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Numerical solution of linear control systems using interpolation scaling functions

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Abstract The current paper proposes a technique for the numerical solution of linear control systems. The method is based on Galerkin method, which uses the interpolating scaling functions. For a highly accurate connection between functions and their derivatives, an operational matrix for the derivatives is established to reduce the problem to a set of algebraic equations. Several test problems are given, and the numerical results are reported to show the accuracy and efficiency of this method.

Keywords. Linear control systems, Galerkin method, Interpolating scaling functions, operational matrix.

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1. Introduction

The main purpose of this manuscript is to study numerical method based on interpolating scaling functions for the solution of the following linear optimal control problem (OCP)

$$\dot{x} = Ax(t) + Bu(t), \quad x(t_0) = x_0,$$
$$J = \frac{1}{2}x(t_f)^T Sx(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left(x^T Px + 2x^T Qu + u^T Ru\right) dt, \tag{1.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times n}$. The control $u(t)$ is an admissible control if it is piecewise continuous in t for $t \in [t_0, t_f]$. These values belong to a given closed subset U of $\mathbb{R}^+$. The input $u(t)$ is derived by minimizing the quadratic performance index J, where $S \in \mathbb{R}^{n\times n}$, $P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times m}$ are positive semi-definite matrices and $R \in \mathbb{R}^{m\times m}$ is positive definite matrix.

Optimal control theory are encountered in various fields such as engineering, economics, aerospace, chemical engineering, robotic and finance. We know, it is difficult to solve generally optimal control problems. Thus, the key to solve many of these real

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world problems are numerical methods. In order to solve linear quadratic OCPs, various numerical approaches are proposed by researchers. Yousefi et al. presented the He's variational iteration method [23] for the linear optimal control problem. Also see [7] for the use of the Adomian decomposition method for solving this equation. In [5], homotopy perturbation method was applied to solve optimal control problems. Also some other methods are used for solving this problem to transform the new problem such as converting the problem to differential inclusion form [11], or measure space and then solved in [4], genetic algorithm, and Others deal with the optimal control problem directly. For example see [6, 8, 9, 10, 16, 20, 22] and the references therein.

Over the last decade, wavelets have found applications in numerous areas of mathematics, engineering, computer science, statistics, physics, etc [19]. Multiwavelets are revealed to possess several advantages with respect to scalar wavelets. The reason of their success is due to the fact that, unlike scalar wavelets, multiwavelets can be constructed with several simultaneous properties, such as orthogonality, symmetry, having high numbers of vanish moments and closed form [18, 17]. In this work, we use the interpolating scaling functions which are introduced by Alpert [1, 2]. In addition to the simultaneous properties which proposed for multiwavelets, interpolating scaling functions have interpolating property. This feature reduces the time and computation cost. Also operational matrix of derivative is derived in [14, 3] helps to save computing time. So, we use the interpolating scaling functions to solve Eq. (1.1). The outline of this paper is as follows. In Section 2, we describe the basic formulation of the interpolating scaling functions required for our subsequent development. In Section 3 the proposed method is used to approximate the solution of the problem. As a result a set of algebraic equations are formed and a solution of the considered problem is introduced. In Section 4, we report our numerical findings and demonstrate the accuracy of the proposed numerical scheme.

2. The interpolating scaling functions

Suppose P_r be the Legendre polynomial of order r where r is a fixed nonnegative integer number, and let τ_k for $k = 0, \cdots, r-1$ denote the roots of P_r. The interpolating scaling functions (ISF) are given by [14]

$$\phi^k(t) := \begin{cases} \sqrt{\frac{2}{\omega_k}} L_k(2t-1), & t \in [0,1], \\ 0, & \text{otherwise}, \end{cases}$$

where ω_k are the Gauss–Legendre quadrature weights given as

$$\omega_k = \frac{2}{r P'_r(\tau_k) P_{r-1}(\tau_k)},$$

and $L_k(t)$ are the Lagrange interpolating polynomials given as [15]

$$L_k(t) = \prod_{i=0,\ i\neq k}^{r-1} \left(\frac{t-\tau_i}{\tau_k-\tau_i} \right).$$

for $k = 0, \cdots, r-1$. Now we can expand any polynomials g on $[0,1)$ of degree less than r by using the functions $\phi^0, \cdots, \phi^{r-1}$ as

$$g(t) = \sum_{k=0}^{r-1} d_k \phi^k(t),$$

where the coefficients d_k are given by

$$d_k = \sqrt{\frac{\omega_k}{2}} g(\hat{\tau}_k), \qquad k = 0, \cdots, r-1,$$

where

$$\hat{\tau}_k = \frac{\tau_k + 1}{2}.$$

Let

$$\phi_{nl}^k(t) := 2^{\frac{n}{2}} \phi^k(2^n t - l), \quad k = 0, \cdots, r-1, \; l = 0, \cdots, 2^n - 1, \tag{2.1}$$

where n is a fixed nonnegative integer number, then we have the following orthonormality conditions [14]

$$\int_0^1 \phi_{nl}^k(t) \phi_{n\acute{l}}^{\acute{k}}(t) dt = \delta_{l\acute{l}} \delta_{k\acute{k}}, \quad k, \acute{k} = 0, \cdots, r-1, \quad l, \acute{l} = 0, \cdots, 2^n - 1.$$

2.1. **The function approximation.** For any two fixed nonnegative integer numbers r and n, the function $f(t) \in L^2[0,1]$ represented by ISF expansion as

$$f(t) \approx \sum_{k=0}^{r-1} \sum_{l=0}^{2^n-1} s_{nl}^k \phi_{nl}^k(t) = S^T \Phi(t), \tag{2.2}$$

where

$$S = \left[s_{n0}^0, \ldots, s_{n0}^{r-1} | s_{n1}^0, \ldots, s_{n1}^{r-1} | \ldots | s_{n,2^n-1}^0, \ldots, s_{n,2^n-1}^{r-1}\right]^T, \tag{2.3}$$
$$\Phi(t) = \left[\phi_{n0}^0(t), \ldots, \phi_{n0}^{r-1}(t) | \phi_{n1}^0(t), \ldots, \phi_{n1}^{r-1}(t) | \ldots | \phi_{n,2^n-1}^0(t), \ldots, \phi_{n,2^n-1}^{r-1}(t)\right]^T,$$

and the coefficients c_{nl}^k are computed as

$$s_{nl}^k = \int_0^1 f(t) \phi_{nl}^k(t) dt = \int_{h_l}^{h_{l+1}} f(t) \phi_{nl}^k(t) dt,$$

where

$$h_l = \frac{l}{2^n}, \qquad l = 0, \cdots, 2^n - 1.$$

These coefficients are computed by using Gauss-Legendre quadrature as

$$s_{nl}^k = 2^{-n/2} \sqrt{\frac{\omega_k}{2}} f(2^{-n}(\hat{\tau}_k + l)), \; k = 0, \ldots, r-1, \; l = 0, \ldots, 2^n - 1. \tag{2.4}$$

Also the function $g(x,t) \in L^2([0,1] \times [0,1])$, represented by ISF expansion as

$$g(x,t) \approx \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \Phi_i(x) \Phi_j(t) = \Phi^T(x) G \Phi(t), \tag{2.5}$$

where G is an $N \times N$ matrix as

$$G = \begin{bmatrix} g_{11} & \cdots & g_{1N} \\ \vdots & & \vdots \\ g_{N1} & \cdots & g_{NN} \end{bmatrix},$$

where $N = r2^n$ and

$$g_{i,j} = \int_0^1 \int_0^1 g(x,t)\Phi_i(t)\Phi_j(x)dtdx, \quad i,j = 1,2,\ldots,N,$$

so that, by applyeing the method of [14] we get

$$g_{i,j} = 2^{-n}\sqrt{\frac{\omega_k}{2}}\sqrt{\frac{\omega_{k'}}{2}} g\left(2^{-n}(\hat{\tau}_k + l), 2^{-n}(\hat{\tau}_{k'} + l')\right), \tag{2.6}$$

where $l = \frac{i-k-1}{r}$ and $l' = \frac{j-k'-1}{r}$, $k, k' = 0, 1, \cdots, r-1$.

2.2. **The operational matrix of the derivative.** Suppose that the derivative of $f(t)$ in (2.2) given by

$$\frac{d}{dt} f(t) = \sum_{k=0}^{r-1} \sum_{l=0}^{2^n-1} \tilde{s}_{nl}^k \phi_{nl}^k(t) = \tilde{S}^T \Phi(t), \tag{2.7}$$

where $\tilde{S}$ is a vector defined similarly to (2.3). We obtain the relation between S and $\tilde{S}$ by

$$\tilde{S} = DS, \tag{2.8}$$

where D is the operational matrix of the derivatives [12] for the ISFs. Using Eq. (2.7) we get $\tilde{s}_{nl}^k$ as

$$\tilde{s}_{nl}^k = \int_{h_l}^{h_{l+1}} \phi_{nl}^k(t) \left(\frac{d}{dt} f(t)\right) dt, \ k = 0, \ldots, r-1, \ l = 0, \ldots, 2^n - 1.$$

By integration by parts from the above integral, we get

$$\tilde{s}_{nl}^k = \left[f(t)\phi_{nl}^k(t)\right]_{h_l}^{h_{l+1}} - \int_{h_l}^{h_{l+1}} f(t) \left(\frac{d}{dt}\phi_{nl}^k(t)\right) dt.$$

Using (2.1) and (2.2), we get

$$\tilde{s}_{nl}^k = 2^{(n/2)} \left[f(h_{l+1})\phi^k(1) - f(h_l)\phi^k(0)\right] - 2^n \sum_{i=0}^{r-1} q_{ki} s_{nl}^i, \tag{2.9}$$

where

$$q_{ki} = \int_0^1 \phi^i(t) \left(\frac{d}{dt}\phi^k(t)\right) dt, \quad k, i = 0, 1, \cdots, r-1.$$

Employing the Gauss-Legendre quadrature formula, we obtain

$$q_{ki} = \sqrt{\frac{\omega_i}{2}} \frac{d}{dt}\phi^k(\hat{\tau}_i), \quad k, i = 0, 1, \cdots, r-1.$$

To evaluate $f(h_l)$ and $f(h_{l+1})$ we use the average of left and right limits on h_l as

$$f(h_l)=\frac{1}{2}\left(\sum_{i=0}^{r-1}s_{n,l-1}^{i}\phi_{n,l-1}^{i}(h_l)+\sum_{i=0}^{r-1}s_{nl}^{i}\phi_{nl}^{i}(h_l)\right),\qquad l=1,\cdots,2^n-1. \tag{2.10}$$

Using (2.1), we can express Eq. (2.10) as

$$f(h_l)=2^{\frac{n}{2}}\frac{1}{2}\left(\sum_{i=0}^{r-1}s_{n,l-1}^{i}\phi^{i}(1)+\sum_{i=0}^{r-1}s_{nl}^{i}\phi^{i}(0)\right),\qquad l=1,\cdots,2^n-1. \tag{2.11}$$

Also, to evaluate the values of function f at the points h_0 and h_{2^n} we have

$$f(h_0)=\sum_{i=0}^{r-1}s_{n0}^{i}\phi_{n0}^{i}(h_0)=2^{\frac{n}{2}}\sum_{i=0}^{r-1}s_{n0}^{i}\phi^{i}(0), \tag{2.12}$$

$$f(h_{2^n})=\sum_{i=0}^{r-1}s_{n,2^n-1}^{i}\phi_{n,2^n-1}^{i}(h_{2^n})=2^{\frac{n}{2}}\sum_{i=0}^{r-1}s_{n,2^n-1}^{i}\phi^{i}(1). \tag{2.13}$$

Substituting (2.11)–(2.13) in Eq. (2.9), we obtain

$$\begin{aligned}
\tilde{s}_{n0}^{k}&=2^n\left[\sum_{i=0}^{r-1}\left(\frac{1}{2}\phi^{i}(1)\phi^{k}(1)-\phi^{i}(0)\phi^{k}(0)-q_{ki}\right)s_{n0}^{i}+\sum_{i=0}^{r-1}\frac{1}{2}\phi^{i}(0)\phi^{k}(1)s_{n1}^{i}\right],\\
\tilde{s}_{nl}^{k}&=2^n\left[\sum_{i=0}^{r-1}\left(-\frac{1}{2}\phi^{i}(1)\phi^{k}(0)\right)s_{n,l-1}^{i}+\sum_{i=0}^{r-1}\left(\frac{1}{2}\phi^{i}(1)\phi^{k}(1)-\frac{1}{2}\phi^{i}(0)\phi^{k}(0)-q_{ki}\right)s_{nl}^{i}\right.\\
&\quad\left.+\sum_{i=0}^{r-1}\frac{1}{2}\phi^{i}(0)\phi^{k}(1)s_{n,l+1}^{i}\right],\quad l=0,\cdots,2^n-2,\\
\tilde{s}_{n,2^n-1}^{k}&=2^n\left[\sum_{i-0}^{r-1}-\frac{1}{2}\phi^{i}(1)\phi^{k}(0)s_{n,2^n-2}^{i}+\sum_{i=0}^{r-1}\left(\phi^{i}(1)\phi^{k}(1)-\frac{1}{2}\phi^{i}(0)\phi^{k}(0)-q_{ki}\right)s_{n,2^n-1}^{i}\right].
\end{aligned}$$

From the above equations, the matrix D can be expressed as a block tridiagonal matrix as

$$D=2^n\begin{bmatrix}
\underline{R} & H & & & & \\
-H^T & R & H & & & \\
 & \ddots & \ddots & \ddots & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & -H^T & R & H\\
 & & & & -H^T & \overline{R}
\end{bmatrix},$$

where, each block is an $r\times r$ matrix and for $k,i=1,...,r$, we have

$$[\underline{R}]_{ki}=\frac{1}{2}\phi^i(1)\phi^k(1)-\phi^i(0)\phi^k(0)-q_{ki},$$
$$[R]_{ki}=\frac{1}{2}\phi^i(1)\phi^k(1)-\frac{1}{2}\phi^i(0)\phi^k(0)-q_{ki},$$
$$[\overline{R}]_{ki}=\phi^i(1)\phi^k(1)-\frac{1}{2}\phi^i(0)\phi^k(0)-q_{ki},$$
$$[H]_{ki}=\frac{1}{2}\phi^i(0)\phi^k(1).$$

Since Eqs. (2.9)–(2.13) are exact for polynomials up to degree $r-1$ so the operational matrix of the derivative is exact for polynomials up to degree $r-1$.

3. Description of the numerical methods

Let us begin to solve the following linear optimal control problem (OCP). For this purpose, we consider Pontryagin's maximum principle (PMP) for system (1.1) and achieve the optimal control law $u^*(t)=-k(t)x(t)$. According to the PMP, one has

$$\dot{\lambda}=-\frac{\partial H}{\partial x}=-Px-Qu-A^T\lambda, \tag{3.1}$$
$$\frac{\partial H}{\partial u}=Q^Tx+Ru+B^T\lambda=0, \tag{3.2}$$

where H is Hamiltonian of the system (1.1), so

$$H(x,u,\lambda,t)=\frac{1}{2}\left(x^TPx+2x^TQu+u^TRu\right)+\lambda^T\left(Ax+Bu\right), \tag{3.3}$$

also $\lambda\in\mathbb{R}^n$ is co-state vector. The optimal control is computed by

$$u^*=-R^{-1}Q^Tx-R^{-1}B^T\lambda, \tag{3.4}$$

where λ and x are the solution of the following Hamiltonian system

$$\begin{cases}\dot{x}=[A-BR^{-1}Q^T]x-BR^{-1}B^T\lambda,\\ \dot{\lambda}=[-P+QR^{-1}Q^T]x+[QR^{-1}B^T-A^T]\lambda,\end{cases} \tag{3.5}$$

with the initial condition $x(t_0)=x_0$ and the terminal condition $\lambda(t_f)=Sx(t_f)$ [13]. This system of equation is linear, so we obtain

$$\begin{pmatrix}x\\ \lambda\end{pmatrix}=\begin{pmatrix}F(t,t_f) & G(t,t_f)\\ L(t,t_f) & M(t,t_f)\end{pmatrix}\begin{pmatrix}x(t_f)\\ \lambda(t_f)\end{pmatrix}, \tag{3.6}$$

where F, G, L and M are $n\times n$ matrices. Applying the terminal condition and so

$$\begin{aligned}x(t)&=(F+GS)x(t_f),\\ \lambda(t)&=(L+MS)x(t_f).\end{aligned} \tag{3.7}$$

If $F+GS$ is invertible, we get

$$\lambda(t)=\underbrace{(L+MS)(F+GS)^{-1}}_{Y(t,t_f)}x(t). \tag{3.8}$$

After differentiating with respect to the t from Eq.(3.8), we have

$$\dot{\lambda}(t) = \dot{Y}(t,t_f)x(t) + Y(t,t_f)\dot{x}(t). \tag{3.9}$$

Using equations (3.5) and (3.8) from [13], one can write

$$\dot{Y} = (YB+Q)R^{-1}(B^TY+Q^T) - YA - A^TY - P, \tag{3.10}$$

also by applying the optimal control law (3.4), we have

$$u^*(t) = -R^{-1}Q^Tx - R^{-1}B^TY(t,t_f)x(t). \tag{3.11}$$

According to equation (3.8) the terminal conditions for equation (3.10) are

$$\begin{array}{ll} F(t_f,t_f) = I, & G(t_f,t_f) = 0, \\ L(t_f,t_f) = 0, & M(t_f,t_f) = I. \end{array} \tag{3.12}$$

By considering the following variables

$$\begin{aligned} V(t) &= F(t,t_f) + G(t,t_f)S, \\ W(t) &= L(t,t_f) + M(t,t_f)S, \end{aligned} \tag{3.13}$$

and substituting these new variables into (3.7) and then into (3.5), one has

$$\begin{cases} \dot{V}(t) = \left(A - BR^{-1}Q^T\right)V(t) - BR^{-1}B^TW(t), \\ \dot{W}(t) = \left(-P + QR^{-1}Q^T\right)V(t) + \left(QR^{-1}B^T - A^T\right)W(t), \end{cases} \tag{3.14}$$

with conditions $V(t_f) = I$ and $W(t_f) = S$.

Assume that we expand $V(t)$ and $W(t)$ using interpolating scaling functions as

$$\begin{aligned} V(t) &\approx \mathbf{V}\Phi(t), \\ W(t) &\approx \mathbf{W}\Phi(t), \end{aligned} \tag{3.15}$$

where $\mathbf{V}$ and $\mathbf{W}$ are the $(n \times N)$ unknown vector. By differentiating from both sides of Eq. (3.15), and using Eq. (2.10), we get

$$\begin{aligned} \dot{V}(t) &\approx \mathbf{V}D\Phi(t), \\ \dot{W}(t) &\approx \mathbf{W}D\Phi(t). \end{aligned} \tag{3.16}$$

Now replacing (3.15) and (3.16) in (3.14) yields

$$\begin{cases} \left[\mathbf{V}D - \left(A - BR^{-1}Q^T\right)\mathbf{V} + BR^{-1}B^T\mathbf{W}\right]\Phi(t) = 0, \\ \left[\mathbf{V}D - \left(-P + QR^{-1}Q^T\right)\mathbf{V} - \left(QR^{-1}B^T - A^T\right)\mathbf{W}\right]\Phi(t) = 0. \end{cases} \tag{3.17}$$

The entries of vector $\Phi(t)$ are independent, so we obtain

$$\begin{cases} \mathbf{V}D - \left(A - BR^{-1}Q^T\right)\mathbf{V} + BR^{-1}B^T\mathbf{W} = 0, \\ \mathbf{V}D - \left(-P + QR^{-1}Q^T\right)\mathbf{V} - \left(QR^{-1}B^T - A^T\right)\mathbf{W} = 0, \end{cases} \tag{3.18}$$

Eq. (3.18) gives a system of linear equations with $n \times N$ equations and $n \times N$ unknowns. This system of equations can be solved to find $\mathbf{V}$ and $\mathbf{W}$. So the unknown function $u(t)$ and $x(t)$ may be found using Eq. (3.15).

4. Numerical Experiments

In this section, some numerical examples are presented to illustrate the validity and the merits of the new technique. We report l_∞ error of the solution that is defined as

$$L_\infty = \max_{0\le i\le 10} |u_i - \tilde{u}_i|,$$

where u_i and $\tilde{u}_i$ are the exact and computed values of the solution u at the points $t_i = \frac{i}{10}$, $i = 0, 1, \cdots, 10$, respectively.

Example 4.1. Consider a single-input scalar system as follows:

$$\begin{aligned} &\dot{x} = -x(t) + u(t),\\ &J = 1/2\int_0^1 (x^2(t) + u^2(t))dt, \end{aligned} \tag{4.1}$$

with initial condition

$$x(0) = 1. \tag{4.2}$$

The exact solution of this problem is [13]

$$\begin{cases} x(t) = \cosh(\sqrt{2}t) + \beta \sin h(\sqrt{2}t),\\ u(t) = (1+\sqrt{2}\beta)\cosh(\sqrt{2}t) + (\sqrt{2}+\beta)\sinh(\sqrt{2}t), \end{cases}$$

where

$$\beta = -\frac{\cosh(\sqrt{2}t) + \sqrt{2}\sinh(\sqrt{2}t)}{\sqrt{2}\cosh(\sqrt{2}t) + \sinh(\sqrt{2}t)}.$$

After using the method proposed in the previous section, we obtain

$$\begin{cases} \dot{V}(t) = -V(t) - W(t),\\ \dot{W}(t) = -V(t) + W(t),\\ V(0) = 1, \quad W(1) = 0, \end{cases}$$

where, the following optimal control law may be computed by

$$u^*(t) = -W(t).$$

Table 1, 2 consist of L_∞ norm of example 1 for $N = \{6, 8\}$. Figure 1 shows the plot of approximate solutions of Eq (4.1) also Figure 2 demonstrates the plot of absolute errors.

Table 1. Ł$_\infty$ error of $x(t)$ for various values of N for Example (4.1).

t	0	0.2	0.4	0.6	0.8	1
$N=6$	0	4.68e-6	3.62e-6	3.05e-6	2.77e-6	2.32e-6
$N=8$	0	1.78e-8	1.44e-8	1.22e-8	1.09e-8	9.60e-9

TABLE 2. L_∞ error of $\lambda(t)$ for various values of N for Example (4.1).

t	0	0.2	0.4	0.6	0.8	1
$N = 6$	1.39e-6	6.80e-7	4.27e-8	7.91e-7	1.68e-6	0
$N = 8$	5.46e-9	2.58e-9	4.13e-10	1.88e-9	5.0e-9	0

FIGURE 1. Plot of approximation solutions for example 1 (left $x(t)$) (right $\lambda(t)$)

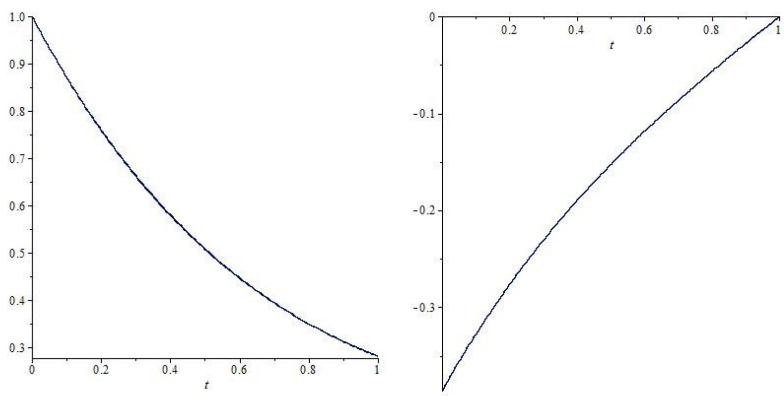
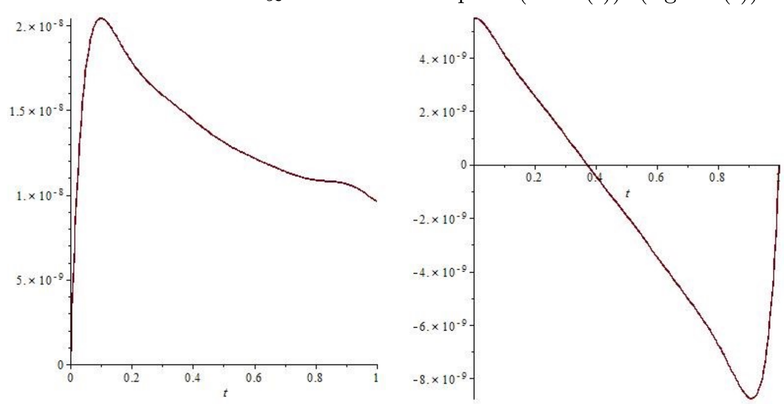


FIGURE 2. Plot of L_∞ errors for example 1 (left $x(t)$) (right $\lambda(t)$)



Example 4.2. Consider a single-input scalar system as follows:

$$\dot{x}(t) = u(t),$$

$$J = \int_0^1 (x^2(t) + u^2(t))dt, \tag{4.3}$$

with the initial condition

$$x(0) = 0. \tag{4.4}$$

For this example, the analytical solutio is [21]

$$\begin{cases} x(t) = \frac{e(e^{t}-e^{(}-t))}{2(e^2-1)}, \\ u(t) = \frac{e(e^{t}+e^{(}-t))}{2(e^2-1)}. \end{cases}$$

Therefore, we should have

$$\begin{cases} \dot{V}(t) = -\frac{1}{2}W(t), \\ \dot{W}(t) = -2V(t), \\ V(0) = 0, \quad W(1) = 0. \end{cases}$$

Also, we can obtain the following optimal control law

$$u^*(t) = -W/2,$$

Table 3, 4 consist of L_∞ norm of example 2 for $N = \{6, 8\}$. Figure 3 shows the plot of approximate solutions of Eq (4.3) also Figure 4 demonstrates the plot of absolute errors.

Table 3. Ł$_\infty$ error of $x(t)$ for various values of N for Example (4.3).

t	0	0.2	0.4	0.6	0.8	1
$N = 6$	0	4.12e-7	3.08e-7	2.33e-7	1.72e-7	1.01e-7
$N = 8$	0	7.96e-10	6.25e-10	4.78e-10	3.50e-10	2.24e-10

Table 4. Ł$_\infty$ error of $\lambda(t)$ for various values of N for Example (4.3).

t	0	0.2	0.4	0.6	0.8	1
$N = 6$	5.37e-7	4.78e-7	4.05e-7	3.50e-7	3.19e-6	0
$N = 8$	1.07e-9	9.29e-10	7.88e-10	6.79e-10	5.97e-10	0

5. Conclusion

In this paper, we presented a numerical scheme for solving optimal control problems. This technique is based on interpolating scaling functions and Galerkinmethod. The method tested on several examples taken from the literature to observe the efficiency of the new technique. The numerical results given in the previous section demonstrate the accuracy of this scheme. The obtained results show that this techniques can solve the problem effectively. Because of interpolating property of scaling functions, this system of equations are solved rapidly by using this method. We used Maple to solve this system of equations.

FIGURE 3. Plot of approximation solutions for example 2 (right $x(t)$) (left $\lambda(t)$)

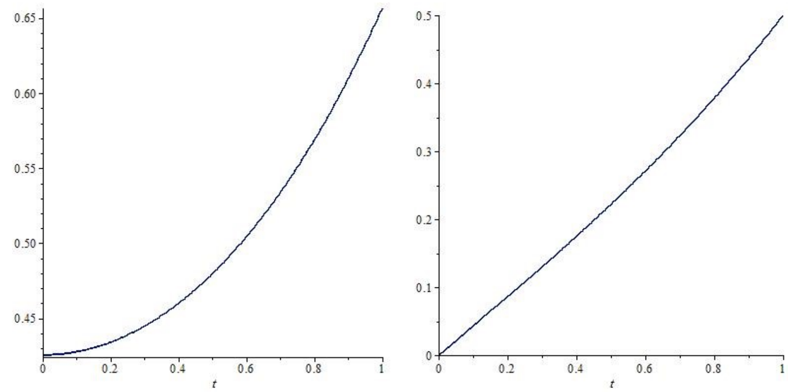
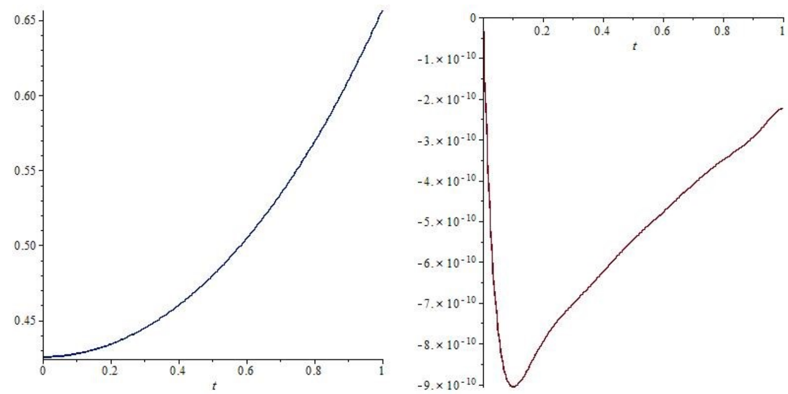


FIGURE 4. Plot of L_∞ errors for example 2 (right $x(t)$) (left $\lambda(t)$)



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