and t, to obtain them. Also, from (1.4) it is obvious that $y(0, 0) = \frac{\partial^2 f}{\partial t \partial x}(0, 0)/K(0, 0, 0, 0)$ with the assumption $K(0, 0, 0, 0) \neq 0$.
Now, to obtain the initial values for next stages, that is $y(x_i, t_1)$ and $y(x_i, t_2)$ for $i = 3, \ldots, N$, first we set $t = t_1$ in (1.2) and we substitute the outer integral by trapezoidal rule and the inner integral by described multistep method. Therefore, we obtain the values $y(x_i, t_1)$, $i = 3, \ldots, N$. Then, we set $t = t_2$ in (1.2) and we substitute the outer integral by Simpson's rule and the inner integral by described multi-step method. Therefore, we obtain the values

$y(x_i, t_2)$, $i = 3, \ldots, N$. Now, we have the whole of the initial values which we need for applying the multi-step method for each step.

4. Convergence Analysis

In this section, we present an error bound for the approximate solution and we obtain a convergence order for the method. To this end, we consider the equation:

$$\int_c^t \int_a^x K(x,t,z,s)y(z,s)dzds = f(x,t), \quad (x,t) \in J := [a,b] \times [c,d]. \tag{4.1}$$

At the step (i,j), by setting $x = x_i$ and $t = t_j$ and substitution the integrals by quadrature rules, we have:

$$\sum_{r=0}^{j} \sum_{l=0}^{i} w_{jr} w_{il} K(x_i, t_j, x_l, t_r) y_{l,r} + O(h^\nu) + O(k^\mu) = f_{i,j}, \tag{4.2}$$

where $y_{l,r} = y(x_l, t_r)$, $f_{i,j} = f(x_i, t_j)$ for $i = 1, \ldots, N$, $j = 1, \ldots, M$ and ν, μ are the order of quadrature rules with respect to x and t, respectively.
Suppose that, Y is the exact solution of the equation:

$$\sum_{r=0}^{j} \sum_{l=0}^{i} w_{jr} w_{il} K(x_i, t_j, x_l, t_r) Y_{l,r} = f_{i,j}. \tag{4.3}$$

Then we have the following result.

Theorem 4.1. *Let*

(i) $\mid y_{p,q} - Y_{p,q} \mid = max \mid y_{i,j} - Y_{i,j} \mid$, *for* $1 \le i \le N$, $1 \le j \le M$,

(ii) $K(x,t,z,s) \in C(J \times J)$,

(iii) $\gamma = sup \mid K(x,t,z,s) \mid$ *for* $x, z \in [a,b]$, $t, s \in [c,d]$,

$$\eta = \mid w_{qq} w_{pp} K(x_p, t_q, x_p, t_q) \mid \text{ and}$$

$$\xi = (p-1)h(q-1)k + w_{pp}(q-1)k + w_{qq}(p-1)h.$$

Then

$$\mid y_{p,q} - Y_{p,q} \mid \le \frac{\mid O(h^\nu) \mid + \mid O(k^\mu) \mid}{\eta - \gamma\xi}. \tag{4.4}$$

Proof. By subtracting (4.3) from (4.2) for $i = p$ and $j = q$, we obtain:

$$\sum_{r=0}^{q} \sum_{l=0}^{p} w_{qr} w_{pl} K(x_p, t_q, x_l, t_r)(y_{l,r} - Y_{l,r}) + O(h^\nu) + O(k^\mu) = 0. \tag{4.5}$$

Thus

$$w_{qq} w_{pp} K(x_p, t_q, x_p, t_q)(y_{pq} - Y_{pq}) + \sum_{r=0}^{q-1} \sum_{l=0}^{p-1} w_{qr} w_{pl} K(x_p, t_q, x_l, t_r)(y_{l,r} - Y_{l,r})$$

$$+\sum_{r=0}^{q-1} w_{qr}w_{pp}K(x_p,t_q,x_p,t_r)(y_{p,r}-Y_{p,r})$$

$$+\sum_{l=0}^{p-1} w_{qq}w_{pl}K(x_p,t_q,x_l,t_q)(y_{l,q}-Y_{l,q})+O(h^{\nu})+O(k^{\mu})=0, \tag{4.6}$$

and from the assumptions (i) and (iii) it follows

$$\mid w_{qq}w_{pp}K(x_p,t_q,x_p,t_q)\mid\mid y_{pq}-Y_{pq}\mid\leq\gamma\mid y_{pq}-Y_{pq}\mid(p-1)h(q-1)k$$

$$+\gamma\mid y_{pq}-Y_{pq}\mid w_{pp}(q-1)k$$

$$+\gamma\mid(y_{pq}-Y_{pq})\mid w_{qq}(p-1)h+\mid O(h^{\nu})\mid+\mid O(k^{\mu})\mid. \tag{4.7}$$

In this inequality we used this fact that for every Newton-Cotes method we have $\sum_{i=0}^{s} w_{si} = sh$, where $h = s_{i+1} - s_i$.
Therefore, from (4.7) we obtain

$$\mid y_{p,q}-Y_{p,q}\mid\leq\frac{\mid O(h^{\nu})\mid+\mid O(k^{\mu})\mid}{\eta-\gamma\xi}. \tag{4.8}$$

So the theorem is proved. □

The following results can be obtained from the above theorem.

Corollary 4.2. *The approximate solution obtained by described multi-step method tends to the exact solution at the mesh points if h and k tends to zero.*

Corollary 4.3. *The order of convergence of the method is* $\min\{\nu, \mu\}$.

5. Numerical examples

In this section, we give some numerical examples to clarify the efficiency of the presented method. As mentioned previously, we shall use the 3/8-Simpson's rule as the main repeated formula combined with a 4-step and a 5-step end formulae over even and odd subintervals, respectively. We show the coefficients of 4-step and 5-step rules by α_i and β_i, respectively. By choosing the α_i and β_i such that the quadrature rules are exact for polynomials of degree at most three, we will have three arbitrary parameters, which with convergence considerations, we will obtain ([9]):

$$\begin{cases}\alpha_0=0.39740084,\\ \alpha_1=1.07706330,\\ \alpha_2=1.05107170,\\ \alpha_3=1.07706330,\\ \alpha_4=0.39740084.\end{cases}\quad and\quad\begin{cases}\beta_0=0.43284615,\\ \beta_1=1.02180185,\\ \beta_2=0.95933115,\\ \beta_3=1.24606743,\\ \beta_4=0.87843368,\\ \beta_5=0.46151978.\end{cases}$$

To implement the described method we need the starting values. It can be determined by one-step methods, such as Runge-Kutta methods. Here, we have used the exact starting values of solution.

Example 5.1. As the first example, consider the equation [15]:

$$\int_0^t \int_0^x (sin(x+z) + sin(t+s) + 3)y(z,s)dzds = f(x,t), \quad (x,t) \in [0,2] \times [0,2],$$

where $f(x,t)$ is selected such that the exact solution to be $y(x,t) = cos(x+t)$. In this example, we obtain the results of Table 1. To compare, we report the results of [15] in Table 2.

TABLE 1. Absolute errors of Example 5.1.

(x,t)	$h = k = 0.05$	$h = k = 0.025$	$h = k = 0.0125$
(1,1)	0.2337e -4	0.4858e -5	0.1198e -7
(1,2)	0.1568e -4	0.1373e -5	0.5574e -7
(2,1)	0.1568e -4	0.1373e -5	0.5574e -7
(2,2)	0.2106e -3	0.1797e -4	0.2095e -5

TABLE 2. Results of [15] with Eulers method for Example 5.1.

(x,t)	$h = k = 0.05$	$h = k = 0.025$	$h = k = 0.0125$
(1,1)	0.4060e -1	0.2270e -1	0.1130e -1
(1,2)	0.1230e -1	0.6760e -2	0.3530e -2
(2,1)	0.1230e -1	0.6760e -2	0.3530e -2
(2,2)	0.4060e -1	0.1940e -1	0.9460e -2

Example 5.2. Consider the following two-dimensional VIE for $(x,t) \in [0,1] \times [0,1]$

$$\int_0^t \int_0^x (xz^2 + cos(s))y(z,s)dzds = \frac{1}{4}x^5 - \frac{1}{4}x^5 cos(t) + \frac{1}{4}x^2 sin^2(t),$$

with exact solution $y(x,t) = xsin(t)$. Table 3 shows the numerical result for this example with $h = k = 0.01$.

TABLE 3. Computational results of Example 5.2 for $h = k = 0.01$.

(x,t)	$h = k = 0.01$
(0.1,0.1)	0.7880e -8
(0.3,0.2)	0.4585e -7
(0.4,0.9)	0.1667e -7
(0.5,0.6)	0.2546e -6
(0.7,0.2)	0.3969e -7
(0.8,0.8)	0.5226e -6
(1,1)	0.1244e -5

Example 5.3. Consider the third example in the form

$$\int_0^t \int_0^x (sin(tz)+1)y(z,s)dzds = \frac{x^2t^2+2sin(xt)-2xtcos(xt)}{2t^2}sin(t), \quad (x,t)\in[0,1]\times[0,1],$$

whose exact solution is $y(x,t) = xcos(t)$. Table 4 shows the absolute errors at the points $(x,t) = (2^{-i}, 2^{-i})$ for $i = 0, 1, ..., 7$.

TABLE 4. Computational results of Example 5.3 for $h, k = \frac{1}{2^7}$ at the points $(x,t) = (2^{-i}, 2^{-i})$.

$(x,t) = (2^{-i}, 2^{-i})$	$h = k = \frac{1}{2^7}$
$i = 0$	0.5027e -6
$i = 1$	0.6169e -6
$i = 2$	0.1929e -6
$i = 3$	0.1517e -6
$i = 4$	0.2254e -5
$i = 5$	0.1570e -4
$i = 6$	0.1906e -5
$i = 7$	0.2384e -6

Example 5.4. Consider the following 2D-NVIEs

$$\int_0^t \int_0^x \frac{6}{1+z+s}u^3(z,s)dzds = 2xt^3+\frac{1}{2}(6x^2+12x)t^2+2x^3t+6x^2t+6xt, \quad (x,t)\in[0,1]\times[0,1].$$

which has the exact solution as $u(x,t) = x+t+1$. To solve this equation, we substitute $y(x,t) = u^3(x,t)$ to get a linear equation.

$$\int_0^t \int_0^x \frac{6}{1+z+s}y(z,s)dzds = 2xt^3+\frac{1}{2}(6x^2+12x)t^2+2x^3t+6x^2t+6xt, \quad (x,t)\in[0,1]\times[0,1].$$

Then we apply the method presented in this paper to obtain the results of Table 5. According to this Table, we see that the results improve when the step-length h and k decrease.

Example 5.5. Finally, let us consider [14]:

$$\int_0^t \int_0^x 2e^{x+t}u^3(z,s)dzds = \frac{1}{9}(e^{x+t}-e^{4x+t}-e^{x+7t}+e^{4x+7t}), \quad (x,t)\in[0,1]\times[0,1].$$

The exact solution is $u(x,t) = e^{x+2t}$. Similar to the previous example, we substitute $y(x,t) = u^3(x,t)$ to get a linear equation.
Table 6 shows the absolute errors at some points using the presented method together with the results obtained by the method of [14] .

Comparing the results listed in this table, shows the high accuracy of the presented method with respect to the method of [14]

TABLE 5. Computational results of Example 5.4 for different values of h, k.

(x,t)	$h=k=0.05$	$h=k=0.01$
(0.1,0.1)	0.3715e -2	0.5400e -6
(0.2,0.3)	0.4380e -3	0.4086e -6
(0.4,0.7)	0.1592e -3	0.9868e -6
(0.5,0.5)	0.7246e -6	0.5612e -6
(0.6,0.3)	0.1084e -3	0.5864e -6
(0.8,0.8)	0.1019e -5	0.7435e -6
(1,0.9)	0.6840e -6	0.2859e -6

TABLE 6. Computational results of Example 5.5.

$(x,t)=(2^{-i},2^{-i})$	Present Method with $h=k=\frac{1}{2^7}$	Method of [14] with $m=64$
$i=1$	0.1295e -4	0.1000
$i=2$	0.3583e -5	0.4600e -1
$i=3$	0.4207e -5	0.2900e -1
$i=4$	0.1060e -2	0.2300e -1
$i=5$	0.2038e -3	0.2000e -1
$i=6$	0.3896e -2	0.3100e -1

6. Conclusion

In this paper, we extended multi-step methods to solve two-dimensional nonlinear Volterra integral equations (2D-NVIEs) of the first kind. We converted a 2D-NVIE of the first kind to a one-dimensional VIE of the first kind and then we solved the resulted equations by using multi-step methods. The numerical results confirm the convergence and stability of the method. It seems that the presented method can be applied to other equations such as integro-differential equations.

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